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Design of Reflection–Locked Laser Frequency Control Systems: Part I., Error Sensing and Feedback Actuators

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Abstract

Elements of laser frequency and phase error sensing by the reflection RF sideband technique (sometimes called Pound–Drever method) and feedback correction of these errors are discussed. The effective loop transfer function from optical phase to electrical error signal (i.e. demodulator output voltage) is derived. Phase, frequency, force and position actuators used commonly in effecting feedback are also characterized. Baseband electronic amplifiers, compensators, and loop tailoring are to be discussed in Part II.

1 Introduction

Laser frequency and phase stabilization to passive optical cavities, and of optical cavities to one another, have been developed intensively in the evolution of the 40m prototype interferometer, and are integral features of planned LIGO interferometers. In this document we hope to collect observations and elements of theory which may assist in practical future application of these techniques.

The laser emits light of frequency $\omega_L = \pi kcn/l$ where k is a large integer, c is the speed of light in vacuum, n is the mean refractive index of the material (plasma, glass, crystal, etc.) inside the laser cavity, and l is the length of the laser cavity. Either n or l may be altered, by naturally occurring forces or by deliberate action, to effect a change in ω_L. A further apparent Doppler frequency shift $\omega'_L = \omega_L \dot{d}/c$ occurs when the optical path d between the laser and the reference changes with time. We seek to detect these deviations by comparing the instantaneous laser frequency with the resonant frequency of a passive optical cavity, and to apply servo feedback to actuators which either correct the laser or induce the cavity to follow it.

2 Frequency and Phase Sensing

The currently preferred means for sensing the relative frequency or phase deviation between laser and reference cavity is described in [5]. Consider schematic Figure 1, showing a laser source and a passive optical reference cavity. The cavity mirrors have (complex) field reflectivities r_1, r_2 and transmissions t_1, t_2; the power reflectivities and transmissions are thus $R_i = |r_i|^2$ and $T_i = |t_i|^2$.

2.1 Input Field

Following [6], the phase of the laser field is modulated at a radio frequency ω_m high compared to frequencies at which servo corrections are required, and also well above the the *cavity knee frequency* $\omega_c = 1/2\tau_e$, where $\tau_e = 2l/c(1-\sqrt{R_1 R_2})$ is the *cavity storage time.*

The phase modulated laser field can be expanded as

$$\begin{aligned} E_L(t) &= E_L e^{-j(\omega_L t + \phi_L(t) + m \sin \omega_m t)} \qquad (1) \\ &= E_L e^{-j\phi_L(t)} \Big\{ \; J_0(m) e^{-j\omega_L t} \\ &\quad + J_1(m) \left[e^{-j(\omega_L + \omega_m)t} - e^{-j(\omega_L - \omega_m)t} \right] \\ &\quad + J_2(m) [\cdots] + \cdots \Big\} \end{aligned}$$

where m is the *modulation index* and $J_i(m)$ is the i^{th} Bessel function of the first kind, evaluated at m. By our choice of a high ω_m we ensure that the J_1 and higher terms in the field will oscillate at a frequencies outside the resonant passband of the cavity if the fundamental (J_0)

component is resonant. In subsequent analysis, then, we may approximate the field within the cavity as though only the J_0 component existed in the input field[1].

The incident laser mode is not, in general, perfectly matched to the cavity. Define the *modematching parameter* β as the overlap integral between the laser field's spatial distribution and the TEM_{00} mode of the cavity. $\beta = 1$ corresponds to a perfectly aligned wavefront with perfect curvature arriving at the cavity input coupler. Since the higher spatial modes are orthogonal, there can be no interference between the field in these modes (which carry $1-\beta^2$ of the power) and that in the proper mode, and we can treat the unmatched portion as though it were a distinct source of light on our detector whose phase is inconsequential[2].

2.2 Steady–State Cavity Field

Assume we hold $\omega_L = \omega_C =$ the resonant frequency of the cavity for some period of time, as might a servo system. As long as $\phi_L(t)$ is zero or varies slowly, on timescales much longer than τ_e, light builds up in the cavity by resonance until the internal field of the cavity reaches the asymptotic value

$$E_{int} = \beta J_0(m) \mathcal{A} E_L, \tag{2}$$

where we have defined

$$\mathcal{A} \equiv \frac{|t_1|^2 |r_2|}{1 - |r_1||r_2|},$$

a *cavity matching factor* [4]. The forward leakage field transmitted through M2 is $E_T = r_2 E_{int}$, and the return leakage field transmitted through M1 back toward the laser is $E_C = t_1 E_{int}$. In addition to the return leakage, fraction $r_1 E_L$ of the input is promptly reflected by input coupler M1 (of which, as we have described, only $\beta J_0((m)$ is in the same spatial and temporal mode as E_C). On resonance

[1]This is computationally convenient, but by no means necessary. An ω_m smaller than or comparable to the cavity width can also be used [?].

