LIGO-T890008-00-R

#80

List of 5/01/89

Steps Toward Optimal Design of Vibration Isolating Stacks

Peter R. Saulson

April 21, 1989

1 Summary

I construct a general one-dimensional model for evaluating the vibration isolation provided by a stack of n layers, from which a pendulum is suspended. When the mass m of each stack layer is less than that of the test mass, M, the isolation (above the resonances) is poorer by a factor

$$P = \left(\frac{n + \frac{M}{m}}{n+1}\right)^n .$$

I suggest a method for optimizing isolation at a particular frequency, subject to a constraint on the total mass of the stack.

2 Model

I examine the vibration isolation expected from a pendulum hung from a stack. So far, the analysis has only been performed using Newton's Law for lumped elements, moving in one dimension. In this approximation, it is easy to write down a general form for the coupled equations of motion of the masses.

I specialize the analysis to the case of a single pendulum, suspended from a stack of n layers. The layers of the stack are identical to each other, characterized by masses m connected by springs k. The pendulum has mass M and spring K, or a resonant frequency $\omega_p \equiv \sqrt{\frac{K}{M}}$.

First, I consider the question of how the isolation depends on the mass m of the stack elements, for a fixed number of layers n. For definiteness, it is necessary to specify how the value of the spring constant k should scale with m. I chose the requirement that the ratio of the spring constant to the total mass $M_T \equiv M + nm$ stay fixed at a chosen value

$$\frac{k}{M_T} \equiv {\omega_0}^2.$$

This is equivalent to keeping a fixed amount of sag in a vertical spring.

I cast Newton's Laws in the frequency domain by assuming a harmonic time dependence $e^{i\omega t}$ for all quantities. The general form of the coupled equations of motion is then

$$\mathbf{D}\ \mathbf{x} = \mathbf{y}.$$

Here, $\mathbf{x}$ is the column vector of mass coordinates $(x_1, x_2, \ldots, x_n, x_p)^T$, and $\mathbf{y}$ is the scaled input motion $(x_i\frac{k}{m}, 0, 0, \ldots)^T$. (The symbol ^{T} denotes the transpose.) The dynamics matrix $\mathbf{D}$ has the form

$$\mathbf{D} = \begin{pmatrix} 2a-\omega^2 & -a & 0 & 0 & \ldots & 0 \\ -a & 2a-\omega^2 & -a & 0 & \ldots & 0 \\ 0 & -a & 2a-\omega^2 & -a & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & -a & a+b-\omega^2 & -b \\ 0 & \ldots & 0 & 0 & -{\omega_p}^2 & {\omega_p}^2-\omega^2 \end{pmatrix}.$$

Here

$$a \equiv \frac{k}{m} = {\omega_0}^2\Big(n + \frac{M}{m}\Big),$$

and

$$b \equiv {\omega_p}^2\frac{M}{m}.$$

For ease of calculation, I have chosen not to include any damping in the model. This will not be important for any of the results I want to derive from the model. Note that the matrix $\mathbf{D}$ is of tridiagonal form (i.e. only the main diagonal and diagonals adjacent to it have non-zero elements). This comes about because in a stack, each mass is connected only to its nearest neighbors.

Our chief interest is in the transfer function $\frac{x_p}{x_i}$, which gives the fraction of input vibration which reaches the test mass. We can solve for it using, for example, Cramer's Rule, which yields

$$\frac{x_p}{x_i} = \frac{det(\mathbf{N})}{det(\mathbf{D})}.$$

The matrix $\mathbf{N}$ is derived from $\mathbf{D}$ by replacing its last column with the column vector $\mathbf{y}$.

The determinant of the matrix $\mathbf{N}$ consists only of a single term, ${\omega_0}^{2n}(n+\frac{M}{m})^n{\omega_p}^2$. One way to prove this is by invoking two theorems of the theory of determinants. First, the determinant of a matrix with one column which has only one non-zero element is equal to that element times its minor (i.e., times the determinant of the matrix derived by excising both the row and the column containing the non-zero element). This smaller matrix is of upper triangular form, since all of the elements below the main diagonal are zero. (This is because the matrix $\mathbf{D}$ is tridiagonal.) Another theorem states that the determinant of an upper triangular matrix is equal to the product of its diagonal elements. This completes the proof.

The determinant of the dynamics matrix $\mathbf{D}$ is not quite so simple. It is a polynomial in ω of degree $2(n+1)$. The general form is complicated, but if we are only interested in the high frequency limit, then we can approximate it, up to a sign, by its leading term, $\omega^{2(n+1)}$.

Thus we can write

$$\lim_{\omega\to\infty}\frac{x_p}{x_i}=\frac{{\omega_0}^{2n}(n+\frac{M}{m})^n{\omega_p}^2}{\omega^{2(n+1)}}.$$

This approximation is useful starting at frequencies not much above the highest resonance.

