THE RESPONSE OF A FREE MASS GRAVITATIONAL WAVE ANTENNA

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ABSTRACT

The geodesic equation for light travelling under the influence of a weak gravitational wave is solved for the time taken by the light to travel to a distant point and to return, as a function of the frequency, polarization and direction of incidence of the gravitational wave. The result is used to derive the response of gravitaional wave detectors which use optical interfeometry to sense the motion of free test masses. Both delay line and optical cavity based interferometers show increasing sensitivity with increased storage time, until the storage time reaches $(4f)^{-1}$, where f is the gravitational wave frequency. A simple version of a scheme for increasing the sensitivity still further by interchanging the light between the two arms of the detector in synchrony with the gravitational wave is also analyzed.

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I. INTRODUCTION

A new generation of interferometric gravitational wave antennae is under intensive development. Interferometric antennae have several potential advantages. Among these are the potential for reaching very high sensitivities through the use of kilometer-scale baselines and high laser power for the interferometry, a much lower quantum limit than bar antennae, and inherently broadband response.

In its simplest form an interferometric antenna consists of three test masses. One, at the origin, carries a beam splitter and two mirrors which define one end of each of the optical pathes. The other masses, located a distance L from the origin along the x and y axes, respectively, carry mirrors which define the far ends of the optical pathes. The measurable effect of the wave is to alter (by different amounts) the time taken by the light to travel down the two arms and to return. When the two beams are recombined at the beamsplitter, a difference in return times results in a phase difference which can be observed, for example, as a change in intensity.

Previous derivations of the response of an interferometric antenna [1] made the assumption that the light travel time in the interferometer was short compared with the period (or other characteristic time) of the gravitatational wave. With the prospect of kilometer-scale baselines and long light storage times, this assumption becomes inappropriate. In this paper we derive the response of an interferometric antenna to a gravitational wave as a function of frequency, angle of incidence, and polarization.

II. SINGLE PASS INTERFEROMETER

The simplest interferometer is a Michelson interferometer in which the light makes one round trip down the arms before recombination. The difference in travel times for the two arms for this case represents the building block from which we will calculate the time diference for more complicated (more realistic) interferometer configurations.

In this section we solve for the time difference due to a sinusoidal gravitational wave ("transfer function") and for the response to a delta function gravitational wave ("impulse response"). Both calculations ~~assume that the wave is weak so that a calculation~~ are accurate to first order in h, the metric perturbation. ~~is sufficient. In fact, use of~~ the ~~exact solution for the sinusoidal case would not improve the accuracy of the calculation. Fourier decomposition of the incident gravitational wave is valid only to first order in h, so that higher order terms in the antennae response depend on the nonlinerities inherent in the wave as well as those in the detector. In any case,~~ since one expects even the strongest waves incident on the earth to have $h \leq 10^{-16}$ ~~in the kilohertz region,~~ 10^{-10} a first order calculation is sufficient.

Is it necessary to justify first-order calculation?

A. Transfer function

We consider a sinusoidal plane gravitational wave incident on the detector as shown in figure 1. A weak gravitational wave on a flat background can be represented by the metric

$$g_{ij} = \eta_{ij} + h_{ij}(t,\vec{r})$$

where η_{ij} is the Minkowski metric, and $h_{ij}(t,\vec{r})$ are the components of the wave. By a suitable gauge transformation, h_{ij} can always be written as a transverse traceless tensor. If we orient the two arms of the interferometer along the x

and y axes, then the only components of interest are h_{xx} and h_{yy}. These can be written

$$h_{xx} = h\left[\cos 2\Omega\,(\cos^2\varphi - \sin^2\varphi\cos^2\theta) - \sin 2\Omega\,\sin^2\varphi\cos\theta\right]\exp(i\vec{k}\cdot\vec{r} - i\omega t)$$

$$\equiv h\,\alpha_{xx}\exp(i\vec{k}\cdot\vec{r} - i\omega t) \qquad (1)$$

$$h_{yy} = h\left[\cos 2\Omega\,(\sin^2\varphi - \cos^2\varphi\cos^2\theta) + \sin 2\Omega\,\sin^2\varphi\cos\theta\right]\exp(i\vec{k}\cdot\vec{r} - i\omega t)$$

$$\equiv h\,\alpha_{yy}\exp(i\vec{k}\cdot\vec{r} - i\omega t)$$

where h is the strain of the wave in the plane perpendicular to the propagation direction and $\vec{k}$ is the wave vector,

$$\vec{k} = \frac{\omega}{c}(\sin\theta\sin\varphi\;\hat{x} - \sin\theta\cos\varphi\;\hat{y} + \cos\theta\;\hat{z}) \qquad (2)$$

The polarization is specified by the angle Ω.

