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Calculation of Optical Parameters for the 40m Power Recycling Interferometer

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Contents

1 Purpose

The purpose of this document is to provide definitions, calculations and numerical values of the optical parameters for the 40m power recycling interferometer.

2 Scope

This document summarizes the calculations of the power in the cavity fundamental mode as it propagates through the recycling interferometer. The results of the calculations are the numerical values for the recycling and input mirror reflectivities and prediction for various optical parameters of the 40m recycling interferometer.

3 Summary of Recombined 40m Parameters

3.1 Ideal Arm Cavity Reflectivity

If all the light entering the arm cavity is in the cavity fundamental mode the reflectivity of the cavity is defined entirely by reflectivities of the front and end mirrors.

If the transmissivity and losses of the front mirror are T_a and L_a the reflectivity is

$$R_a = 1 - L_a - T_a.$$

The amplitude reflectivity is $r_a = \sqrt{R_a}$. Corresponding parameters for the end mirror are labeled with the subscript b.

The amplitude reflectivity of the arm cavity is

$$g = \frac{r_a - (1 - L_a)r_b}{1 - r_a r_b}. \tag{1}$$

The power reflectivity is

$$G = g^2.$$

The situation when $g = 0$ corresponds to optimal coupling. Cases when $g < 0$ and $g > 0$ are called overcoupling and undercoupling correspondingly. The optimal coupling takes place when

$$r_a = (1 - L_a)r_b. \tag{2}$$

For high reflectivities we can make an approximation

$$r_a \approx 1 - \frac{L_a + T_a}{2}.$$

The approximate condition for the optimal coupling is

$$T_a \approx T_b + L_a + L_b. \tag{3}$$

The approximation above is good for very low transmissivities and losses, which is the case for the 40m arm cavities. In fact, the error of the approximation is of the order of T_a or L_a, whichever is larger.

In the case of the recycling cavity the situation is different. As will be shown in Sect.4.4 the transmissivity of the recycling mirror is of the order of 20% and the approximation is no longer valid. This is why we will use the exact formula similar to the eq. (2) to define the optimal coupling for the recycling cavity instead of the widely used approximate condition, eq. (3).

The recombined 40m interferometer arm cavities are close to optimal coupling and reflect very little light. The transmissivities of the front and end mirrors are

$$T_a = 280 \text{ ppm} \tag{4}$$

$$T_b = 12 \text{ ppm} \tag{5}$$

The losses in the coatings are determined from ring-down measurements and split equally between the two mirrors $(L_a = L_b)$.

3.2 Losses in the Arm Cavity

One of the most important parameters in the calculations is the losses in the arm cavities. These are mostly losses in the coatings. Unlike the transmissivities the losses depend on cleanliness of the surface of the mirror, on alignment of the arm cavities and etc. The losses affect the arm cavity reflectivity.

The following table illustrates the dependence of the 40m arm cavity parameters on losses in the mirror coatings.

Table 1. 40m Arm Cavity Parameters

L (ppm)	g	G	τ (msec)
40	-0.505	0.255	1.37
60	-0.359	0.129	1.24
80	-0.240	0.057	1.13
100	-0.138	0.019	1.03
120	-0.053	0.0028	0.957
140	0.021	0.0004	0.890
160	0.085	0.0072	0.832

Negative values of g correspond to overcoupling and positive values correspond to undercoupling. We see that with the losses somewhere near to 120 ppm the cavity becomes optimally coupled.

The losses per mirror for the 40m recombined interferometer are about 100 ppm. From the table we see that the ideal arm cavity reflectivity is

$$G = 0.019.$$

The sharpness of the cavity fringes is defined by finesse. First define, F, sometimes called the "coefficient of finesse". This is a function of mirror reflectivities

$$F = \frac{4 r_a r_b}{(1 - r_a r_b)^2}.$$

Finesse is

$$\text{Finesse} = \frac{\pi\sqrt{F}}{2}.$$

Recombined 40m arm cavities had finesse of about 12800.

