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Dissipation Dilution

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1 Introduction

1.1 Purpose and Scope

This document reviews the theory of dissipation dilution, with particular emphasis on how it can be calculated as part of a normal mode analysis.

1.2 Version History

5/5/07, -00draft20070505: Initial draft, with text recycled from note “Supplement to Section 2.2.7 of T020205-02 (draft of 13 Sep 2006)” of 10 Jan 2007 to Glasgow IGR Group by Mark Barton.

5/9/07, -00draft20070509: Lots more stuff. Circulated to IGR.

5/20/07: -00draft20070520: Added content to Introduction. Clarification and correction of Section 7 (discussion of Cagnoli et al.).

6/5/08: -00 final. Added table of contents.

1.3 Applicable Documents

- Models of the Advanced LIGO Suspensions in Mathematica™ (LIGO-T020205-02).
- G. Cagnoli et al., Physics Letters A 272 (2000): 39 – 45
- G. Gonzalez and P. Saulson, J. Acoust. So. Am., 96: 207-212 (1994)
- I.R. Titze and E.J. Hunter, “Normal vibration frequencies of the vocal ligament”, J. Acoust. Soc. Am. 115 (5), 2264-9.

2 Introduction

Dissipation dilution is the name given to the phenomenon whereby a mechanical oscillator made from elastic elements with loss angle ϕ can have a quality factor rather greater than the value that might naively be expected, i.e., $Q = 1/\phi$. To achieve their budgeted thermal noise, interferometric gravity wave detectors rely heavily on dissipation dilution in the pendulum suspensions, both for the pendulum mode of the test mass and the violin modes of the suspension wires/fibres.

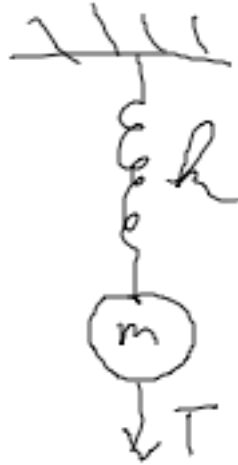
The calculation of dissipation dilution for the key systems of interest has been thrashed out in the literature (Gonzalez and Saulson, Cagnoli et al., etc) and a number of initially divergent ways of approaching the calculation have been reconciled. However there is still some lingering confusion as to why dissipation dilution happens in the first place. To most people in the field, it is plausible that a pendulum should exhibit dissipation dilution, because the role of gravity is obvious, but it is not so clear that a violin mode should behave the same way. Thus, if a novel system needs to be analyzed there is still a risk of another round of errors.

In particular Rahul Kumar, Alan Cumming and Calum Torrie at IGR have been using the finite-element program ANSYS to look at more complicated systems such as fused silica ribbons with realistic (tapered) neck shapes. This poses a fresh set of challenges, because it turns out that normal mode analysis, which is the basis of such programs, throws away the information required to determine whether dissipation dilution will occur, and at a fairly early step in the processing.

This document gives an overview of the two related but interestingly different ways that dissipation dilution can occur, how these are confused with each other and the non-diluted case by normal mode analysis, and how the necessary information can potentially be recovered. The method described here has been implemented in the Mathematica pendulum modeling toolkit by Mark Barton and the manual for that software (T020505-02) gives further detail, although with less emphasis on specific physical systems.

3 Basic Theory

The following toy model illustrates the fundamental mechanism behind one of the two major classes of dissipation dilution. Consider a lossy spring with unstretched length l_0 and spring constant $k = k_0(1 + i\phi)$ attached to a mass m , with a lossless longitudinal tensioning force T (possibly equal to mg but possibly from some other mechanism altogether):



The force from the spring is

$$F = -k(l - l_0) \quad (1)$$

At equilibrium

$$F + T = 0 \quad (2)$$

so

$$l_{eq} = l - l_0 = \frac{T}{k} \quad (3)$$

In the following subsections we calculate the energy loss for three different ways of driving this system: vertically, horizontally (exactly) and along an arc.