3 Penalty for Small Masses

One way to interpret this result is to consider what happens if we leave the number of stack layers, n, fixed, and compare the transfer function for different values of the mass ratio $\frac{M}{m}$. A convenient measure is the penalty P, defined as the ratio of the transfer function with arbitrary $\frac{M}{m}$ to the transfer function with mass ratio equal to unity. In the region above the highest resonance of either system, the penalty is just a constant, given by

$$P=\left(\frac{n+\frac{M}{m}}{n+1}\right)^n.$$

For example, for a 4-layer stack, using stack masses one tenth as large as the pendulum mass gives isolation poorer by just over a factor of 60, for all frequencies above the highest resonance of the low-mass stack.

Whether or not this penalty is too high to pay for the benefit of lower stack mass can only be considered in the context of a total system design, and then only by exercising wise judgement.

To help visualize the performance of these two stack design examples, I show graphs of the transfer functions, evaluated using the exact form for the denominator, $det(\mathbf{D})$.

4 Optimal Stack Design for Fixed Total Mass

We can also consider stack design given different constraints. For example, instead of fixing the number of layers we could instead fix the total mass, and look for optimum isolation as the number of layers is varied. The problem so stated is not well-posed, but we can make it so by looking for the number of layers which gives the greatest isolation at a particular frequency, ω_c. For a sufficiently high frequency, it is easy to show that dividing the same mass into more layers improves the isolation. (One way to do this is to treat the high frequency expression for the transfer function as a function of n evaluated at a fixed ω_c, and examine its derivative with respect to n.)

For a large enough number of layers, the isolation at the reference frequency will start to degrade again, as the band of resonances approaches ω_c. This shows that there exists an optimum n.

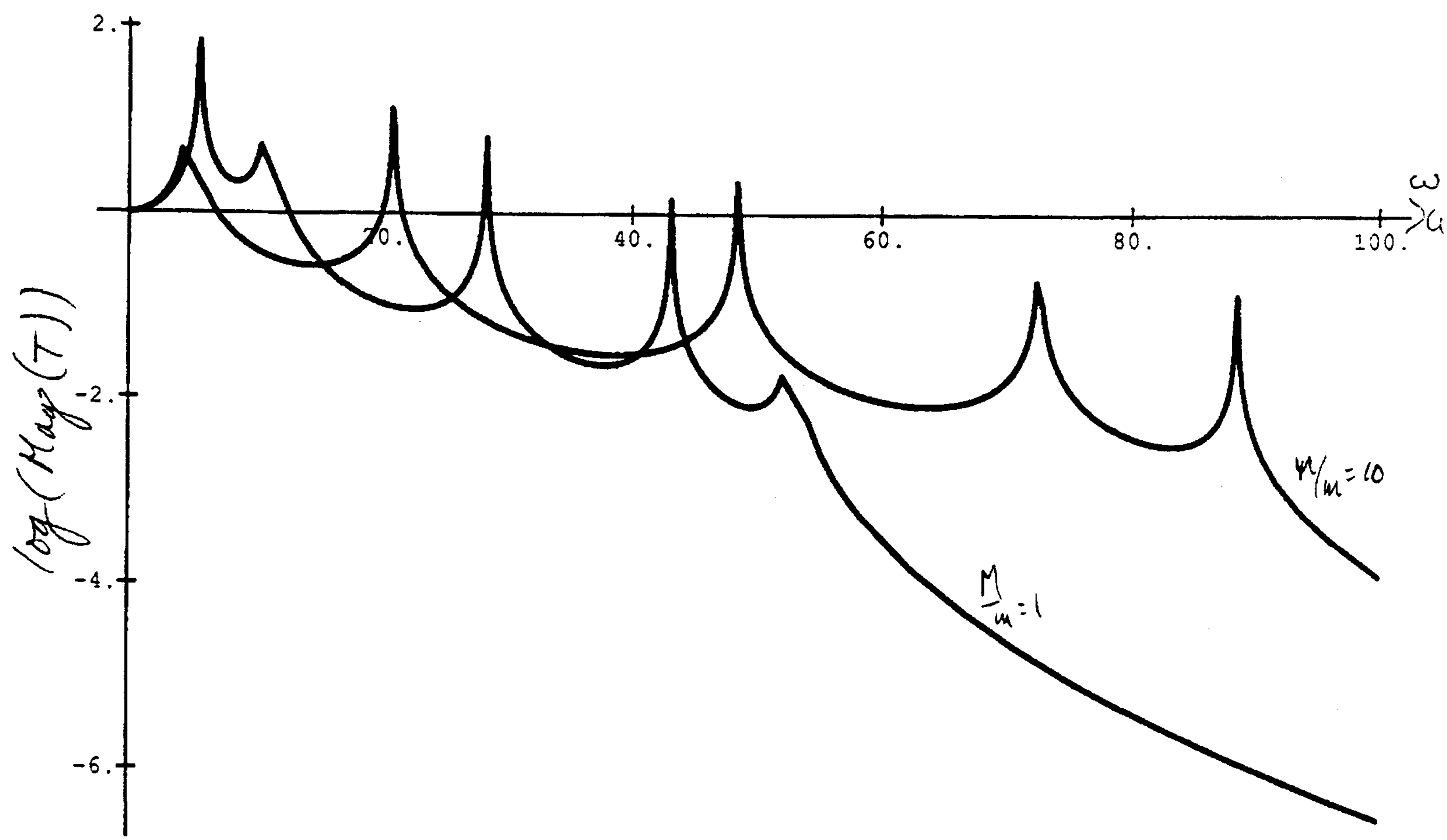
Unfortunately, this penalty for very large n depends on the detailed structure of $det(\mathbf{D})$, not just on its leading term. Therefore, using just the high frequency limit to the transfer function will not lead to the correct n. (It does formally give a value for n_{opt}, but one which is much to large.) I don't know of any way to solve the problem short of explicit evaluation of the transfer function for a variety of layer numbers.