3.3 Storage Time and Corner Frequency

The time scale for storage time and corner frequency is set by the one-way traveling time, which for $L_0 = 38.2$ m is equal to

$$T = \frac{L_0}{c} = 0.127 \ \ \mu\text{sec}.$$

The amplitude storage time is defined as the time required for the amplitude of light inside the cavity to decrease by a factor of e. It is given by the formula

$$\tau = \frac{2T}{\ln\left(\frac{1}{r_a r_b}\right)}.$$

The recombined 40m arm cavity shows storage time

$$\tau = 1084 \ \ \mu\text{sec}.$$

The corner frequency in rad/sec is

$$\omega_c = \frac{1}{\tau}.$$

Often the corner frequency in Hz is more suitable, $f_c = \omega_c/2\pi$. Recombined 40m arm cavity corner frequency is

$$f_c = 147 \ \ \text{Hz}.$$

3.4 Modematching Coefficient

In the case of perfect modematching the arm reflects a fraction, G, of the light power back to the beam splitter. In practice the transverse field entering the cavity almost never coincides with the fundamental mode of the cavity. Suppose the incoming field overlaps with the cavity mode by μ. In other words, the value of the overlap integral of the normalized incoming field and the normalized cavity mode is equal to μ. Then the amount of power that the incoming light has in the cavity mode is

$$M = \mu^2.$$

This is the modematching coefficient. Note, that $M \leq 1$.

For 1 Watt of incident power M Watt is in the right mode. The rest of the power, $(1 - M)$, is in the orthogonal mode. The light in the right mode makes it into the cavity and a fraction of it leaks out of the cavity back to the laser. The light in the orthogonal mode is reflected by the front mirror. The total power of light reflected by the arm cavity is

$$R_{arm} = MG + (1 - M)R_a. \tag{6}$$

One can see that for $M = 1$ (perfect modematching) real cavity reflectivity coincides with the ideal cavity reflectivity. On the other hand, when $M = 0$ (all the light is in the orthogonal mode) the arm cavity reflectivity is equal to the reflectivity of the front mirror.

The modematching for the recycling interferometer will be discussed below.

3.5 Visibility

The modematching coefficient is the only quantity we need in order to characterize coupling of light to the arm cavity. This quantity is difficult to measure. Instead, one measures the visibility of the cavity and then calculates the modematching coefficient.

The visibility of the arm cavity is defined as

$$V = \frac{P_{max} - P_{min}}{P_{max}},$$

where P_{max} and P_{min} are maximum and minimum powers reflected by the arm.

The minimum reflected power is achieved when the arm cavity is on resonance. Correspondingly, the maximum power is reflected by the arm cavity when it is off resonance.

Let the incident power be P_0. The amount of power in the carrier is $P_0 J_0^2$, where $J_0(\Gamma)$ is a function of the modulation index Γ. The rest of the power, $P_0(1 - J_0^2)$, is in the sidebands. We assume that none of the sidebands can resonate in the arms. When the arm cavity is at resonance the reflected power is

$$P_{min} = P_0 J_0^2 R_{arm} + P_0(1 - J_0^2) R_a.$$

When the cavity is off resonance all the power is reflected back by the front mirror

$$P_{max} = P_0 R_a.$$

Therefore

$$V = J_0^2 \left(1 - \frac{R_{arm}(M)}{R_a}\right).$$

One consequence of this expression is that the maximum visibility that can ever be achieved is equal to J_0^2.

We can now relate visibility and modematching coefficient using eq. (6)

$$V = J_0^2 M \left(1 - \frac{G}{R_a}\right).$$

Finding the modematching coefficient from this equation requires the knowledge of the modulation depth, Γ. Usually we want to avoid measuring the power in the sidebands at the same time when we measure the modematching index. One way of doing this is to go to low modulation. At the low modulation the visibility becomes roughly independent of the modulation index

$$V \approx M \left(1 - \frac{G}{R_a}\right).$$

To illustrate these formulas we calculate the low modulation visibility of the 40m arm cavity for different values of the modematching coefficient. The transmissivities of the front and end mirrors are the same as in Sect.3.1. The losses are chosen to be $L = 100$ ppm (see [2]) and the modulation depth is $\Gamma = 0.8$

Table 2. 40m Arm Cavity Visibility

M	V	R_{arm}
0.5	0.49	0.51
0.6	0.59	0.41
0.7	0.69	0.31
0.8	0.78	0.21
0.9	0.88	0.12
1.0	0.98	0.019

At the present we have visibility of about $V = 70\%$. Therefore the modematching coefficient is about 0.7. In the past the modematching coefficients as high as 0.9 were achieved.

