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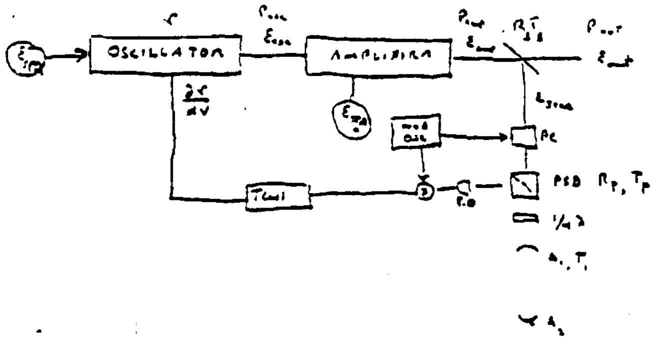
# FREQUENCY STABILIZATION FUNDAMENTAL NOISE TERMS




$\nu =$ laser frequency

$f =$ noise frequency

Basic noise terms

Spontaneous emission in oscillator Townes limit

$\nu^2(f) = \frac{\nu}{2\pi S Q}$

$S = \#$ of stored photons in mode

$Q =$ quality factor of oscillator cavity

$= \frac{4\pi\nu}{(T+A)} \frac{\ell n}{c}$

$= 2\pi\nu\tau_{en\ st}$

$n =$ index of oscillator cavity

$A =$ loss in cavity

$\ell =$ geometric path length in oscillator

$T =$ output coupler transmission

$\tau_{en\ st} = \frac{2\ell n}{c(T+A)}$

$S$ in terms of measurable, the output power

$$S = \frac{2P_{out}}{h\nu}\left(\frac{\ell}{cT}\right) \cong \frac{P_{osc}}{h\nu}\tau_{en\ st}$$

$$\underset{\text{Townes}}{\nu^2(f)} = \frac{h\nu}{P_{osc}(2\pi\tau_{en\ st})^2}$$

Line width of oscillator 1/2 power full width (Gaussian phase wander assumed)

$$\Delta\nu(1/2\text{power}) = \frac{h\nu}{2\pi P_{osc}\tau^2_{en\ st}}$$

Reexpressing the noise spectral density in terms of the intrinsic line width

$$\underset{\text{Townes}}{\nu^2(f)} = \frac{\Delta\nu(1/2\text{power})}{2\pi}$$

Argon laser

$\ell = 150cm$

$T = 0.1$

$A = 0.01$

$\lambda = 5.145 \times 10^{-5} cm$

$P_{osc} = 5watts$

$n = 1$

$\tau_{en\ st} = 9 \times 10^{-8} sec$

$\Delta\nu(\frac{1}{2}\text{power}) = 1.5 \times 10^{-6} Hz$

$\nu(f) = 5 \times 10^{-4}\ Hz^{1/2}$

Nd: YAG miser ring (at present)

$n = 1.82$ $\tau_{\text{store}} = \frac{n\ell}{Tc} = 4.5 \times 10^{-9} sec$
$\ell = 1.1cm$
$T = .015$
$P_{\text{osc}} = 5mw$ $\Delta\nu(\frac{1}{2}\text{power}) = 0.29Hz$
$\nu = 2.83 \times 10^{14} Hz$ $\nu(f) = .22Hz^{1/2}$

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The power spectral density of spontaneous emission noise due to spontaneous emission in the amplifier

$P_{\text{rms}_{\text{noise}}} = hr\Delta B$ $\Delta B =$ bandwidth of the amplifier

This produces a net phase wander in the output of the amplifier

$$\Delta\varphi^2_{\text{rms}} = \frac{hr\Delta B}{P_{\text{osc}}}$$ if uniformly distributed over the bandwidth

$$\varphi^2(f) = \frac{h\nu}{P_{\text{osc}}} \qquad \nu^2(f) \cong f^2 \frac{h\nu}{P_{\text{osc}}}$$

$$\underset{\text{amplifier}}{\nu(f)} \sim f \left(\frac{h\nu}{P_{\text{osc}}}\right)^{1/2}$$

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The intrinsic noise from the shot noise fluctuations at the cavity stabilizer detector

Sensitivity to frequency fluctuations

$$I_{\text{sig}} = R_s 4\left[1 - \frac{4(A_1 + A_2)}{T_1}(1 - A_2)\right]^{1/2} T_P R_p \; I_o \; J_o(\Gamma) J_1(\Gamma)$$

$$\sin\left[\frac{16\pi\ell}{c}\left(\frac{1}{T_1(1 - \frac{A_1+A_2}{T_1})}\right) \Delta\nu\right] \sin\omega_m t$$

$$I_{\text{DC}} = R_s T_p R_P I_o J_o^2(\Gamma)\left[1 - \frac{4(A_1 + A_2)}{T_1}\right]$$

