LASER INTERFEROMETER GRAVITATIONAL WAVE OBSERVATORY
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CALIFORNIA INSTITUTE OF TECHNOLOGY
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Document Type **LIGO-T960040-00- D** Mar. 11, 96

RESPONSE OF PENDULUM TO MOTION OF SUSPENSION POINT

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Distribution of this draft:

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1 HORIZONTAL MOTION OF SUSPENSION POINT

Let us think about the pendulum system shaken horizontally as shown in Fig. 1 The mass (M in mass) is suspended by the wire (l in length) at the release point above the center of mass by d, and x_1, x_2, θ, and θ' denotes the horizontal displacement of the suspension point, horizontal displacement of the center of the mass, the pitch angle of the mass, and the pitch angle of the wire, respectively. The mass is assumed to be pseudo-critically damped by the suspension's sensor.

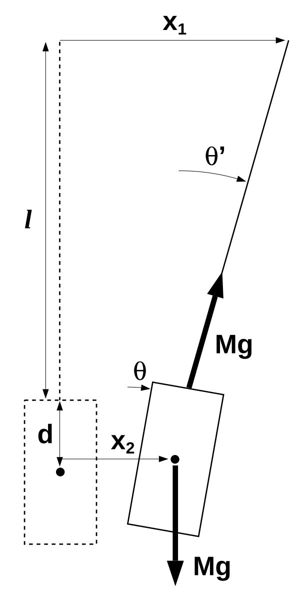

Figure 1: Response of the pendulum system to the horizontal motion of the suspension point.

The equation of motion of the mass for x_2 and θ are, respectively:

$$g\theta' - \tilde{k}_x \cdot \frac{d}{dt}(x_2 - x_1) = \frac{d^2 x_2}{dt^2}$$

$$g(\theta' - \theta)d - \tilde{k}_\theta \cdot \frac{d\theta}{dt} = I \cdot \frac{d^2\theta}{dt^2},$$

where I is a normalized moment of inertia (moment of inertia divided by mass), g is the acceleration of gravity, $\tilde{k}_x$ and $\tilde{k}_\theta$ are low-pass-filter operators for damping force along x and θ, respectively, and $\theta' = \dfrac{x_1 - (x_2 + d\theta)}{l}$ is the geometrical relation. It should be noted that the damping term for x comes from x_2-x_1, whereas that for θ comes from only θ, because the suspension's sensors moves together with the suspension point. The wire resonances are not incorporated here.

By taking the Laplace transform on the equations, we obtain the following transfer functions:

$$\frac{X_2}{X_1} = \frac{Il K_x s^3 + (Ig + lK_xK_\theta)s^2 + g(ldK_x + d^2K_x + K_\theta)s + dg^2}{Ils^4 + l(IK_x + K_\theta)s^3 + (Ig + dgl + d^2g + lK_xK_\theta)s^2 + g(ldK_x + d^2K_x + K_\theta)s + dg^2} ,$$

$$\frac{\Theta}{X_1} = \frac{dgs^2}{Ils^4 + l(IK_x + K_\theta)s^3 + (Ig + dgl + d^2g + lK_xK_\theta)s^2 + g(ldK_x + d^2K_x + K_\theta)s + dg^2} ,$$

where $X_1(s)$, $X_2(s)$, $\Theta(s)$, $K_x(s)$, and $K_\theta(s)$ are the Laplace transforms of $x_1(t)$, $x_2(t)$, $\theta(t)$, $\tilde{k}_x(t)$, and $\tilde{k}_\theta(t)$, respectively.

The parameters in our case are:

- $I = \frac{D^2}{16} + \frac{L^2}{12} = 4.740 \times 10^{-3} m^2$ (D=0.25m: diameter, L=0.1m: Thickness)
- $l = 0.45 m$ (The pendulum frequency is 0.74Hz.)
- $d = 0.0068 m$ (The pitch frequency is 0.60 Hz.)
- $K_x = \frac{a}{(s-p_1)(s-p_2)\dots(s-p_n)}$;

  $a = 8.980 \times 10^{16}$, $p_n = -1.688 \pm 75.19j$, $-4.898 \pm 67.83j$, $-7.634 \pm 53.83j$, $-9.623 \pm 34.56j$, $10.66 \pm 11.91j$, (Chebychev 10 pole, 1 dB, 12Hz; Gain at DC is 4 for pseudo-critical damping.)
- $K_\theta = \frac{a}{(s-p_1)(s-p_2)\dots(s-p_n)}$;

  $a = 1.122 \times 10^{14}$, $p_n = -1.688 \pm 75.19j$, $-4.898 \pm 67.83j$, $-7.634 \pm 53.83j$, $-9.623 \pm 34.56j$, $-10.66 \pm 11.91j$, (Chebychev 10 pole, 1 dB, 12Hz; Gain at DC is 0.005 for pseudo-critical damping.)