3.1 Vertical motion.

Let the mass move or be driven vertically, i.e., longitudinally with respect to the spring, with amplitude A_L and angular frequency ω (not necessarily at a resonance for now). Then in phasor space where real amplitudes are proportional to $\cos \omega t$ and imaginary ones to $\sin \omega t$, the force is

$$F = -A_L k_0 (1 + i\phi) \quad (4)$$

and the velocity is

$$v = i\omega A_L \quad (5)$$

so the power output of the spring (i.e., the rate of energy loss) is

$$P = Fv = A_L^2 k_0 \phi \omega \quad (6)$$

The maximum stored energy is

$$E = \frac{1}{2} k_0 A_L^2 \quad (7)$$

If we now suppose that ω is a resonance after all and any drive is switched off then there will be exponential decay with time constant $\frac{2}{\phi\omega}$ in energy or $\frac{1}{\phi\omega}$ in amplitude, that is, the quality factor will be $Q = 1/\phi$. This is the baseline result without dissipation dilution.

3.2 Motion on a circular arc (ideally stiff pendulum)

If the longitudinal stiffness of the pendulum approaches infinity, the mass is constrained to move along the arc of a circle with radius l_{eq} centred on the suspension point. Trivially then, the length of the spring is not changed and so there is no dissipation. The radial component of the gravitational force is

$$F_R = T \sin \theta \approx \frac{Tx}{l_{eq}} \quad (8)$$

i.e., there is an effective spring constant of

$$k_R = \frac{T}{l_{eq}} \quad (9)$$

3.3 (Exactly) horizontal motion (constrained or ideally compliant pendulum)

Conversely, if the longitudinal stiffness of the pendulum approaches zero, the mass will tend to move in a horizontal line. The limiting case can also be achieved directly by imposing a frictionless constraint. Let the mass be driven a distance x sideways. To first order the tension is constant, and so to first order there is a restoring force along the horizontal line

$$F_T = T \sin \left(\frac{x}{l_{eq}} \right) \approx \frac{Tx}{l_{eq}} \quad (10)$$

i.e., an effective spring constant of

$$k_T = \frac{T}{l_{eq}} \quad (11)$$

It will be significant later that this spring constant is numerically equal to the one in the previous section (Eq. (9)), despite having been produced by a somewhat different mechanism. Since the gravitational potential energy is constant and all the energy flow is in and out of the spring, it is tempting but incorrect to suppose that this spring constant would have the same loss angle associated with it as that, ϕ , of the underlying spring. To see what really happens, consider the length of the spring as a function of time:

$$l = \sqrt{l_{eq}^2 + x^2} \approx l_{eq} + \frac{x^2}{2l_{eq}} \quad (12)$$

If we now let the motion be oscillatory with amplitude A_T and angular frequency ω , and avoid phasors for the moment:

$$l \approx l_{eq} + \frac{(A_T \cos \omega t)^2}{2l_{eq}} \approx l_{eq} + \frac{A_T^2}{2l_{eq}} \cos 2\omega t \quad (13)$$

So introducing phasors at 2ω , the oscillatory part of the spring motion is

$$l_{T(spring)} = \frac{A_T^2}{2l_{eq}} \quad (14)$$

and the oscillatory part of the force on the spring (*not* along the horizontal line) as

$$F_{T(spring)} = -\frac{A_T^2}{2l_{eq}} k_0 (1 + i\phi) \quad (15)$$

Thus, using the same argument as before (keeping in mind that the velocity is now $v_{T(spring)} = 2i\omega l_{T(spring)}$), the power dissipated is

$$P_T = \frac{A_T^4 k_0 \phi \omega}{2l_{eq}^2} \quad (16)$$

(In principle this calculation could also be done using the force and displacement in the horizontal line, but this would be inconvenient because there is a nonlinear relationship between the spring picture and the horizontal picture, which makes the use of phasors difficult.) The striking result is that the dissipation is fourth order in amplitude. By contrast, the maximum stored energy is still second order:

$$\begin{aligned} E_T &= \frac{1}{2} k_0 \left(\sqrt{A_T^2 + l_{eq}^2} - l_0 \right)^2 - \frac{1}{2} k_0 (l_{eq} - l_0)^2 \\ &= \frac{1}{2} k_0 \left(l_{eq} \left(1 + \frac{A_T^2}{2l_{eq}^2} \right) - l_0 \right)^2 - \frac{1}{2} k_0 (l_{eq} - l_0)^2 \\ &= \frac{1}{2} k_0 \left(1 - \frac{l_0}{l_{eq}} \right) A_T^2 + O[A_T^4] \\ &= \frac{T}{2l_{eq}} A_T^2 = \frac{k_T}{2} A_T^2 \end{aligned} \quad (17)$$