Simplify by assuming $T_1 > A_1 + A_2 \longrightarrow$ which is not necessarily the best thing to do to keep storage time in reference cavity high

$$\tau_{en\ st} = \frac{2}{T_1\left(1 - (\frac{A_1+A_2}{T_1})^2\right)} \frac{\ell}{c}$$

$T_P = R_P \sim 1$

With these assumptions

$$I_{\rm DL} = R_s I_o J_o^2(\Gamma)$$
$$\underset{\substack{\rm at \\ \rm resonance}}{I_{\rm SIG}} \cong 32\pi R_s I_o J_o(\Gamma) J_1(\Gamma)\ \tau_{\rm st}\Delta\nu \sin\omega_m t$$

Look at shot noise limit

$$\underset{\rm noise}{\dot{n}^2(f)} = \frac{2\ R_s I_o J_o^2(\Gamma)\eta}{h\nu}$$

$$\underset{\rm sig}{\dot{n}^2(f)} = \eta^2 (32\pi)^2 \frac{R_s^2 I_o^2 J_o^2(\Gamma) J_1^2(\Gamma)}{(h\nu)^2} \tau_{\rm st}^2 \nu^2(f) \frac{1}{2} \leftarrow \langle \sin^2\omega_m t \rangle$$

Value of $\nu^2(f)$ limited by shot noise $\underset{\rm shot}{\dot{n}^2(f)} = \underset{\rm signal}{\dot{n}^2(f)}$

$$\underset{\rm shot}{\nu(f)} = \frac{1}{16\pi}\left(\frac{h\nu}{\eta P_{\rm amp} R_s}\right)^{1/2} \frac{1}{J_1(\Gamma)\tau_{\rm en\ st}} \qquad \underset{\rm max}{J_1(\Gamma)} \sim \frac{1}{\sqrt{3}}$$

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Example suppose cavity storage time

$\ell = 1$ meter $\quad T \sim 1 \times 10^{-2} \quad A \sim 1 \times 10^{-4} \quad \tau_{\rm EN} \simeq \frac{2}{T_1}\frac{\ell}{c} \simeq 6.6 \times 10^{-7} sec$

$R_s = \underset{\text{glass uncoated}}{4\times 10^{-2}} \quad P = 10$ watts $\quad \lambda = 5.145 \times 10^{-5} cm$

$P_{\rm det} = 400 mw \quad \nu = 5.83 \times 10^{14} Hz \quad \eta = .8$

$\nu(f) \simeq 5.7 \times 10^{-5} \quad Hz^{1/2}$

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Assumption for feedback system

$$\underset{\rm output}{\nu(f)} = \underset{\rm shot}{\nu^2(f)} + \frac{\underset{\rm amp}{\nu^2(f)} + \underset{\rm osc}{\nu^2(f)}}{(1+k_{fg})^2}$$

To achieve the shot noise limit

$$\underset{\rm output}{\nu(f)} \cong \underset{\rm shot}{\nu(f)} + \frac{\left(\underset{\rm osc}{\nu^2(f)} + \underset{\rm amp}{\nu^2(f)}\right)}{k_{fg}(\omega)} \qquad k_{fg}(0) \sim \Delta B_{\rm loop}$$

In $4Km$ system if one desires $5 \times 10^{-4} sec$ storage time with mirrors having $A \sim 1 \times 10^{-4}$

$$\tau_{\text{en st}} = \frac{2}{T_1}\frac{\ell}{c} \qquad T_1 = \frac{2\tau_t}{\tau_{\text{en st}}} = 5.33 \times 10^{-2} \qquad \tau_t = 1.33 \times 10^{-5} sec = 13\mu sec$$

Power lost in cavity $= \frac{4(A_1+A_2)}{T_1} = 1.5 \times 10^{-2}$

If recycling is done these mirrors limit recycling gain to 30 in power for two cavities, $h$ sensitivity increase of $5-6$

If long cavities are used for frequency stabilization in a final stage one must decide on the power dedicated to the frequency lock

Without recycling: No strong case for limiting the power to the lock system $R_s = 1.5 \times 10^{-2}$ is not unreasonable

$P_{\text{in}} = 10 watts$ power/arm $= 5 watts$ shot noise in a FOX/LI system same as direct cavity interrogation system within factor of 2

Bandwidth is limited to $\frac{1}{2}\nu_{fs} \sim 18KHz$

$$\nu(f)_{\text{shot}} \simeq \frac{1}{16\pi}\left(\frac{h\nu}{\eta P_{\text{arm}} R_s}\right)^{1/2} \frac{1}{J_1(\Gamma)\tau_{\text{en st}}}$$
$$\simeq 1.3 \times 10^{-7} Hz^{1/2}$$

$J_1(\Gamma)$ does not appear in gravity wave shot noise for sym FOX/LI system. so one could make $J_1(\Gamma) = \frac{1}{\sqrt{3}}$