Because of the extreme complexity of the equations, the transfer function was calculated using the “Block diagram method” in simulink of matlab. Fig. 2 shows the block diagram which reflects the two equations of motion for x and θ. Inport, Outport, and Outport1 represent X_1, X_2, and Θ, respectively. Θ is multiplied by 0.01 (1cm) at the last stage to produce the cavity length change.

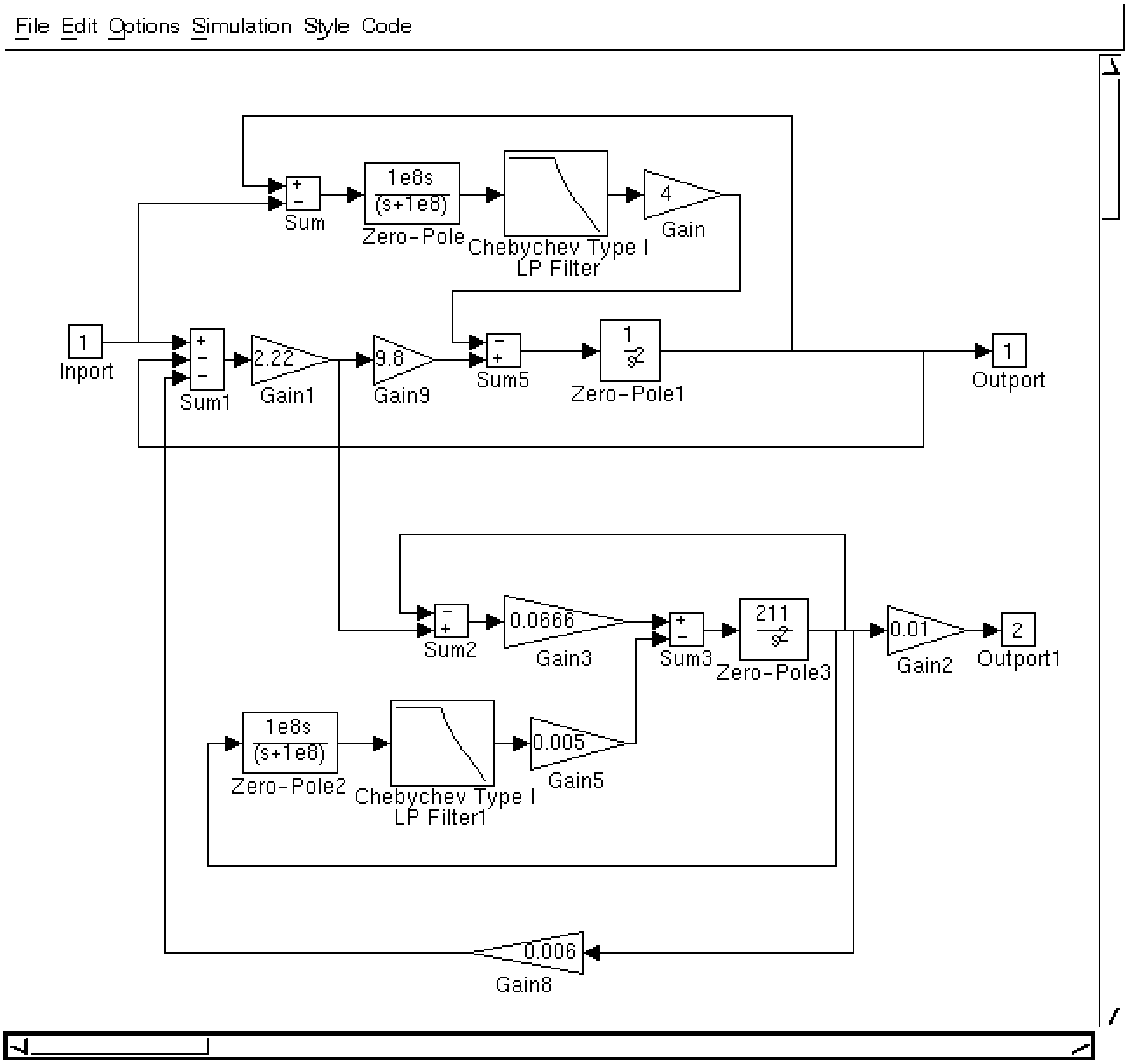

Figure 2: Block diagram reflecting the equation of motion of the pendulum system with horizontal motion of the suspension point.