[2]This simplification can be violated if the detector or intervening optics mix spatial modes, destroying the orthogonality. In this case the mismatched light can introduce coherent terms, which may carry noise.

this component will be exactly in antiphase with E_C, and, depending on the ratio $|r_1|^2/\mathcal{A}$, may be equal, larger (*undercoupled*) or smaller (*overcoupled*) than E_C.

2.3 Phase Error Signal

Now consider a small perturbation of the laser phase, $\delta\phi_L(t) = \phi_0$. For a short time ($\ll \tau_e$) the cavity field is unresponsive constant, so the two modematched reflected field components are no longer precisely in antiphase but are now instantaneously offset by relative angle $\phi = \phi_L - \phi_C = \phi_0 + m \sin \omega_m t$ (Figure 2). By the law of cosines, the squared resultant of the two reflected components is

$$|E_R|^2 = \beta^2 J_0^2(m)|E_L|^2 + |E_C|^2 - 2\beta J_0(m)|E_L||E_C| \cos(\phi_0 + m \sin \omega_m t) \quad (3)$$

2.4 Time Dependence of the Reflected Field

The internal cavity field decays and is replaced by fresh light bearing the new phase ϕ_0. Since the input frequency is still $\omega_0 = \omega_c$, the final state will have exactly the same internal field amplitude as the initial state. The cavity field phase $\phi_C(t)$ evolves[3] as

$$2\tau_e \dot{\phi}_C(t) = \phi_L(t) - \phi_C(t)$$

so that the relative phase $\phi \equiv \phi_L - \phi_C$ obeys

$$2\tau_e \dot{\phi} + \phi = 2\tau_e \dot{\phi}_L \,. \quad (4)$$

The Laplace transform solution, which we may regard as the transfer function between the laser field phase and the resulting phase difference between laser and cavity fields, is simply

$$\frac{\phi(s)}{\phi_L(s)} = \frac{2\tau_e s}{1 + 2\tau_e s} \,. \quad (5)$$

[3] assuming $\tau_e \gg 2l/c$

2.5 Detection and Demodulation

The resultant field E_R, along with the improperly matched and sideband power totally reflected by the input coupler M1, is directed by a circulator or Faraday device onto photodiode D1, producing photocurrent (following Equations 2 and 3)

$$i_p(t) = I_0 \left\{ 1 - \beta^2 + \beta^2 |r_1|^2 + \beta^2 \mathcal{A}^2 J_0^2 - 2\beta^2 \mathcal{A} J_0 \cos(\phi + m \sin \omega_m t) \right\}$$

where we have introduced the convenient parameter $I_0 \equiv e\eta P / h\nu_0$, the D.C. photocurrent measured when the cavity is far from resonance, for incident laser power P and detector quantum efficiency η.

In the limits $|r_1|^2 \simeq 1$ (the input coupler is highly reflective) and $\phi \ll \pi$ (the phase error is a small one), the detected photocurrent reduces to

$$\frac{i_p(t)}{I_0} \simeq 1 + \beta^2 \mathcal{A}^2 J_0(m)^2 - 2\beta^2 \mathcal{A} J_0(m) \left[\cos(m \sin \omega_m t) - \phi \sin(m \sin \omega_m t) \right]. \qquad (6)$$

Again referring to Figure 1, this photocurrent flows through impedance $\mathrm{Z_R}$, is preamplified by an RF preamplifier with voltage gain $\mathrm{G_R}$ and fed into a mixer. The mixer is simultaneously driven by a sample of the phase modulation waveform, adjusted in RF phase θ_R by a phase shifter. The mixer output voltage is integrated by a lowpass filter whose time constant T exceeds the period of the moduluation but is shorter than the cavity storage time ($\omega_m^{-1} \ll T \ll \tau_e$). This gives the instantaneous error signal voltage