If system is recycled it becomes more complicated loss is not equal to $R_s$ but rather $R_s(1 - J_o(\Gamma))$ $J_o(\Gamma) = 0.2$

when

so optimization must be done on $\Gamma$ in modulator $J_1(\Gamma)_{\max} = \frac{1}{\sqrt{3}}$

But if frequency lock power to be maximized in a recycled system

$$R_s \sim \frac{4(A_1+A_2)}{T_1}$$

then

$$P_{\text{arm}} = \frac{P_{\text{in}}}{2 \times 16\left(\frac{A_1+A_2}{T_1}\right)} \simeq 8.33 P_{\text{in}}$$

main beam splitter↑ ↖total loss

So a recycled system with max signal into cavity lock system of the example parameters

$\nu(f) \simeq 3.4 \times 10^{-8} Hz^{1/2}$

To get to this with a loop gain of $1 \times 10^4$ limited by cavity free spectral gain requires the stabilization to be good to

$\nu(f)_{\text{source}} < 3 \times 10^{-4} Hz^{1/2}$ Clear that one must cascade frequency control loops with intermediate cavities

Given the $\nu(f) \sim 10^4 Hz_2^{1/2}$ in a typical argon laser, Nd:YAG at $\nu(f) \sim 10^1 Hz^{1/2}$ would also require intermediate stages of frequency locking

What is required?

Using transfer function of $FP$ to gravity wave at $\omega \to 0$

$$\left.\frac{\Delta\varphi}{\Delta h_g}\right|_{FP} = \frac{4\pi c\tau_{en\ st}}{\lambda}$$

The phase change at the recombined output of the interferometer due to a frequency shift

$$\Delta\varphi_1 \cong \frac{16\pi}{T_1}\frac{\ell_1}{c}\Delta\nu \qquad \Delta\varphi_2 \cong \frac{16\pi}{T_2}\frac{\ell_2}{c}\Delta\nu$$

The frequency shift is common mode to the two cavities. Let the two cavities have almost the same length and almost the same transmission for the input mirror (could do this with loss as well but the result will be the same)

$$T_1 = T - \frac{\Delta T}{2} \qquad T_2 = T + \frac{\Delta T}{2}$$

$$\ell_1 = \ell + \frac{\Delta\ell}{2} \qquad \ell_2 = \ell - \frac{\Delta\ell}{2}$$

$$\Delta\varphi_{\text{total}} = \Delta\varphi_1 - \Delta\varphi_2 = \frac{16\pi\ell_o}{cT}\left[\frac{\Delta\ell}{\ell} + \frac{\Delta T}{T}\right]\Delta\nu$$

Converting to $h_g$ through transfer function

$$\Delta h_g = \frac{2\Delta\nu}{\nu}\left[\frac{\Delta\ell}{\ell} + \frac{\Delta T}{T}\right] \qquad \text{using} \qquad \tau_{en\ st} \sim \frac{2}{T}\frac{\ell}{c}$$

Converting to amplitude spectral density

$$\underset{\text{noise}}{h(f)} = \frac{2\nu(f)}{\nu}\left[\frac{\Delta\ell}{\ell} + \frac{\Delta T}{T}\right]$$

prior formula gave (bluebook)

$$h(f) = 2\left(\frac{\Delta\ell}{\ell} + \frac{\Delta F}{F}\right)\frac{\nu(f)}{\nu}$$

↖ finesse balance

Estimate of what is required

Depends on ability to match finesse or $T$. Maximum length unbalance that can be used to compensate finesse unbalance is determined by instrumentation chamber size

$\Delta\ell < 3\,meters$ adjustment  max $\frac{\Delta\ell}{\ell} \sim 10^{-3}$

$\Delta T/T$ depends on mirror coating uniformity could be $10^{-2} \to 10^{-3}$

Optimistic case would be

$$\frac{\Delta\ell}{\ell} + \frac{\Delta T}{T} \to 0 \qquad \text{more sensible is few}\times 10^{-3} \qquad \text{say } \; 2\times 10^{-3}$$

For $h_{rms} \sim 10^{-21}$ at $1KHz$ $\quad h(f) \sim 3\times 10^{-23}$

$$\nu(f) < \frac{\nu h(f)}{2[\frac{\Delta\ell}{\ell} + \frac{\Delta T}{T}]} \simeq \frac{6\times 10^{14}\times 3\times 10^{-23}}{2\times 2\times 10^{-3}} \simeq 4.5\times 10^{-6} Hz^{1/2}$$

Not clear that electronic subtraction of the cavity from signals helps here once one has made such a good optical balance. One would have to do better in subtraction than in the balance condition i.e. a part $10^3$ subtraction since the frequency noise in the cavities individually is allowed to be $10^3$ times larger.