The two transfer functions are shown in Fig. 3. It should be noted that the shallow slope below 12 Hz is caused by the suspension's sensor which moves as x_1.

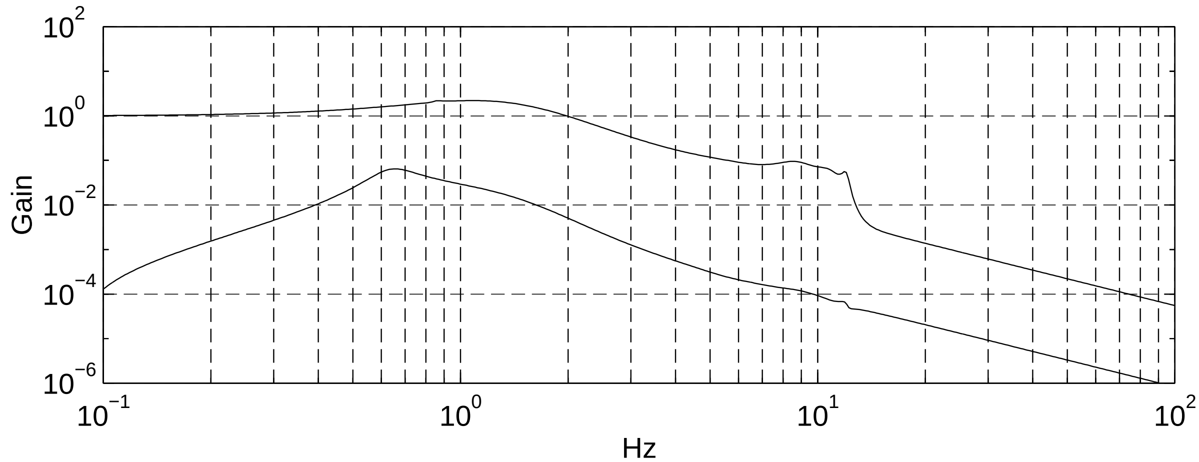

Figure 3: Transfer function from the horizontal displacement of the suspension position to the cavity length variation via the horizontal motion (Upper) and via the pitch angle change with assumed d of 1 cm (Lower).

2 VERTICAL MOTION OF THE SUSPENSION POINT

Let us think about the pendulum system shaken vertically as shown in Fig. 4. The mass (M in mass) is suspended by the wire (*l* in length) at the release point above the center of mass by d and horizontally off the center of mass by r. The mass is then held up right by the torque Mgr. z_1, z_2, θ, and MT denotes the vertical displacement of the suspension point, vertical displacement of the center of the mass, the pitch angle of the mass, and the tension of the wire, respectively. The mass is assumed to be pseudo-critically damped in pitch angle by the suspension's sensor. It should be noted that the center of mass doesn't move horizontally to the first order, because the wire is up straight to the first order.

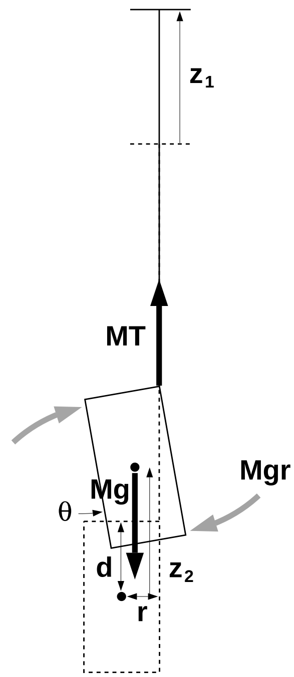

Figure 4: Response of the pendulum system to the vertical motion of the suspension point.

