Isolation Stacks
Preliminary Design Methodology

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Abstract

A simple preliminary design methodology for multi-stage isolation stacks has been developed. It makes use of one-dimensional approximations leading to closed-form design equations. The method provides information on the relative effectiveness of different designs (in the vertical direction only) and can be used for preliminary design purposes and to gain insight in the stack design problem. Horizontal transmissibility is not addressed by this technique so that final evaluation and design refinements must be based on 3-dimensional MATLAB simulations.

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1. One-Dimensional Isolation Stack Analysis

Seen as a one-dimensional system **for the axial direction**, an *n*-stage isolation stack reduces to the chain of masses M_i and springs k_i shown in Fig. 1. An uncoupled natural frequency f_i can be defined for each stage as the natural frequency of that stage taken out of the stack, i.e. $f_i = (1/2\pi)\sqrt{(k_i/M_i)}$.

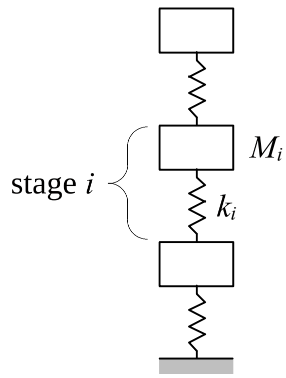

Figure 1: one dimensional approximation of isolation stack.

Using linear system theory and assuming structural damping only, the absolute vertical transmissibility T_{zz} of the stack is asymptotically given by the chart of Fig. 2.

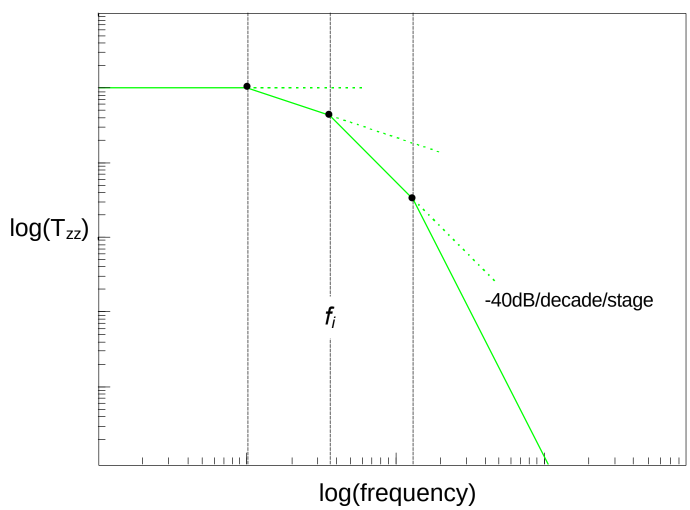

Figure 2: approximate asymptotic T_{zz} transmissibility of multi-stage isolation stack (not to scale).

Each stage *i* in the stack generates a -40dB/decade roll-off, starting at the cutoff frequency f_i. The DC transmissibility is of course equal to unity. Because of resonances, the actual stack transmissibility usually lies above the asymptotic approximation. Away from resonant frequencies however, the error made by the approximation is extremely small. To illustrate this, Fig. 3 compares the computed vertical response of the MIT prototype stack (using 3D model but fixed properties) to the asymptotic approximation (uncoupled natural frequencies of the stages from the floor up are 21.3, 17.4, 15.1, and 17.1 Hz, based on DC stiffnesses).