$$\begin{aligned} V_m &= -\frac{8}{\pi} \mathrm{Z_R G_R G_m} \beta^2 \mathcal{A} J_0 \, \phi_0 \\ &\quad \times \frac{1}{T} \int_0^T \sin(m \sin \omega_m t') \sin(\omega_m t' + \Phi_\mathrm{R}) \, dt', \\ &= -\frac{8}{\pi} \mathrm{Z_R G_R G_m} \beta^2 \mathcal{A} \cos\theta_\mathrm{R} J_0(m) J_1(m) \, \phi_0 \qquad (7) \end{aligned}$$

where we include the conversion gain of the mixer $\mathrm{G_m}$ (generally about 1/2) and a correction factor $(4/\pi)$ for the property that mixers typically demodulate by a nearly square waveform, rather than the sinusoid we use in the integration[4] [7]. We have omitted all terms in the photocurrent oscillating at frequencies other than ω_m; these all average to zero after the demodulation, but can in practice add noise and interference to the signal.

Assembling the optical phase response with the photodetection and demodulation results, the overall transfer function between laser phase and demodulator output voltage is

$$\frac{V_m}{\phi_L}(s) = -\frac{8}{\pi}\mathrm{Z_R G_R G_m}\,\beta^2 \mathcal{A}\cos\theta_\mathrm{R} J_0(m) J_1(m)\left(\frac{2\tau_e s}{1+2\tau_e s}\right). \quad (8)$$

[4]In principle we should also fold the impulse response of functions $\mathrm{Z_R}$ and $\mathrm{G_R}$ inside the integral, to allow for RF bandpass filtering which is usually featured by these elements. It is equivalent, and often more convenient, to compute the frequency responses $\mathrm{G_R(s)}$ and $\mathrm{Z_R(s)}$ and frequency translate them from ω_m to DC, treating them with the other compensation downstream of the mixer.

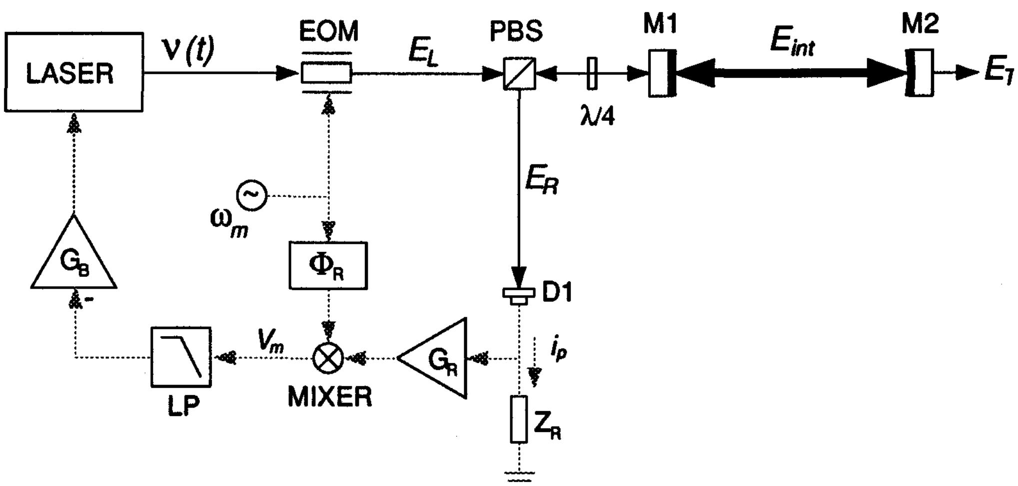

Figure 1: Semi–schematic of a system for laser frequency stabilization. In this example, the laser's frequency ν is controlled in response to a phase/frequency error signal derived from the output e_1 of photodetector D1. The photocurrent detected by D1 is proportional to the squared modulus of reflected field E_R, which is the resultant of the incident field E_L and the stored field in the cavity E_C (see Figure 2). E_L is phase modulated at radio frequency ω_m; the Fourier component at ω_m in the detected signal e_1 is selected out by homodyne RF detection, using a double–balanced mixer whose reference is derived from the phase modulating oscillator. The result is lowpass–filtered to remove the images at $2\omega_m$ and above, and fed into the frequency servo as its error signal.