The equation of motion of the system for z_2 and θ are, respectively:

$$T - g = \frac{d^2 z_2}{dt^2}$$

$$(r - \theta d)T - gr - \tilde{k}_\theta \cdot \frac{d\theta}{dt} = I \cdot \frac{d^2\theta}{dt^2},$$

where I is a normalized moment of inertia (moment of inertia divided by mass), g is acceleration of gravity, $\tilde{k}_\theta$ is a low-pass-filter operator for damping force along θ, respectively, and $z_2 = z_1 - \theta r$ is the geometrical relation.

By taking the Laplace transform on them and approximating the convolution term as $\Theta(s) \otimes T(s) \approx \Theta(s) \cdot g$, we obtain the following transfer functions:

$$\frac{Z_2}{Z_1} = \frac{Is^2 + K_\theta s + dg}{(I + r^2)s^2 + K_\theta s + dg},$$

$$\frac{\Theta}{Z_1} = \frac{rs^2}{(I + r^2)s^2 + K_\theta s + dg},$$

where $Z_1(s)$, $Z_2(s)$, $\Theta(s)$, and $K_\theta(s)$ are Laplace transforms of $z_1(t)$, $z_2(t)$, $\theta(t)$, and $\tilde{k}_\theta(t)$, respectively.

The actual transfer functions should also contain a pair of poles due to the vertical resonance of the pendulum, the resonance frequency of which is $\omega_v = \sqrt{\frac{AY}{Ml}}$, where A is the cross section of the wire and Y is Young's modulus of the wire. Therefore the overall transfer functions are:

$$\left.\frac{Z_2}{Z_1}\right|_{total} = \frac{Is^2 + K_\theta s + dg}{(I + r^2)s^2 + K_\theta s + dg} \cdot \frac{\omega_v^2}{s^2 + \frac{\omega_v}{Q_v}s + \omega_v^2},$$

$$\left.\frac{\Theta}{Z_1}\right|_{total} = \frac{rs^2}{(I + r^2)s^2 + K_\theta s + dg} \cdot \frac{\omega_v^2}{s^2 + \frac{\omega_v}{Q_v}s + \omega_v^2},$$

where Q_v is a quality factor of the vertical resonance.

The parameters in our case are:

- $I = \frac{D^2}{16} + \frac{L^2}{12} = 4.740 \times 10^{-3} m^2$ (D=0.25m: diameter, L=0.1m: Thickness)
- $l = 0.45m$ (The pendulum frequency is 0.74Hz.)
- $d = 0.0068m$ (The pitch frequency is 0.60 Hz.)
- $r = 0.0068 \times 10^{-3} m$ (The initial misbalance is 1mrad.)
- $K_\theta = \frac{a}{(s - p_1)(s - p_2)\dots(s - p_n)}$;

  $a = 1.122 \times 10^{14}$, $p_n = -1.688 \pm 75.19j, -4.898 \pm 67.83j, -7.634 \pm 53.83j,$
  $-9.623 \pm 34.56j, -10.66 \pm 11.91j$, (Chebychev 10 pole, 1 dB, 12Hz; Gain at DC is 0.005 for pseudo-critical damping.)
- $\omega_v = 11Hz$
- $Q_v = 4000$

The transfer function was calculated using the "Block diagram method" in simulink of matlab again. Fig. 2 shows the block diagram which reflects the equations of motion and the vertical resonance. Inport, Outport, and Outport1 represent Z_1, Z_2, and Θ, respectively. Z_1 is multiplied by

3.1×10^{-4} (0.31 m rad) to convert vertical motion to cavity length change due to the earth's curvature. Θ is multiplied by 0.01 (1cm) at the last stage to produce the cavity length change.