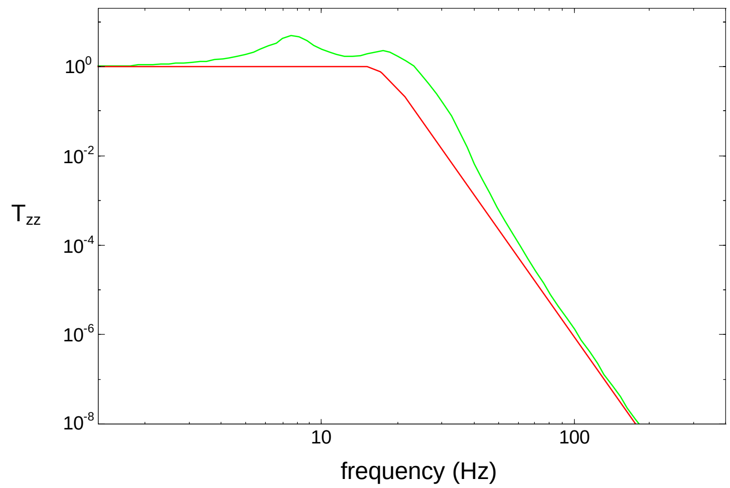

Figure 3: MIT stack with fixed (DC) properties; asymptotic approximation ( ) for T_{zz} compared to 3D model prediction ( ).

At any frequency f above all stage's uncoupled natural frequencies f_i, the approximation gives

$$T_{zz(f)} = \frac{f_1^2 \cdot f_2^3 \cdots f_n^2}{f^n}\,, \tag{1}$$

where n is the number of stages and Πf_i is the product of the stages' natural frequencies f_i, $i=1, \ldots, n$. This formula shows that performance is determined exclusively by the stages' uncoupled frequencies. Relative masses of isolated body to stack elements, mass distribution, stiffnesses, etc. have no *direct* effect on isolation performance.

Using this same formula (1), we can derive an upper bound on the performance of a stack using a given spring design. Assuming a linear spring with maximum allowable axial deflection δ_{max}, the natural frequency of an optimal single stage stack (spring loaded at its maximum) is

$$f_{min} = (1/2\pi)\,\sqrt{(g/\delta_{max})}\,, \tag{2}$$

where g is the acceleration of gravity and f_{min} is in Hz. For a multi-stage stack, this last relation holds only for the upper stage. The static load on other stages is more than the weight of that stage so that uncoupled natural frequencies are larger than f_{min}. Using f_{min} for all stages in Eq. (1) instead of the actual values gives the best conceivable performance of an n-stage stack using those springs:

$$T_{zz(f)} > \left(g \,/\, (4\pi^2 f^2 \delta_{max}) \right)^n. \tag{3}$$

Equation (3) confirms that the axial performance of an n-stage isolation stack is bounded by the available axial deflection of the springs. Spring design efforts should therefore concentrate on providing reasonably sized springs with the largest possible allowable axial deflection (avoiding yield, excessive creep, and instabilities). Note that spring rates or load capabilities are not directly involved: the stack simply uses more or less springs depending on their load capability.

For non-linear springs, we define $d_{max}= P_{max}/k_{ax}$, based on the allowable static load P_{max} and the tangent axial stiffness k_{ax} at maximum loading. Note that in that case d_{max} does not correspond to the actual deflection of the spring under the load P_{max}.

For springs with frequency dependent stiffness, estimates of transmissibility at a single frequency can be obtained by defining k_{ax} as the spring stiffness at the frequency of interest instead of the DC stiffness. For the prototype stack of Fig. 3 for example, the stiffness of the VITON springs at 100 Hz is 2.7834 times larger than at DC. The corrected stage frequencies are then $\sqrt{2.7834}=1.6684$ times larger than the DC estimates. This leads to an increase in the estimated $log_{10}(T_{zz})$ equal to $2\cdot\ log_{10}\ (1.6684^4)=1.7784$. Figure 4 compares this corrected estimate with the results from a 3D MATLAB simulation using frequency dependent properties.

