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# Effects of using the wrong antenna pattern on sensitivity and parameter estimation

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Let's take a simple example of a constant signal $S$ modified by an antenna gain factor $G_i$ and noise $n_i$ (variance $\sigma^2$) to give an apparent strain $h_i$ in an interferometer so that

$$h_i = G_i S + n_i \,, \tag{1}$$

where the subscript $i$ represents the $i$th sample in time. Note the signal is constant but the gain varies with time due e.g., to Earth's rotation.

Conventionally we would determine the maximum likelihood estimator for $S$ from

$$\chi^2 = \sum \frac{(h_i - G_i S)^2}{\sigma^2} \,. \tag{2}$$

Setting $d\chi^2/dS = 0$ we get the maximum likelihood estimator for the signal

$$\hat{S} = \frac{\sum h_i G_i}{\sum G_i^2} \,. \tag{3}$$

But what happens if we calculate this using the *wrong* antenna gain factor, $W_i$? We get an estimator that is

$$\hat{S}_W = \frac{\sum h_i W_i}{\sum W_i^2} \tag{4}$$

$$= \frac{\sum (G_i S + n_i) W_i}{\sum W_i^2} \tag{5}$$

$$= \frac{1}{\sum W_i^2} \left( S \sum G_i W_i + \sum W_i n_i \right) \,. \tag{6}$$

The expectation value for this estimator is

$$\langle \hat{S}_W \rangle = S \frac{\sum G_i W_i}{\sum W_i^2} \,. \tag{7}$$

Clearly, if $W_i = G_i$ we get the unbiased maximum-likelihood estimator of $S$. However, if $W_i \neq G_i$, there is a bias in the estimator given by

$$\text{bias} = \langle \hat{S}_W \rangle - S = S \left( \frac{\sum G_i W_i}{\sum W_i^2} - 1 \right) \,. \tag{8}$$

To see the effect of this on sensitivity, we need to calculate the variance of the estimator:

$$\text{var}(\hat{S}_W) = \left\langle \left( \hat{S}_W - \langle \hat{S}_W \rangle \right)^2 \right\rangle \tag{9}$$

$$= \left\langle \left( \frac{1}{\sum W_i^2} \left( S \sum G_i W_i + \sum W_i n_i \right) - S \frac{\sum G_i W_i}{\sum W_i^2} \right)^2 \right\rangle \tag{10}$$

$$= \left\langle \left( \frac{1}{\sum W_k^2} \right)^2 \sum W_i n_i \sum W_j n_j \right\rangle \qquad (11)$$

$$= \left( \frac{1}{\sum W_k^2} \right)^2 \sum_{i,j} W_i W_j \langle n_i n_j \rangle \qquad (12)$$

$$= \left( \frac{1}{\sum W_k^2} \right)^2 \sum_{i,j} W_i W_j \, \sigma^2 \, \delta_{ij} \qquad (13)$$

$$= \frac{\sigma^2}{\sum W_i^2} \, . \qquad (14)$$

The signal-to-noise ratio of the 'wrong' method is therefore

$$\mathrm{snr}_W = \frac{\langle \hat{S}_W \rangle}{\sqrt{\mathrm{var}(\hat{S}_W)}} \qquad (15)$$

$$= \frac{S}{\sigma} \frac{\sum G_i W_i}{\sqrt{\sum W_i^2}} \, . \qquad (16)$$

Again, if $W_i = G_i$ we get the 'best' signal-to-noise ratio

$$\mathrm{snr} = \frac{S}{\sigma} \sqrt{\sum G_i^2} \, . \qquad (17)$$

The reduction in signal-to-noise ratio from using the wrong weights is therefore

$$r = 1 - \frac{\mathrm{snr}_W}{\mathrm{snr}} = 1 - \frac{\sum G_i W_i}{\sqrt{\sum G_i^2} \sqrt{\sum W_i^2}} \, , \qquad (18)$$

where the last term is (sort of) the correlation coefficient between the two antenna gain factors.