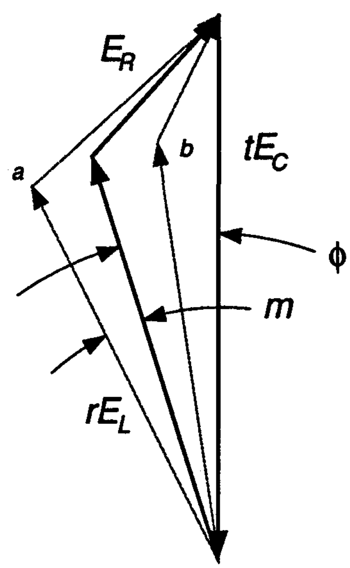

Figure 2: Phasor diagram of the optical field components striking detector D1 in Figure 1. The common overall time dependence $e^{j2\pi\nu t}$ has been suppressed so we can view the field vectors in a stationary frame. The detector photocurrent is proportional to the squared modulus of reflected field E_R, which is the resultant of the incident field rE_L and the stored field in the cavity tE_C, where r and t are, respectively, the amplitude reflectivity and transmission of M1. E_L is phase modulated at radio frequency ω_m by peak angle m, the *modulation index*, that is, $E_L(t) = E_0 \exp[jm \sin(\omega_m t)]$. The squared resultant field E_R contains a Fourier component at ω_m proportional to the phase difference ϕ between the cavity and laser fields.

3 Feedback Actuators

A Optical Phase and Frequency Noise Terminology

Deviations of the laser frequency or phase are usually expressed in terms of the difference between the laser parameters and those of an "ideal" laser, whose output field $E_i(t) = E_0 \sin(2\pi\nu_0 t)$ is a perfect sine wave with phase $\phi = 0$ at time $t = 0$. The actual laser's deviation from this standard can then be expressed either in terms of frequency, as in

$$E(t) = E_0 \sin[2\pi(\nu_0 + \delta\nu(t))\, t],$$

or as phase fluctuations, as in

$$E(t) = E_0 \sin(2\pi\nu_0 t + \delta\phi(t)).$$

The two measures of deviation are related by

$$\delta\phi(t) = 2\pi \int_0^t \delta\nu(t')\, dt'$$

and are effectively interchangeable, although we will mostly use the frequency deviation for convenience. The *fractional frequency deviation* is defined as $y(t) \equiv (\nu(t) - \nu_0)/\nu_0$. In practice, since time and frequency measurements are intrinsically relative, we will often define the resonant frequency of a reference cavity as ν_0 for simplicity, neglecting the intrinsic noise of the reference with respect to "absolute" time or frequency.

Random variations of the laser phase or frequency are described by statistical measures which relate the characteristic timescale of the typical deviations to their magnitude. One common time–domain measure, the *Allan variance,* $\sigma_y^2(\tau)$, describes the typical amount by which the fractional frequency y wanders between two measurements as a function of the time τ between the measurements, and is typically employed in the description of accurate clocks [1]. A more accessible measure from the standpoint of control system analysis, and the typical quantity measured in the laboratory, is the frequency–domain *power spectral density* or *PSD* of y, $S_y(f)$.

The power spectral density[5] of the random process $y(t)$ is defined as

$$S_y(f) = \lim_{T\to\infty} \frac{1}{T} \left| \int_{-T}^{T} y(t)\, e^{-2\pi j f t}\, dt \right|^2$$

and is usually estimated in the laboratory by squaring and averaging together several successive discrete Fourier transforms $\tilde{y}(f)$ of the variable $y(t)$ to approximate the infinite integral [2]. If, as in this case, the variable y is dimensionless, S_y has units of Hz^{-1}. One straighforward interpretation of this quantity is as follows: if the signal $y(t)$ is passed through a bandpass filter of width Δf at center frequency f_0, the mean squared output of the filter will be $\overline{y'^2} = S_y(f_0)\,\Delta f$, where the bar denotes a time average. The RMS value of the filter output is thus $y'_{RMS} = \sqrt{S_y(f_0)\Delta f}$. Another quantity often mentioned is the spectral density of the raw frequency ν, $S_\nu(f) = \nu_0^2 S_y(f)$, which has the confusing units of $\mathrm{Hz}^2/\mathrm{Hz}$. Finally, to add a bit more confusion, the typical quoted quantity is actually the square root of the PSD, which I call the *RPSD* (for "root power spectral density"). The RMS value measured is thus the RPSD multiplied by the square root of the filter bandwidth. An example RPSD spectrum of frequency noise from an argon ion laser is shown in Figure 3.

[5] Here we use the conventional single–sided power spectrum, that is, we define frequency f to be positive.

Figure 3: Typical frequency noise RPSD of a large–frame argon ion laser (running single–frequency, single mode) for fluctuation frequencies up to 2.5 kHz.

B Properties of Optical Cavities

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