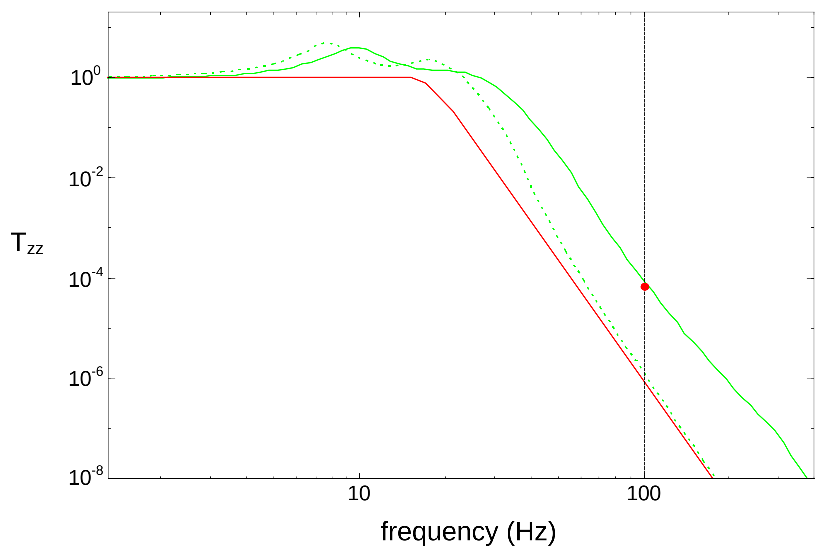

Figure 4: corrected asymptotic estimate for frequency dependent stiffnesses; prediction based on DC stiffnesses ( ) and corrected prediction at 100Hz (•) are compared to 3D simulation results using fixed (---) and frequency dependent ( ) stiffnesses.

For isolation in the shear direction, the springs should also have the smallest possible transverse stiffness. In other words they should be designed to minimize the ratio k_{sh}/k_{ax} of shear to axial stiffness (instabilities will be limiting factors). Note that because the horizontal response of the stack is inherently 3-dimensional, estimates of horizontal transmissibilities cannot be easily obtained and final design selection and refinement must be based on 3-dimensional MATLAB simulations.

2. Mass and Spring Distributions - Closed Form Design Equations

We now consider the design of a multistage stack with multiple legs to support a given top structure (Fig. 5). All legs are assumed identical. Leg elements and top structure are assumed rigid.

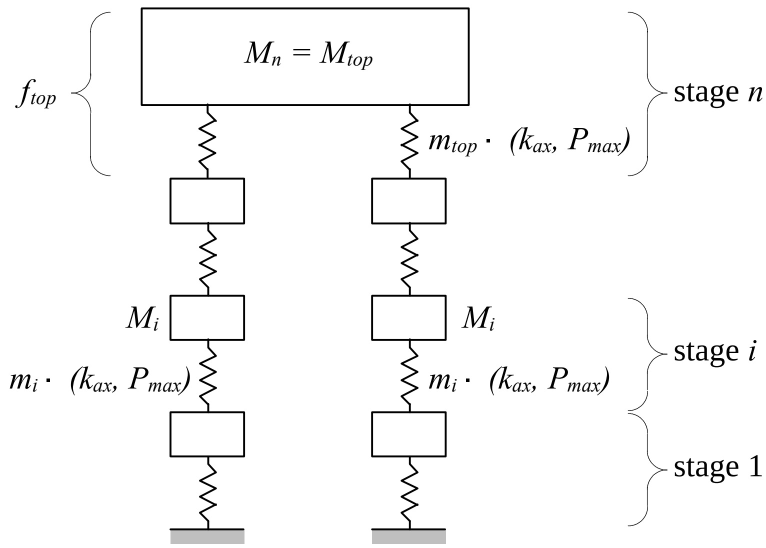

Figure 5: multi-leg, multi-stage stack.

We have shown in the previous section that the stages' uncoupled natural frequencies are the dominant design parameters for an isolation stack. With this in mind, let us solve the following problem: given the type of springs (k_{ax}, P_{max}) to be used, the mass of the top structure M_{top}, the number of legs n_{leg}, the number or stages n and their required minimum characteristic frequencies f_i, what is the lightest actual stack? In other words, for each stage i, design the number of springs m_i per leg and the stack element mass M_i in each leg so that the stage natural frequencies do not exceed required values f_i and the springs are not overloaded. Note that the natural frequency of the upper stage is out of our control: it is determined by the given mass of the upper structure M_{top} and the smallest number of spring that can support it. This design problem can be formulated as:

$$Minimize\ M = M_{top} + n_{leg} \cdot \Sigma M_i , \tag{4}$$

Such that:

$$(1/2\pi)\sqrt{(m_i\, k_{ax} / M_i)} < f_i , \quad i = 1, \ldots, n-1$$

$$g*(M_i + M_{i+1} + \ldots + M_{top}/n_{leg}) < m_i\, P_{max} , \ i = 1, \ldots, n.$$