At frequencies much higher than the suspension frequency, the test masses are free to move and mark inertial reference frames. In the transverse traceless gauge, the spatial coordinates of all three test masses remain fixed as a gravitational wave propagates through the antenna. The light follows a trajectory such that $ds^2 = 0$ where ds is the proper length along the trajectory. This leads to a differential equation for the trajectory of the light travelling along the x axis .

$$c\frac{dt}{dx} = \pm(1 + h_{xx}(x,t))^{1/2} \qquad (3)$$

This can be solved by a series in powers of h

$$t(x) = t^{(0)}(x) + t^{(1)}(x) + \cdots \qquad (4)$$

where $t^{(i)}$ is of order h^i. With this substitution we get

$$c\,\frac{dt^{(0)}}{dx} = \pm 1 \tag{5a}$$

$$c\,\frac{dt^{(1)}}{dx} = \pm\frac{1}{2}h_{xx}(t^{(0)},x,0,0) \tag{5b}$$

Equation (5a) gives the trajectory in the absence of the gravitational wave,

$$t^{(0)}(x) = t_0 + \frac{x}{c} \qquad \text{(outbound)}$$

$$t^{(0)}(x) = t_0 + 2\tau_t - \frac{x}{c} \qquad \text{(inbound)}$$

where t_0 is the time at which the light leaves the origin, and τ_t is the the transit time L/c. Equation (5b) gives the first order correction due to the wave which is the term of interest. Integration of (5b) from $x=0$ to $x=L$ and back gives the arrival time back at the beam splitter

$$t = t_0 + 2\tau_t + \frac{\tau_t h \alpha_{xx}}{2}\exp\left[i\omega(t_0+\tau_t)\right]$$

$$\times\left[e^{\frac{i}{2}(k_x L+\omega\tau_t)}\operatorname{sinc}\frac{1}{2}(k_x L-\omega\tau_t)+e^{\frac{i}{2}(k_x L-\omega\tau_t)}\operatorname{sinc}\frac{1}{2}(k_x L+\omega\tau_t)\right]$$

$$\equiv t_0 + 2\tau_t + \frac{\tau_t h \alpha_{xx}}{2}G(\omega,k_x)\exp\left[i\omega(t_0+\tau_t)\right] \tag{6}$$

where $\operatorname{sinc}x = \sin x / x$. The first two terms represent the arrival time in the absence of a gravitational wave, while the last term represents the effect of the wave. A similar expression is obtained for the y arm. The difference in arrival times is

how did this sign change?

$$\Delta t = \frac{h\tau_t}{2}\exp\left[-i\omega(t_0+\tau_t)\right]\left[\alpha_{xx}G(\omega,k_x) - \alpha_{yy}G(\omega,k_y)\right] \tag{7}$$

which will be detected as a phase difference $\Delta\varphi = 2\pi c\,\Delta t/\lambda$ at a time $t = t_0 + 2\tau_t$.

The transfer function (interferometer output divided by input) is given by

$$H(\omega) \equiv \frac{\Delta\varphi(t)}{h(t)} \tag{8}$$

$$= \frac{\pi c \tau_t}{\lambda}\left[\alpha_{xx} G(\omega,k_x) - \alpha_{yy} G(\omega,k_y)\right] e^{i\omega\tau_t}$$

Here we would show some pictures.

B. Impulse Response

An instructive alternative way of deriving the response of a gravitational wave antenna is to calculate the frequency shift of light traversing the wave. This was done by Burke[3] for one-way trips, and a calculation for round trips was made by Estabrook and Wahlquist[4] and by Hellings[5] in their discussions of gravitational wave detection using Doppler tracking of space probes. They find that light making a round trip along the x axis suffers a frequency shift.

$$\nu(t) = \frac{\nu}{2}\Big[(1+\cos\varphi\sin\theta)h(t,0,0,0) - 2\cos\varphi\sin\theta h(t-\tau, l, 0, 0) \; [illegible]$$

Note that in this picture, the effect depends explicitly only on the values of $h(t,x,y,z)=h(t-\vec{k}\cdot\vec{r})$ at the events where the light encounters one of the mirrors. For b round trips the formula is similar, with additional terms for each reflection. Finally, the net frequency difference between light travelling in the two arms is

$$\Delta\nu(t) = \frac{\nu_0}{2}\Big[\textit{lots of stuff}\Big]$$

To convert to the phase difference between the two light beams, we use the relation

$$\Delta\varphi = \int_{-\infty}^{t} 2\pi \Delta\nu dt$$

This method of calculation makes it particularly simple to find the impulse response of the antenna. Making the substitution $h(t,x)=h\delta(t-\frac{\vec{k}\cdot\vec{r}}{c})$ in equation (XXX) shows that the impulse response is a set of impulses in the instantaneous frequency difference. Since the integral of a unit delta function is a unit step function, it is also apparent that the response of the phase difference to an impulse is a set of steps. The phase difference is non-zero for an interval of length τ_s, and contains b identical subsections, where b is the number of round trips taken by the light. Calculation of the transfer function, either by substitution of a sinusoidal form for $h(t,x)$ or by explicit Fourier transformation, verifies that the result does in fact agree with equation (XXX).