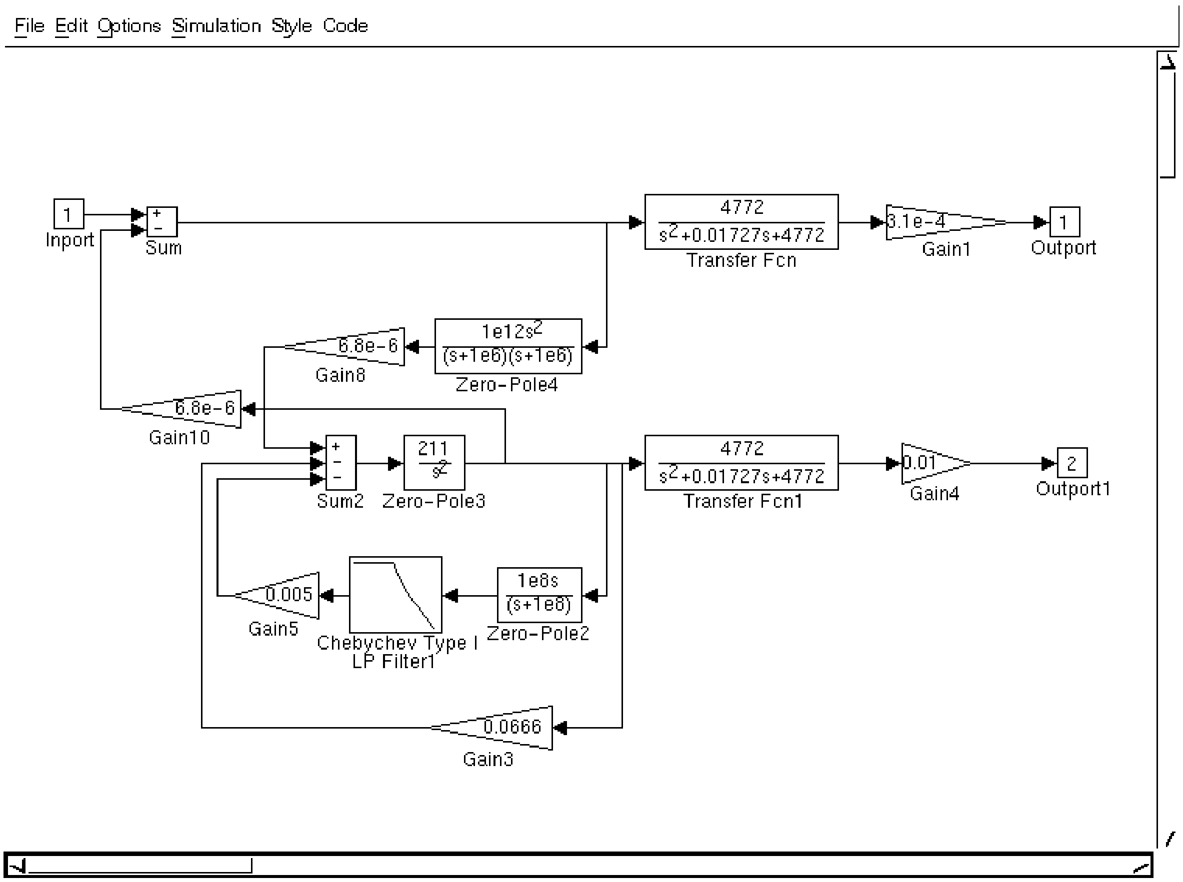

Figure 5: Block diagram of the pendulum system with vertical motion of the suspension point.

The two transfer functions are shown in Fig. 6.

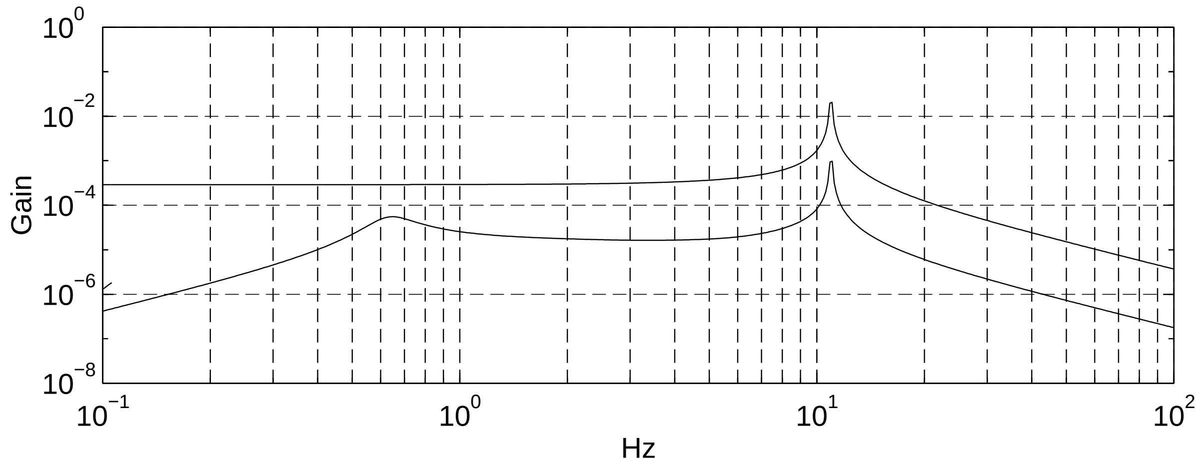

Figure 6: Transfer function from the vertical displacement of the suspension position to the cavity length change via the vertical motion coupling with the earth's curvature (Upper) via the pitch angle change with assumed d of 1 cm (Lower).