We also require that the top structure and each leg element be supported by at least 3 springs (for static equilibrium, since we neglect the springs' bending stiffnesses).
Since decreasing the stages' natural frequencies requires more mass, the lightest stack will exactly satisfy conditions on f_i, we can solve this problem analytically and get

$$m_{top} = max\ (\ ceil(\ gM_{top} /(P_{max} n_{leg})\ ),\ ceil(3/n_{leg})\ ), \tag{5}$$

$$m_i = max\ (\ ceil\ (\ g\ (M_{i+1} + \ldots + M_{top}/n_{leg}) / (P_{max} - k_{ax}\, g/(4\pi^2 f_i^2)\ )\ ),\ 3\ ),\ i=1, \ldots, n-1 \tag{6}$$

where *max(a,b)* returns the largest of *a* and *b*, *ceil* is an operator that rounds to the next larger integer value, and

$$M_i = m_i\, k_{ax} / (4\pi^2 f_i^2). \tag{7}$$

Equations (5) to (7) are used recursively, starting with the top stage to provide masses and number of springs at each stage of the stack.

3. Stack Design Procedure

3.1 Effect of Distribution of Stage Frequencies

Equation (1) shows that the performance of the stack is determined approximately by the product of the stages uncoupled frequencies. For a given performance, we have to ask ourselves whether the weight of the stack will vary significantly when varying individual stage frequencies, keeping their product constant.

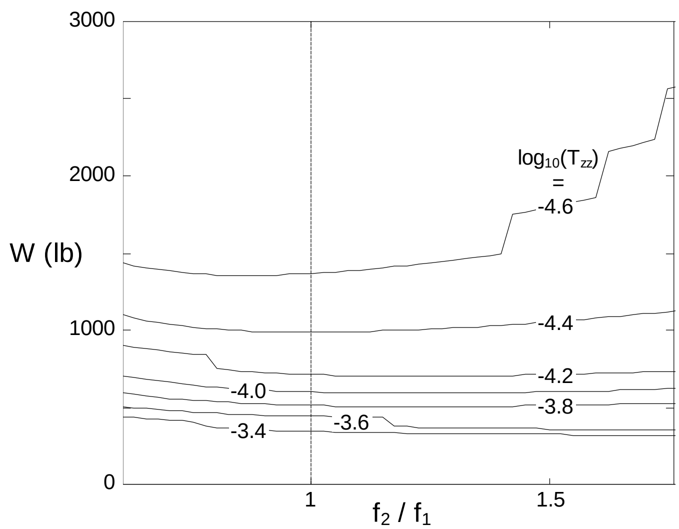

Figure 6: Stack weight as a function of stage frequency ratio for various required performances at 100 Hz (n=3, n_{leg}=3, M_{top} = 78 kg).

To investigate this, we consider the case of a 3-stage, 3-leg stack similar to the MIT prototype. For a given performance, the only design parameter is the ratio of natural frequencies of the 2 lower stages (the frequency of the upper stage is determined by the given mass of the upper structure and the smallest number of spring that can support it). The stack uses VITON springs with k_{ax} = 1709.1 lb/in (at DC; we ignore frequency dependence in this calculation) and P_{max} = 125 lb. Using the design equations of Section 2, we generate stacks of given performance with a ratio f_2/f_1 varying between approximately 0.6 and 1.75, where stage 1 is closest to the ground. Figure 6 shows plots of the stack weight as a function of f_2/f_1, for $log_{10}(T_{zz})$ = -4.6, -4.4, -4.2, , -3.4 at 100 Hz.

The figure shows that - for any given performance level - the weight of the lightest possible stack (minimum point on the curve) is never very different from that of a uniform frequency stack (f_2/f_1=1, i.e. one where all stages have the same uncoupled frequency). In other words, a uniform frequency stack is never much heavier than the optimal stack of identical performance. Also, because the objective function of a minimum weight stack design optimization is discontinuous (due to discrete number of springs, see steps in Fig. 6), solving such a problem is not straightforward and requires specialized discrete techniques or exhaustive search. For these reasons we decided to limit ourselves to uniform frequency stacks.