III. LONGER STORAGE TIME SYSTEMS

The sensitivity of an fixed length interferometer detector can be improved by storing the light in each arm longer than $2\tau_t$. This can be achieved through the use of optical delay lines (suggested for gravitational wave detectors by Weiss [6]) or of optical cavities (suggested by Drever et al [7]). In either case, the total phase difference can be calculated by summing the contribution from each additional round trip.

A. The Optical Delay Line

In an optical delay line, each beam makes b one-way passes in the arm of the interferometer, before being recombined at a beamsplitter. The total phase difference between the two beams of light is simply the sum of the phase differences from each individual pass,

$$\Delta\varphi_b(t) = \sum_{j=0}^{\frac{b}{2}-1} \Delta\varphi(t-2j\tau_t) \tag{12}$$

This can be evaluated and simplified by defining the storage time $\tau_s = b\tau_t$

$$\Delta\varphi_b(t) = \frac{\pi c \tau_t h'}{\lambda}\left[\alpha_{zz} G(\omega,k_z) - \alpha_{yy} G(\omega,k_y)\right]\exp\left[-i\omega(t_0+\frac{\tau_s}{2})\right]\frac{\sin(\omega\tau_s/2)}{\sin\omega\tau_t} \tag{13}$$

The resulting transfer function is

$$H_b(\omega) = \frac{\pi c \tau_t}{\lambda}\left[\alpha_{zz} G(\omega,k_z) - \alpha_{yy} G(\omega,k_y)\right]\exp\left[\frac{i\omega\tau_s}{2}\right]\frac{\sin(\omega\tau_s/2)}{\sin\omega\tau_t} \tag{14}$$

Notice that this function retains a dependence on both τ_t and τ_s.

For normally incident gravitational wave, $\theta = 0°$, $H_b(\omega)$ has a particularly simple form.

$$H_b(\omega;\theta = 0°) = \frac{2\pi c \tau_s}{\lambda}\,\mathrm{sinc}\,\frac{\omega\tau_s}{2}\exp\frac{i\omega\tau_s}{2} \tag{15}$$

Another feature of (14) is that for any fixed angle of incidence and frequency H_b depends on Ω as $\cos 2(\Omega - \Omega_c)$, where Ω_c depends on θ, L and ω.

The effect of increasing the number of bounces in the delay line is shown in Figure XX, for a normally incident wave. For frequencies less than $(2\tau_s)^{-1}$, the sensitivity increases with τ_s. For frequencies above $(2\tau_s)^{-1}$, the sensitivity oscillates between 0 and its maximum value.

The effect of changing the angle of incidence is shown in Figures XX and XX. *The lines move up and down in various ways.* A particularly interesting case, although not necessarily of any practical utility, is the case $\theta = 90°$, $\varphi = 45°$ (incident in the plane of the detector, perpendicular to the bisector between the arms). At very low frequencies, the sensitivity goes to zero. At higher frequencies the sensitivity rises, due to the propagation delays across the interferometer; in effect, the two arms of the interferometer are centered at different locations and respond to the wave with different phases.

B. Optical Cavities

Gravitational wave detectors using optical cavities may be designed in a variety of ways. Current prototypes [8] use the reflected light from the cavities. The reflected light is made up of two parts – light directly reflected from the front mirror and light which has been stored in the cavity and which "leaks" out back toward the light source. The destructive interference between these two pieces is responsible for the reduction in the reflected intensity when the cavity is in resonance. ~~It is~~ only the component of the light which has been in the cavity ~~which~~ is affected by the wave; hence we shall ignore the directly reflected piece and calculate only the phase shift of the light leaking from the cavity. In the absence of a gravitational wave, the "leaked" light is made up of components which have made different numbers of round trips inside the cavity before leaking out

$$\vec{E} = \sum_{n=1}^{\infty} \vec{E}_0 t_1^2 r_1^{n-1} r_2^n e^{i\varphi_n} \tag{16}$$

where $\vec{E}_0$ is the amplitude of the incident wave, r_1 and r_2 are amplitude reflectivities of the cavity mirrors, t_1 is the amplitude transmission of the input mirror and φ_n is the phase acquired by the light in n round trip in the cavity. If the cavity is in resonance φ_n is a multiple of 2π. For high reflectivity mirrors, $1-|r|^2 \ll 1$, the cavity has a storage time

$$\tau_s = \frac{\tau_t}{1-|r_1 r_2|} \tag{17}$$

A gravitational wave will alter the phase shifts experienced by the light in traversing the arms, $\varphi_n = 2N\pi + \delta\varphi_n$. For realistic wave strengths, $\delta\varphi_n$ is very small and we can expand $\exp i\varphi_n \approx 1 + i\delta\varphi_n$. With that simplification it is straightforward to evaluate (16) using the $\delta\varphi_n$'s from section IIa. This leads to a net phase difference between the light leaking from the two interferometer cavities.