I suspect that considerations of engineering complexity might lead to actual designs with fewer layers than this formal "optimum".

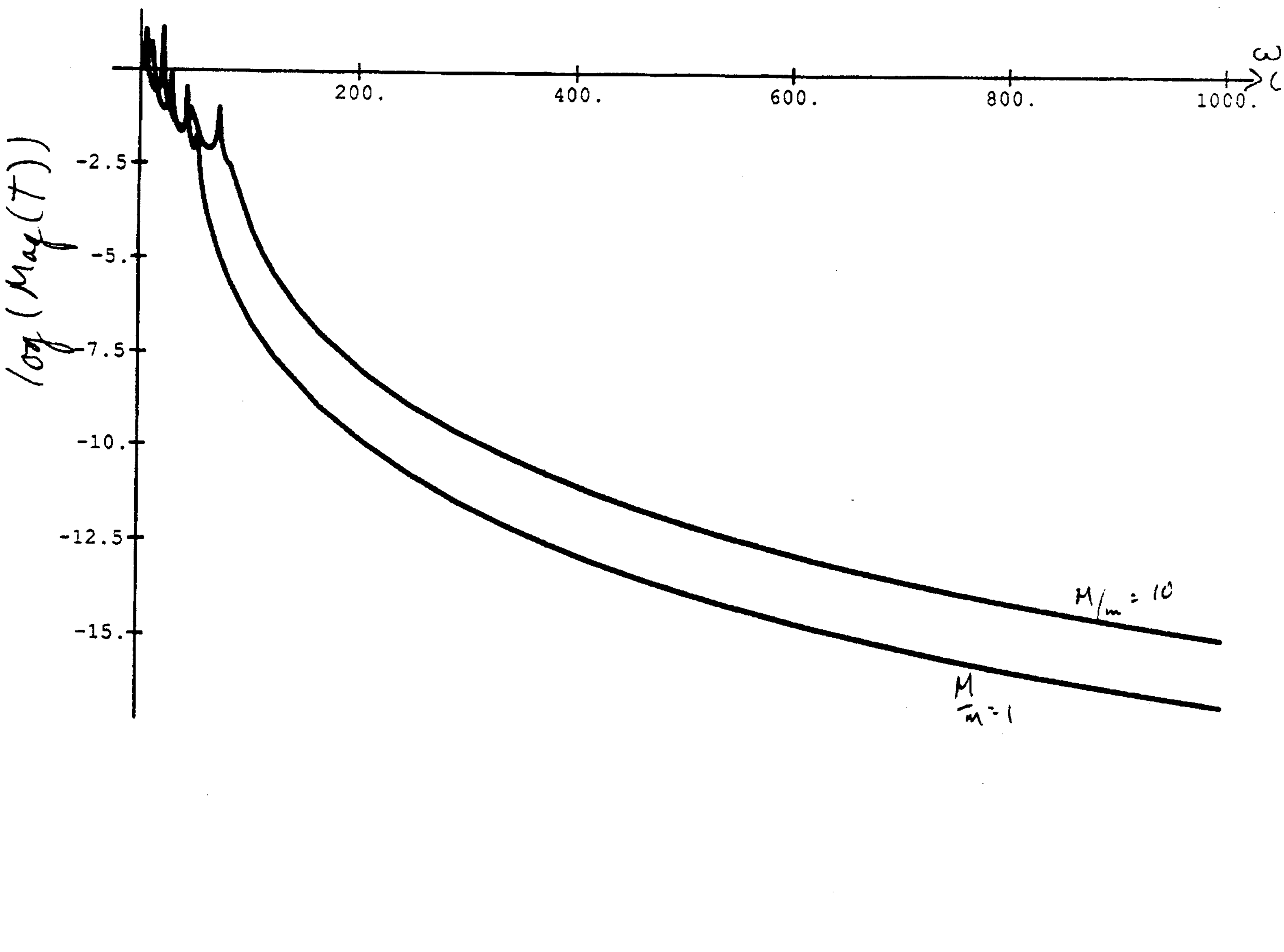
5 Caveats

It may be that a different way of scaling stack spring constants with mass leads to better designs.

These results have been calculated from a one-dimensional model. It is possible, using available software ($MATRIX_X$) and a lot of labor, to treat the coupling between translation and rotation correctly for a real design.



4 layer stack + pendulum



4 layer stack + pendulum