To conclude this section we show that with $M = 0.7$ the total power of light reflected by the recombined 40m arm cavity is

$$R_{arm} = 0.313.$$

Since $G = 0.019$ (see Sect.3.1) most of this reflectivity is due to imperfect modematching.

3.6 Contrast Defect

If the power incident on a beam splitter is P then the combined power reflected by the two arms is $R_{arm}P$. If the beam splitter is at the "dark fringe" almost all this power is directed to the symmetric output, that is back to the laser. A small fraction of this power is directed to the antisymmetric output. Let this fraction be n. The parameter n is closely related to contrast. Thus the power of light at the antisymmetric port photodiode is

$$nR_{arm}P.$$

The rest of the power, namely,

$$(1 - n)R_{arm}P.$$

goes back to the laser.

The parameter n is a measure of incomplete destructive interference at the antisymmetric port photodiode. It is a small fraction of the total power reflected back to the beam splitter by the arms. We assume that this fraction does not change significantly when the total power reflected by the arms increases. The arms of the recycling interferometer will reflect much more light than in the recombined configuration.

Numerical value for n can be found from the measurements of the 40m contrast, see Appendix B.

4 Recycled 40m Optic Transmissions

4.1 Basis for the Calculations

In the preceding sections we defined several parameters and gave their numerical values. Some of these parameters were measured for the current 40m recombined interferometer. We have to extrapolate numerical values of these parameters to the power recycling interferometer.

In this section we justify our choice of these numerical values and use them as input for the calculation of the key parameters and prediction of the performance of the recycling interferometer.

The power recycling interferometer will retain the end mirrors in the arm cavities. Thus the transmissivity of the end mirrors is fixed and equal to

$$T_b = 12 \ \text{ppm}.$$

The transmissivity of the input mirror T_a is to be calculated.

We saw that the losses in the test mass coatings affect strongly the reflectivity of the arm cavity (see Table.1). It is important to accurately know these losses. At the present the losses are roughly $L = 100$ ppm per mirror. We assume that the losses in the coatings of the new input mirrors will be better than that. In fact, the newly coated input test masses are expected to have losses of about 20 - 40 ppm.

For the purpose of the present calculation we assume that the losses are

$$L = 100 \ \text{ppm}.$$

per mirror. Thus will give us some safety margin.

The modematching coefficient at present is roughly $M = 0.7$. Suppose the modematching of light to the arms of the recycling interferometer is not perfect. The fraction of light power in the cavity's fundamental mode is M. The light in the cavity's orthogonal mode will be reflected off the front mirror. This power of light in the orthogonal mode reflected by the arm cavity is equal to

$$(1 - M)R_a \approx 1 - M.$$

This mode will resonate in the recycling cavity since the recycling cavity is degenerate, or near to degenerate. Even after many round trips in the recycling cavity this mode will never make it into the arm cavity. Eventually almost all of this power will leak out of the recycling cavity back to the laser. The diffraction losses for this mode are negligible due to a small number of round trips and the short length of the recycling cavity. Therefore $(1 - M)$ of the total power incident on the recycling mirror will be lost. It does not contribute to the signal but increases the noise in the isolator and the pick-off photodiodes.

The fundamental mode ideally will resonate in both the recycling and arm cavities. In practice the reflective surface of the recycling mirror does not perfectly match the phase front of the fundamental mode. There are a number of reasons for that. One reason is that the arm cavity fundamental mode is slightly different from the ideal (00)-mode used in the calculations of the recycling mirror curvature due to the phase "bumps" on the input mirror surfaces. Another reason is the arms are intrinsically different and one cannot match two different modes with one surface. Not to mention the asymmetry.