$$\Delta\varphi_c(t) = \frac{\pi c \tau_t h}{\lambda}\left[\alpha_{zz} G(\omega,k_z) - \alpha_{yy} G(\omega,k_y)\right] \frac{e^{-i\omega(t_0-\tau_t)}(1-|r_1 r_2|)}{1-2|r_1 r_2| e^{i\omega\tau_t}\cos\omega\tau_t + |r_1 r_2|^2 e^{i2\omega\tau_t}} \quad (18)$$

The transfer function for a cavity based interferometer is

$$H_c(\omega) = \frac{\pi c \tau_t}{\lambda}\left[\alpha_{zz} G(\omega,k_z) - \alpha_{yy} G(\omega,k_y)\right] \frac{e^{i\omega\tau_t}(1-|r_1 r_2|)}{1-2|r_1 r_2| e^{i\omega\tau_t}\cos\omega\tau_t + |r_1 r_2|^2 e^{i2\omega\tau_t}} \quad (19)$$

Transfer functions for normally incident gravitational waves on cavity-based interferometers in figure XX illustrate the increase in sensitivity with storage time. As with delay lines, the sensitivity does not increase significantly for frequencies greater than $(2\tau_t)^{-1}$. The effects of changing the angle of incidence of the wave are shown in Figure XX. *Here the lines don't wiggle as much as the delay line graphs.*

IV. RESONANT INTERCHANGE

A method to further improve the AC response of an interferometer at the expense of the DC response has been proposed by Drever [9]. In this scheme, called resonant interchange or "sloshing", the light is stored alternately in the two arms, switching every τ_s. This allows the light to accumulate additional phase shift for gravitational waves of frequencies near $(2\tau_s)^{-1}$.

One version of this scheme is to use delay lines and to interchange the light in the arms a discreet number of times. If there are M interchanges, the net phase difference acquired by the two beams of light will be

$$\Delta\varphi_{M,b}(t) = \sum_{m=0}^{M-1} (-1)^m \, \Delta\varphi_b(t - m\tau_s) \qquad (20)$$

where $\Delta\varphi_b(t)$ is given by (13). The sum is straightforward to evaluate;

$$) = \frac{\pi c \tau h}{\lambda}\left[\alpha \;\; G(\omega,k \;) - \alpha \;\; G(\omega,k \;)\right] \frac{\sin \omega\tau / 2}{\sin \omega\tau} \; \frac{\sin M(\frac{\pi}{2} - \frac{\omega\tau}{2})}{\sin(\mathrm{pi}/2 - \mathrm{omega}\,\tau\;/2)} \exp i\left[(M-1)\frac{\pi}{2} - \omega(t - (2t)/_s 2)\right]$$

The transfer function $H_{M,b}(\omega)$ is given by

$$H_{M,b}(\omega) = \frac{\pi c \tau_t}{\lambda}\left[\alpha_{xx} G(\omega,k_x) - \alpha_{yy} G(\omega,k_y)\right] \frac{\sin M(\frac{\pi}{2} - \frac{\omega\tau_s}{2})}{\sin(\mathrm{pi}/2 - \mathrm{omega}\,\tau_s/2)} \qquad (22)$$

Notice that this is just the transfer function for a delay line interferometer multiplied by $\frac{\sin M(\frac{\pi}{2} - \frac{\omega\tau_s}{2})}{\sin(\mathrm{pi}/2 - \mathrm{omega}\,\tau_s/2)} \exp\left[i(M-1)(\omega\tau_s + \pi/2)\right]$ Some examples of "sloshing" transfer functions are shown in Figure XX.

Alternative arrangements (Figure ?) include putting the two delay lines inside an optical cavity (Figure ?) and a scheme which uses coupled cavities in the arms (Figure ?). The transfer function for these cases depend on the mirror reflectivities and transmissivities *in a complex way which may be too specialized to be appropriate for Phys. Rev. ... maybe we should just say that they give comparable results. (with a reference to an as yet unwritten internal report, like Kip's preprint series?)*

V. CONCLUSIONS

We have derived the time taken by an electromagnetic wave to travel a distance L and return in the presence of a sinusoidal gravitational wave. This basic result is used to calculate the transfer functions which govern the response of gravitational wave detectors which use interferometry to monitor the separation of free test masses. *Summarize the most interesting results?*

URE 1