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Followup Comments to the Tutorial on Power Spectral Density

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Inspired by questions asked on Monday afternoon, here is some additional information on the measurement and interpretation of power spectral density.

First, an issue of notation, which I'll try to clarify by the example of seismic noise. Typically, measurements are expressed in terms of the *amplitude* spectral density, which is the square root of the power spectral density

$$\tilde{x}(f) = \text{amplitude spectral density of seismic displacement}$$

The amplitude spectral density is related to the power spectral density $P_x(f)$ by

$$\tilde{x}(f) = \sqrt{P_x(f)}$$

where

$$P_x(f) = \lim_{T\to\infty} \frac{2}{T}\left|\int_{-T/2}^{T/2} x(t)e^{-2\pi ift}dt\right|^2$$

$P_x(f)$ is almost (but not quite) the absolute square of the Fourier transform of $x(t)$.

Like the Fourier transform, the power spectral density picks out a frequency, f, and measures the overlap integral between the input data and a sine wave of that frequency. The integral is squared to eliminate sensitivity to phase, which is undefined for pure noise. The dimensions of $P_x(f)$ and $\tilde{x}(f)$ are meter^2/Hz and meter/$\sqrt{\text{Hz}}$, respectively.

How does one measure $\tilde{x}(f)$, and what is its meaning? The measurement might be done by connecting a seismometer to a Fourier transform spectrum analyzer, which computes $P_x(f)$ and can display $\tilde{V}(f)$, the voltage corresponding to x, at many frequencies. The conversion from $\tilde{V}(f)$ to $\tilde{x}(f)$ is done by knowing the transfer function, or conversion factor between displacement input and voltage output, of the seismometer. The spectrum analyzer might be set to display 1000 frequencies equally spaced from 0 Hz (DC) up to 100 Hz, in which case the frequency resolution, or *bandwidth* of the measurement is approximately 0.1 Hz. The bandwidth is approximately the inverse of the measurement time, so it would take ten seconds to accumulate the data for this measurement. No matter what bandwidth is used, the resulting $\tilde{x}(f)$ will be approximately the same, a characteristic of the seismic noise.

Alternatively, the seismometer output can be stored on analog tape and later played back into a spectrum analyzer or a computer system programmed to take Fourier transforms. A computer is needed to analyze long stretches of data for narrow bandwidth measurements.

The spectral density has meaning for any physical quantity that has a spectrum of fluctuations. For example, x may be the the voltage measured across the terminals of a resistor R at temperature T, in which case $P_x(f)$ has units of Volt^2/ Hz. If the fluctuations are from Johnson noise, then $P_x(f)$ has the frequency-independent (white noise) value $4kTRB$, where k is Boltzmann's constant and B is the measurement bandwidth.

The amplitude spectral density can be interpreted as the noise part of a signal-to-noise ratio. For example, if an interferometer output has a spectral density of $1 \cdot 10^{-19}$meter/$\sqrt{\text{Hz}}$ at a frequency of, say, 1 kHz, then the sensitivity for unity signal-to-noise ratio is $1 \cdot 10^{-19}$meter/$\sqrt{\text{Hz}}$ at that frequency. (On a 4-km baseline, this corresponds to a gravity wave strain sensitivity of $\tilde{h}(f) = \tilde{x}(f)/4\text{km}$, or $2.5 \cdot 10^{-23}/\sqrt{\text{Hz}}$.) The measureable displacement, x, that this corresponds to depends on the type of measurement—burst or continuous signals.

Burst signals, such as from the birth of black holes, are of limited duration—usually on the order of milliseconds for the type of signals we expect to detect. The duration t_0 of a burst is related to its characteristic frequency f by $t_0 \approx 1/f$. The sensitivity is related to the spectral density by

$$\text{Bursts}: \quad x = \tilde{x}(f) \cdot \sqrt{f}$$

Taking the above value for $\tilde{x}(f)$ at 1 kHz, the displacement sensitivity to 1 kHz bursts is approximately $x = 1 \cdot 10^{-19}$ meter/$\sqrt{\text{Hz}} \cdot \sqrt{1\ \text{kHz}} = 3 \cdot 10^{-18}$ meter.

For continuous periodic signals, the sensitivity can be improved by averaging over a long stretch of time.

$$\text{Periodic}: \quad x = \frac{\tilde{x}(f)}{\sqrt{T_{\text{meas}}}}$$

where T_{meas} is the duration of the data stream in a search for or measurement of continuous, periodic sources—e.g. 10^7 seconds for an optimal use of 4 months of data, resulting in an unbelievably small displacement sensitivity of $3 \cdot 10^{-22}$ meter. Because T_{meas} can be so much larger than t_0, much smaller displacements from periodic signals can be detected than from burst signals. The strength of gravity waves from periodic signals, however, is often inherently much smaller.

In first approximation, the sensitivity of a gravity wave detector to a given frequency of signal depends on the sum of the contributions from the power spectral densities of the various noise sources at that frequency:

$$\tilde{h}_{\text{sum}}(f) = \sqrt{P_{h_{\text{seismic}}}(f) + P_{h_{\text{shot noise}}}(f) + \ldots}$$

Noise outside of the usual frequency band for detecting gravity waves can also be important. For example, thermal drift associated with temperature fluctuations has significant amplitude mainly at time-scales of several minutes and longer, corresponding to milliHertz frequencies. This will not directly affect the sensitivity to 100 Hz signals. On the other hand, nonlinear couplings, such as drift in the mass position causing a misalignment to optical beams, can open a path for noise to enter in through, say, a coupling between beam wiggle noise and misalignment. That is, the sensitivity at 100 Hz may depend on the product of the wiggle at 100 Hz and the misalignment at 1 milliHertz. The entire frequency band of a power spectral density measurement of noise contains useful information.