The effective loss angle is

$$\phi_{DD} = \frac{A_T^2 \phi \omega l_{eq}}{l_{eq} - l_0} = \frac{A_T^2 \phi \omega l_{eq} k_0}{T} \quad (18)$$

which is amplitude dependent, and tends to zero for small oscillations.

Either the low-compliance or constrained versions of this pendulum will have a horizontal angular frequency

$$\omega_T = \frac{k_T}{m} = \frac{g}{l_{eq}} \quad (19)$$

(i.e., as for a ideally stiff pendulum because of the common restoring force).
 For a free decay of such an oscillation, the fall-off of the energy with time is

$$E_T(t) = \frac{E_T(0)}{1 + \frac{\phi \omega l_{eq} k_0}{T} t} \tag{20}$$

This is rapid initially but *much* slower than exponential for large t .

3.4 Moderately stiff pendulum

For a pendulum that is less than infinitely stiff, but still with a vertical mode higher in frequency than its pendulum mode, the locus of the mass will be approximately circular but with a departure that can be approximated in terms of the centrifugal force. Since the pendulum mode is approximately

$$\theta = \theta_0 \cos \sqrt{\frac{g}{l_{eq}}} t \tag{21}$$

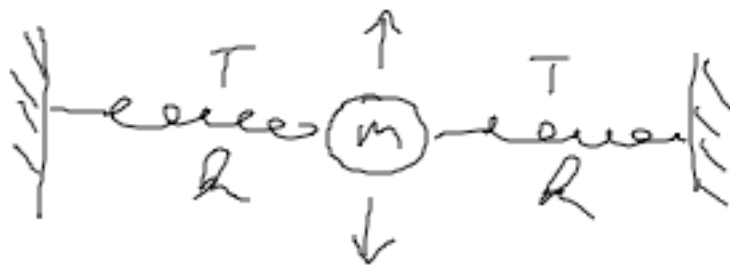
the angular velocity is

$$\dot{\theta} = \theta_0 \sqrt{\frac{g}{l_{eq}}} \sin \sqrt{\frac{g}{l_{eq}}} t \tag{22}$$

and the centrifugal force is

4 The Two Types of Dissipation Dilution

The mechanism in the horizontal motion case above (Section 3.2) is slightly artificial in isolation but is nonetheless important as the simplest representative of one major type of dissipation dilution, which could be called Level I or fourth-order, from the fact that the dissipation is fourth order in the amplitude. A much more realistic example can be constructed by switching off gravity and applying the tensioning force by a second spring opposing the first:



Here the natural motion of the mass really is a straight line transverse to the equilibrium orientation of the springs. The above argument applies to both springs in parallel for double the net effective transverse spring constant: $k_T = 2T / l_{eq}$.) This is already a simple one-bead approximation to a violin mode. Clearly the number of beads can be increased indefinitely until the continuum is reached with the same arguments applying. The continuous case is treated in more detail in Section 6.

Conversely, the mechanism in Section 3.2 is representative of the other major type, which could be called Level II, or higher-order. The fact that there are two different types, or at least degrees, of dilution accounts for a lot of confusion in the literature.

In particular, the role of gravity in a standard pendulum is complex and often misdescribed. For example it is sometimes stated that a pendulum is low loss because it uses a “gravitational spring” (e.g., Saulson 1990). While perhaps arguable, this is misleading, because gravity doesn’t actually supply the restoring force. It *can’t* provide a horizontal restoring force because it acts purely vertically (by definition of vertical). Rather, its role is simply to tension the spring, and the horizontal component of the tension moves the mass back to the centre. Any other method of tensioning the spring to the same value that acts purely vertically gives the same horizontal restoring force and the same horizontal mode frequency. This is the resolution to the curious coincidence in the case of the mass constrained to move purely horizontally in Section 3.2.