3 YAW MOTION OF THE SUSPENSION POINT

The yaw motion of the suspension point naturally causes the yaw motion of the mass with a simple pendulum response: the transfer function is given as follows:

$$\frac{\Phi_2}{\Phi_1} = \frac{{\omega_y}^2}{s^2 + K_\phi s + {\omega_y}^2},$$

where $\Phi_1(s)$ and $\Phi_2(s)$ are Laplace transforms of the yaw angle of the suspension point and mass, respectively, ω_y is the resonance frequency of the yaw mode and Q_ϕ is the quality factor.

The parameters in our case are:

- $\omega_y = 0.5\mathrm{Hz}$
- $K_\theta = \dfrac{a}{(s-p_1)(s-p_2)\dots(s-p_n)}$;

  $a = 6.735 \times 10^{16}$, $p_n = -1.688 \pm 75.19j, -4.898 \pm 67.83j, -7.634 \pm 53.83j,$
  $-9.623 \pm 34.56j, -10.66 \pm 11.91j$, (Chebychev 10 pole, 1 dB, 12Hz; Gain at DC is 3 for pseudo-critical damping.)

Φ_2 is multiplied by 0.01 (1cm) to produce the transfer function from the yaw angle of the suspension point to the cavity length change as shown in Fig. 7. Since it is unlikely that the yaw motion of the suspension point exceeds (the horizontal motion of the suspension point) divided by (1 m), it is fair to say that this mechanism is negligible compared with the simple horizontal-horizontal pendulum response.

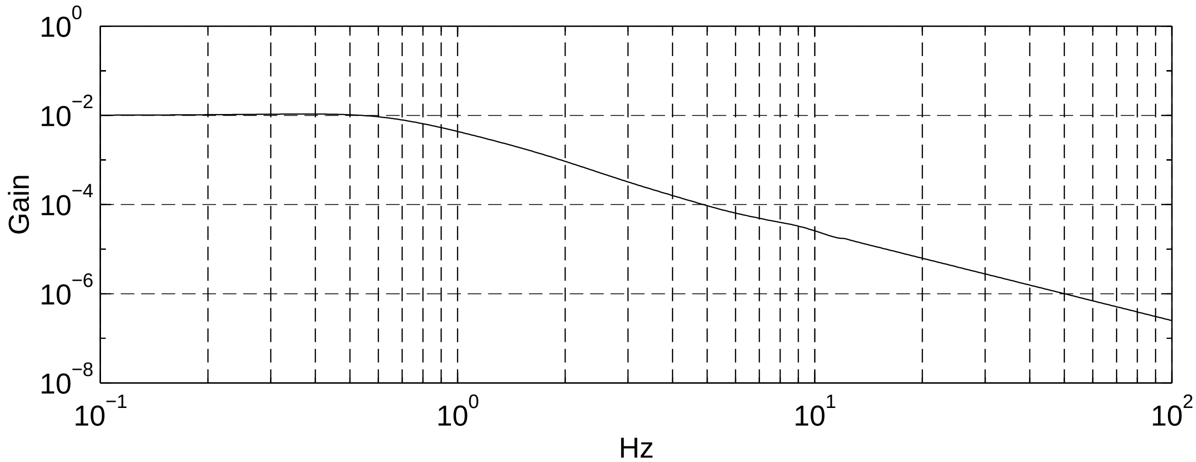

Figure 7: Transfer function from the yaw angle of the suspension point to the cavity length change via the yaw angle change with assumed d of 1 cm.

4 OTHER MOTIONS OF THE SUSPENSION POINT

The other motions of the suspension point do not cause significant changes in the cavity length.

5 CONCLUSION

It was found that in our case the cavity length change caused by motion of the suspension point is dominated by the simple horizontal-horizontal pendulum response except the vertical resonance frequency of the pendulum system.