Perhaps, the most important reason for us is fact that the reflective surface of the recycling mirror is flat as opposed to curved. The losses due to the flatness of the recycling mirror are estimated to be $L_R = 0.006$, see [3]. Thus we see that the modematching in the recycling interferometer is essentially due to the mismatch between the arm cavity and the recycling cavity fundamental modes. For simplicity we assume these modematching losses all appear at the surface of the recycling mirror and slightly exaggerate L_R. For the present calculations we take

$$L_R = 0.01.$$

The contrast defect is another source of losses in the recycling cavity. For the present calculation we assume that the contrast defect is

$$n = 0.04.$$

The reflectivity of the pick-off is defined by the noise considerations in the beam splitter servo. Such calculations will be discussed in a separate technical note. Here we take a very simple approach and find the upper limit for the pick-off reflectivity which still allows the recycling factor of 5.

The calculations show that for the largest value of the pick-off reflectivity for which 5 recycles are still possible is about 4%. For example in the case where there is no coating on the pick-off surface, $R_P = 0.035$, the recycling number of 5 can be achieved with the input mirror reflectivity of T_a = 9400 ppm. On the other hand, with all the other losses specified above there is almost no difference in the numerical value of the input mirror reflectivity when the pick-off reflectivity varies from 0 to 0.01. Beyond the 1% value the parameters of the recycling interferometer become sensitive to the value of the pick-off reflectivity. In what follows we assume that the pick-off surface has the reflectivity

$$R_P = 0.01.$$

4.2 Reflectivity of the Compound Mirror

Viewed from the recycling mirror position the rest of the interferometer looks like a mirror with some amplitude reflectivity ρ. When the arms are on resonance and the beam splitter is at the dark fringe this reflectivity is defined by

$$\rho^2 = (1 - n)G.$$

This compound mirror plays a role of the back mirror in the recycling cavity. The recycling mirror will have to match ρ in order for the recycling cavity to be optimally coupled.

4.3 Recycling Factor

The recycling factor is simply the gain of the recycling cavity

$$N = \left(\frac{t_R t_P}{1 - r_R T_P |\rho|} \right)^2 .$$

It is a function of the recycling mirror reflectivity r_R. This function reaches its maximum when the recycling cavity is optimally coupled

$$r_R = (1 - L_R) T_P |\rho|.$$

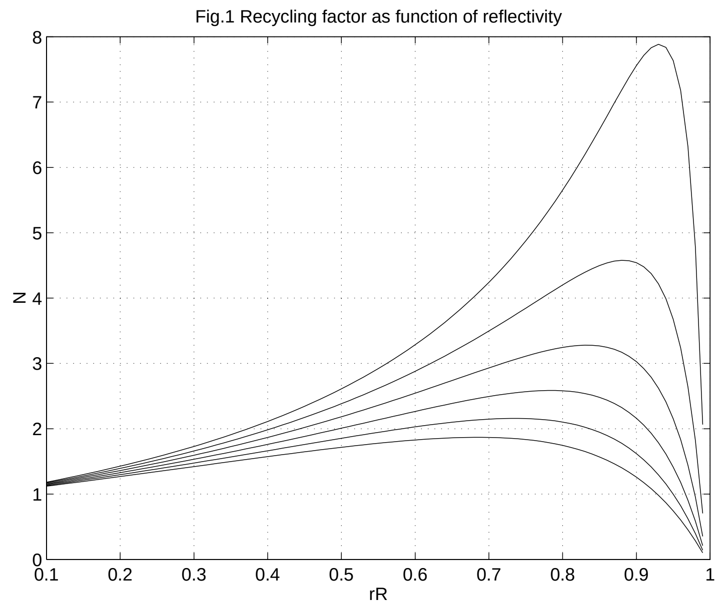


Fig.1 shows the recycling factor as a function of r_R for the values of $|\rho|$ = 0.7, 0.75, 0.8, 0.85, 0.9 and 0.95. The greater the value of ρ the higher the curve.

At optimal coupling N is inversely proportional to the transmissivity of the recycling mirror

$$N_{max} = \frac{(1 - L_R)^2 T_P}{T_R}.$$

This equation defines the recycling mirror reflectivity. Namely, set the desired recycling number, N, and assume optimal coupling, then

$$T_R = \frac{(1 - L_R)^2 T_P}{N}.$$

Knowing T_R we can find the reflectivity of the compound mirror, ρ, and the ideal arm cavity reflectivity, G. To illustrate the formulas we show numerical values of all these parameters for the different choice of the recycling factor in the following table.