3.2 Designing a Uniform Frequency Stack

Assuming uniform stage frequencies, the design of a stack with n_{leg} legs and n stages using springs with axial stiffness k_{ax} and load capacity P_{max} to support a top structure (assumed rigid) of mass M_{top} with a given vertical transmissibility T_{design} at frequency f_{design} is straightforward. It applies the design equations of Section 3 one stage at a time from the top down.

3.2.1 Top Stage

With given springs, the natural frequency of the top stage is out of our control. The minimum number of springs per leg to support the top structure is given by Eq. (5). The natural frequency of the upper stage is then given as

$$f_{top} = \sqrt{(m_{top} n_{leg} k_{ax} / M_{top})}. \qquad (8)$$

3.2.2 Other stages

Using Eq. (1) with this value f_{top} for the top stage natural frequency and equating the transmissibility to the required value T_{design} we get the required natural frequencies of all other stages as

$$f_i = f_{design}^{n/(n-1)} \, T_{design}^{1/(2n-2)} \, f_{top}^{-1/(n-1)} , \quad i=1, \ldots, n-1. \qquad (9)$$

Equations (5) to (7) then give us, stage by stage, the number of springs per leg and the required mass of the leg elements. This procedure was coded into a MATLAB M-file to allow rapid generation of stack designs with various numbers of legs and stages and using different types of springs.

4. Design Example

To illustrate the technique, we redesign the MIT stack A. The vertical transmissibility at 100 Hz is used to rate the design. The VITON springs used in that stack have k_{ax}=1709.1×2.7834=4757.1 lb/in at 100Hz, and P_{max}=125 lbs. The top plate weighs gM_{top}=78.0 lbs. The prototype stack uses 3 legs and 4 stages. Table 1 lists for each stage the element weight (gM_i), the number of springs per leg (m_i), the uncoupled natural frequency (f_i), and the static load per spring (P_i).

i (stage #)	gM_i (lb)	m_i	f_i (Hz)	P_i (lb)
4 (top)	172.	1	28.5	57.
3	220.	3	25.2	92.
2	220.	4	29.0	125.
1 (base)	220.	6	35.5	120.

Total stack weight:	2152.0
Total # springs:	42
$\log_{10}(T_{zz})$ @ 100Hz:	-4.25

Table 1: MIT prototype stack A, simplified analysis based on spring properties at 100 Hz.

Using Eq. (3), with $d_{max} = P_{max}/k_{ax}$ = 0.02628 in., we get the lower bound on stage uncoupled frequencies (for performance at 100 Hz)

$$f_{min}=19.299\ Hz\ , \qquad (10)$$

and the upper bound on performance at 100 Hz for a stack with 3, 4, or 5 stages

$$T_{ZZ(100Hz)} > 5.17\ 10^{-5} \text{ with 3 stages}, \qquad (11)$$

$$1.92\ 10^{-6} \text{ with 4 stages}, \qquad (12)$$

$$7.17\ 10^{-8} \text{ with 5 stages}, \qquad (13)$$

Note that these limits cannot be reached in practice. First, all stages cannot achieve the optimal uncoupled natural frequency f_{min} because they bear the weight of other stages above. It can be shown that a stack that achieves the limit performance has an infinite mass. Also, since each leg has to use at least one spring in the upper stage, the natural frequency of the upper stage cannot reach the lower limit. The springs of the upper stage are far from being loaded to full capacity (see Table 1). Reevaluating the upper bounds (11) to (13) with the actual upper stage frequency (28.5 Hz) gives us more realistic bounds for this particular stack:

$$T_{ZZ(100Hz)} > 1.13\ 10^{-4} \text{ with 3 stages}, \qquad (14)$$

$$4.20\ 10^{-6} \text{ with 4 stages}, \qquad (15)$$

$$1.56\ 10^{-7} \text{ with 5 stages}. \qquad (16)$$

This is a result of the discrete nature of the design problem (integer number of springs).