Table 3. Recycling IFO Parameters for Different N

N	T_R	ρ	G
4	0.243	0.882	0.810
5	0.194	0.910	0.863
6	0.162	0.929	0.898
7	0.139	0.941	0.923
8	0.121	0.951	0.942
9	0.108	0.958	0.957
10	0.097	0.964	0.968

Here G is lowest reflectivity of the arm cavity for which one can still achieve the desired recycling factor. It simply means the optimal coupling. Of course, one can achieve the desired recycling factor with higher reflectivity of the arm but not being optimally coupled.

4.4 Transmissivity of the Recycling Mirror

The goal is to achieve the recycling factor of

$$N_{goal} = 5.$$

(see [3]). From the Table 3. we see that optimal coupling defines the recycling mirror transmissivity as

$$T_R = 0.194.$$

This also requires that the arm cavity reflectivity should be at least

$$G = 0.863.$$

4.5 Transmissivity of the Input Mirror

In Sect.3.1 we saw how to find g and G in terms of T_a and T_b. Now we can invert those formulas and find the transmissivity of the input mirror in terms of g provided we know the parameters of the end mirror.

In the previous section we found the ideal arm cavity reflectivity G. The corresponding amplitude reflectivity is

$$g = -\sqrt{G} = -0.929 \ ,$$

where minus sign is due to the fact that the arm cavity is overcoupled (see the discussion in Sect.3.1). Inverting the eq.(1) we obtain

$$r_a = \frac{g + (1 - L_a) r_b}{1 + g r_b}.$$

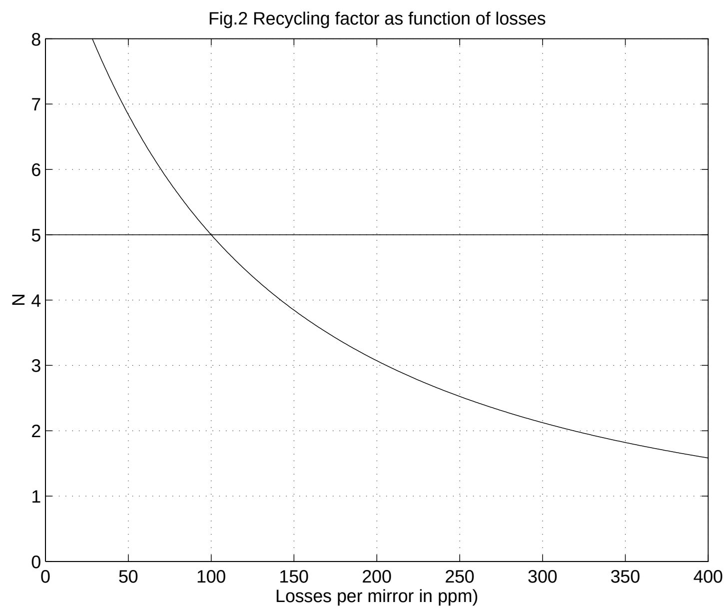
In terms of the power transmissivity the result is

$$T_a = 5750 \ \ \text{ppm}.$$

4.6 Deviations in Parameters

In the previous sections we chose the values for several input parameters in order to calculate the reflectivities of the recycling interferometer. Due to our limited knowledge of these parameters and partially due to the impossibility of predicting accurately their values, these parameters may vary. Thus it is important to analyze what happens if these parameters deviate from the values we specified above.

One of the uncertainties is the losses in the mirror coating L_a. We made a choice that corresponds to the largest losses we have seen so far at the 40m. The reason is as follows. With this choice we expect that the real losses in the mirror coatings will be less than what we specified. In this case the



arm cavity reflectivity will be somewhat higher than the predicted reflectivity. This will affect the optimal coupling of the recycling cavity. The recycling cavity will become overcoupled. Therefore the recycling cavity gain (recycling factor) will be somewhat higher than the number we are aiming for.

If it turns out to be that the real losses in the coatings are higher than what we specified the arm cavity reflectivity will be less than the minimum number, see Sect. 4.1. The recycling cavity becomes undercoupled. One possible way out of this situation is to replace the recycling mirror with the one with lower reflectivity. This way we retain the optimal coupling but lose in the recycling factor. Another way would be to compensate the arm cavity losses by improving the contrast. The latter is usually difficult to implement.

Fig.2 shows the dependence of the recycling factor on the losses L.

The arguments above also apply to the contrast defect or n. However, the uncertainty here is less since we expect n not to vary much.

5 Other Recycled 40m Parameters

In this section we discuss only the shot noise limited performance of the proposed 40m recycling interferometer. Other predictions will be discussed elsewhere. Before we calculate the shot noise limited sensitivity we give definitions to a few more quantities that enter the shot noise calculations.

5.1 Modulation Frequency

The frequency of the RF modulation is based on the length of the recycling cavity

$$l = 2.29 \text{ m}.$$

It is found to be

$$f_{mod} = 32.7 \ \text{MHz}.$$

The details can be found in [4].

5.2 Asymmetry

The asymmetry in the recycling cavity is

$$\delta l = 0.54 \ \text{m}.$$

This is the maximum asymmetry allowed by the vacuum envelope. The asymmetry results in the asymmetry phase

$$\alpha = k_{mod}\delta l = 21.2 \ \text{deg}.$$

5.3 Sideband Recycling Factor

One of the conditions for power recycling is that the sidebands resonate in the recycling cavity. However, they are not amplified by the same amount as the carrier. There are two reasons for that. The first is the asymmetry, and second is that the arm reflectivity is different for the sidebands.

The sideband recycling factor is defined as

$$N_1(\alpha) = \left(\frac{t_R t_P}{1 - r_R T_P r_a \cos\alpha}\right)^2.$$

Then the power in the sidebands in the recycling cavity increases by a factor of $N_1 \cos^2\alpha$. Correspondingly the power in the sidebands going towards the antisymmetric port increases by a factor of $N_1 \sin^2\alpha$.

5.4 Optimal Asymmetry

The value of α for which the sideband power at the antisymmetric port is a maximum is called optimal asymmetry. The optimal asymmetry is achieved at

$$\cos\alpha_{opt} = r_R T_P r_a.$$

For the numerical values for the reflectivities the optimal phase and the asymmetry length are

$$\alpha_{opt} = 28.3 \ \text{deg}, \tag{7}$$

$$\delta l_{opt} = 0.72 \ \text{m}. \tag{8}$$

Neither can we change the nominal asymmetry length nor can we change the modulation frequency to achieve the optimal α with the fixed asymmetry length. Thus we do not consider optimization with respect to α. The gain constraints analyzed in [1] will be discussed elsewhere.

5.5 Shot Noise Limited Sensitivity

Calculation of shot noise limited performance requires knowledge of additional parameters. These are: the power incident on recycling mirror P_{in}, responsivity of the photodiode σ and modulation index Γ.

The power incident on the recycling mirror will be essentially the same as the power incident on a beam splitter in the recombined interferometer, which is

$$P_{in} = 0.1 \ \text{W}.$$

The conversion from power to current at the photodiode depends on the responsivity of the photodiode, which for green light is

$$\sigma = 0.25 \ \text{A/W}.$$

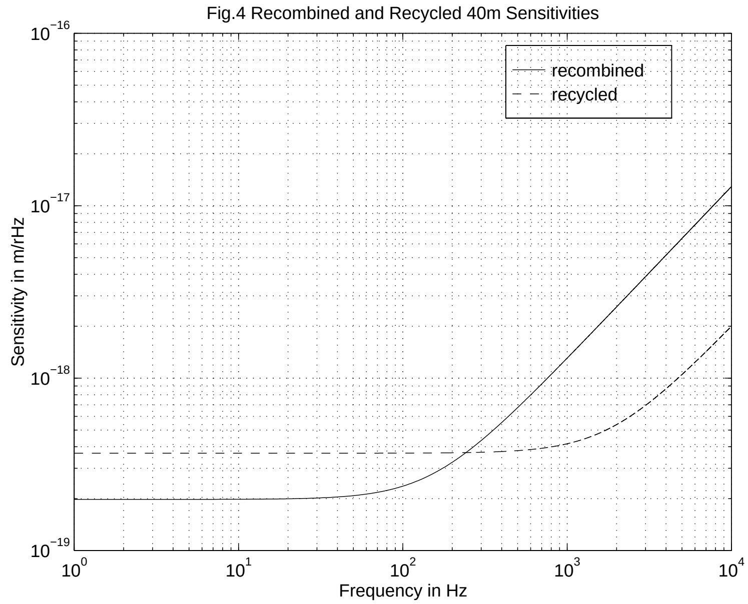
The DC-sensitivity for recycling interferometer is found to be