Let us now apply the design procedure of Section 3.2 to generate stacks with the same performance at 100 Hz as the prototype (i.e. $log_{10}\ (T_{ZZ}) = -4.25$ or $T_{ZZ}=5.62\ 10^{-5}$) and 3, 4, or 5 stages. We immediately recognize that a 3 stage design cannot achieve the required performance (eq. 14). Equation (9) gives us the required stage frequencies for all but the upper stage of the 4 and 5 stage stacks:

$$f_i = 29.74\ Hz \text{ for stages 1,…,3 of the 4-stage stack}, \qquad (17)$$

$$f_i = 40.27\ Hz \text{ for stages 1,…,4 of the 5-stage stack}. \qquad (18)$$

Equations (5) to (7) then lead to the configurations of Tables 2 and 3.

i (stage #)	gM_i (lb)	m_i	f_i (Hz)	P_i (lb)
4 (top)	172.	1	28.5	57.
3	158.	3	29.7	72.
2	158.	3	29.7	124.
1 (base)	316.	6	29.7	115.

Total stack weight:	2067.1
Total # springs:	39
$log_{10}(T_{zz})$ @ 100Hz:	-4.25

Table 2: uniform frequency 4-stage stack design.

i (stage #)	gM_i (lb)	m_i	f_i (Hz)	P_i (lb)
5 (top)	172.	1	28.5	57.
4	86.	3	40.3	48.
3	86.	3	40.3	77.
2	86.	3	40.3	105.
1 (base)	115.	4	40.3	108.

Total stack weight:	1291.8
Total # springs:	42
$\log_{10}(T_{zz})$ @ 100Hz:	-4.25

Table 3: uniform frequency 5-stage stack design.

The 4-stage design is essentially equivalent to the prototype; the total mass is about 4% less and it uses 3 less springs, but the leg elements are not all identical. The 5-stage design is substantially lighter than the prototype (40% savings), uses the same number of springs and provides the same expected performance at 100 Hz.

Figure 7 shows vertical transmissibilities for those 2 designs compared to that of the prototype, as computed with 3-dimensional MATLAB models. As expected, the performance at 100Hz is almost identical for all 3 designs. Also, the performance of the 4 stage design of Table 2 is practically identical to that of the prototype. The 5-stage design has a steeper roll-off but a higher cutoff point, so it performs better above 100 Hz and not as well below that frequency.

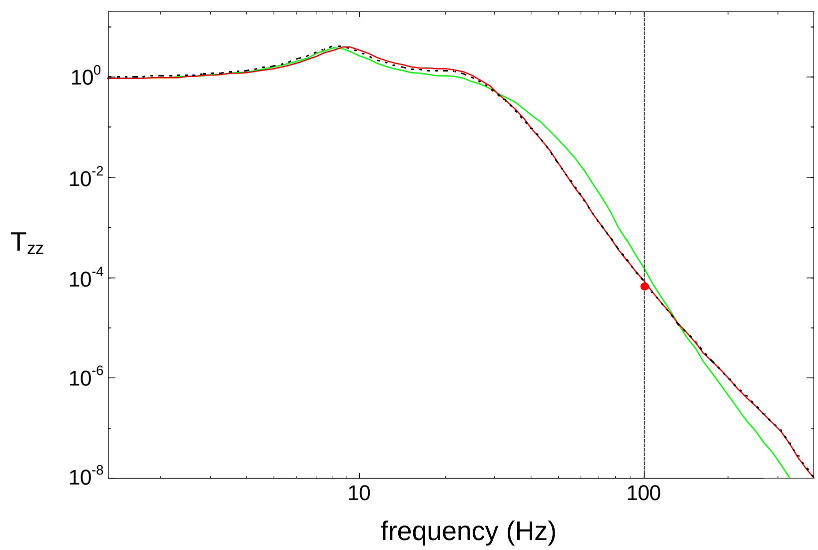

Figure 7: predicted transmissibilities of prototype design (---) compared to 4 ( ) and 5 ( ) stage designs of Tables 2 and 3; the 1-dimensional performance prediction is also shown (•).

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