$$X_{DC} = 3.67 \times 10^{-19} \ \text{m}/\sqrt{\text{Hz}}$$

5.6 Optimal Modulation

The optimal modulation index is found by minimizing X_{DC} with respect to Γ. The minimization is done numerically. For the above choice of parameters the result is

$$\Gamma = 0.81.$$



5.7 The Sensitivity Curve

The frequency dependence of the shot noise limited sensitivity is given by

$$X(f) = X_{DC}\sqrt{1 + f^2/f_c^2}.$$

Fig.3 shows the predicted shot noise sensitivities of the recombined and recycling 40m interferometers.

5.8 Comparison of Recombined and Recycled 40m

The following table summarizes essential differences between 40m recombined and recycling interferometers.

Table 4. 40m Recombined and Recycled Parameters

Parameter	Symbol	Recombined 40m	Recycled 40m
Input mirror transmission (ppm)	T_a	280	5750
End mirror transmission (ppm)	T_b	12	12
Losses per mirror (ppm)	L_a	100	100
Recycling mirror transmission	T_R	-	0.194
Storage time (μsec)	τ	1084	89
Finesse	Finesse	12800	1050
Modematching coefficient	M	0.7	0.7
Visibility	V	0.70	0.09
Modulation frequency (MHz)	f_{mod}	12.3	32.7
Asymmetry length (m)	δl	0.54	0.54
Asymmetry phase (deg)	α	7.8	21.2
Contrast defect	δC	0.02	0.03
Corner frequency (Hz)	f_c	147	1789
DC-Sensitivity (m/rHz)	X_{DC}	1.98×10^{-19}	3.67×10^{-19}

6 Conclusion

This technical document presents the calculations of various parameters of the 40m power recycling interferometer. It is shown that the proposed recycling factor is possible to achieve, provided that all the input parameters vary within the limits described above.

A Measured Contrast

According to [2], the contrast is defined as

$$C = \frac{P_B - P_D}{P_B + P_D},$$

where P_B is power in the bright fringe when both arm cavities are off resonance and P_D is power in the dark fringe when both arms are on resonance. Then the contrast defect is defined as

$$\delta C = 1 - C = \frac{2P_D}{P_B + P_D}.$$

In Sect.3.4 we saw that when the arms are on resonance and the beam splitter is at the dark fringe the power of light at the antisymmetric port photodiode is

$$P_D = nR_{arm}P.$$

When both arms are off resonance the power in the bright fringe is

$$P_B = (1 - n)R_a P.$$

Thus the exact relation between δC and n is

$$\delta C = \frac{2nR_{arm}}{(1 - n)R_a + nR_{arm}}.$$

The contrast defect δC is the parameter we measure in the experiment. However, it is not simple to extrapolate δC from the recombined to the recycling interferometer because the power reflected by the arm cavities is different in both cases. However, we believe that n should be essentially the same for both interferometers since its definition is independent of the power.

Below we give a table that relates δC and n for the recombined 40m interferometer

Table 5. Recombined 40m Contrast Defect

n	δC
0.01	0.006
0.02	0.013
0.03	0.019
0.04	0.026
0.05	0.032
0.06	0.039
0.07	0.046
0.08	0.053
0.09	0.060
0.10	0.067

References

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[3] J. Logan and M. Rakhmanov, Proposal for Recycling the 40m Interferometer.

[4] J. Logan and M. Rakhmanov, Calculation of the Modulation Frequency for the 40m Power Recycling Interferometer, LIGO-T960013-D.