Similarly, any method of tensioning the spring would give *at least* Level I dilution. The standard pendulum improves on this by allowing the mass to rise slightly and reduce the extension of the spring even further, giving Level II, but this is incidental to the horizontal restoring force.

Even more confusion arises from the fact that many calculations are done in the normal mode formalism, and it turns out that this formalism cannot reliably distinguish the two cases! See the next section for details.

5 Damping and Normal Mode Formalism

Many calculations of the properties of suspensions for gravitational wave detectors are done in the normal mode approximation because there are useful tools such as finite-element analysis programs for applying the formalism to complex systems. Unfortunately, by default the normal mode formalism does not preserve enough information to track to track whether there will be dilution at all, much less whether it will be Level I or Level II.

5.1 The Problem

The core of the problem can be seen by further analyzing the simple pendulum introduced above. The normal mode formalism only “sees” the horizontal force and is oblivious to higher order subtleties such as whether the motion is really along a line or along an arc. In fact it will freely interchange the two scenarios depending on the coordinates used for the analysis!

To see this, note that an eigenmode is always linear in the coordinates. If a standard pendulum is analyzed in x , z coordinates, the pendulum mode will be of the form

$$x = x_0 \cos \omega t \quad (23)$$

(See Table 1 for all the key partial results from analysis of a pendulum – whether standard or constrained doesn’t matter – in two different coordinate systems.) Interpreted literally, this is purely horizontal motion as in the constrained pendulum of Section 3.2. Conversely, if a constrained pendulum is analyzed in r , θ coordinates, the pendulum mode will be

$$\theta = \theta_0 \cos \omega t \quad (24)$$

and this is motion along an arc.

Now at first sight this might not seem so serious because any level of dissipation will do, and we don't really need a report on which is which. The trouble is that because it isn't tracking second order displacements in enough detail to distinguish a line from an arc, and thus Level I from Level II, neither is it doing it in enough detail to distinguish dilution from non-dilution. In fact it's not tracking second order displacements at all – it's implicitly replacing both types of pendulum by a horizontal spring attaching the mass to the equilibrium position. See for example the “Stiffness matrix” row of Table 1. In either coordinate system the (1,1) element is diluted, but the (2,2) element isn't. No information is retained as to which is which.

Worse, one obvious approach doesn't work. Substituting either of the above eigenmode expressions into the full potential energy expression will give the same parabolic well as a function of (linear) amplitude, but in one case (x - the line) it will be through varying the elastic term, whereas in the other (θ - the arc) it will be from varying the gravitational term. See rows labeled “Horizontal mode gravitational energy” and “Horizontal mode elastic energy” in Table 1. So this tells us only about the coordinate system and not at all about the original source of the elasticity or the damping behaviour thereof.

Table 1: Key results for normal mode analysis of a pendulum in two different coordinate systems

Coordinates	x, z (positive down)	θ, r
Potential energy	$\frac{1}{2}k(\sqrt{x^2 + z^2} - l_0)^2 - mgz$	$\frac{1}{2}k(r - l_0)^2 - mgr \cos \theta$
Equilibrium position	$x = 0, z = l_{eq} = l_0 + \frac{mg}{k}$	$\theta = 0, r = l_{eq} = l_0 + \frac{mg}{k}$
Kinetic energy	$\frac{1}{2}m(\dot{x}^2 + \dot{z}^2)$	$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$
Stiffness matrix	$\begin{pmatrix} \frac{mg}{l_{eq}} & 0 \\ 0 & k \end{pmatrix}$	$\begin{pmatrix} mgl_{eq} & 0 \\ 0 & k \end{pmatrix}$
Mass matrix	$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$	$\begin{pmatrix} ml_{eq}^2 & 0 \\ 0 & m \end{pmatrix}$
Stiffness*Inv(Mass)	$\begin{pmatrix} \frac{g}{l_{eq}} & 0 \\ 0 & \frac{k}{m} \end{pmatrix}$	$\begin{pmatrix} \frac{g}{l_{eq}} & 0 \\ 0 & \frac{k}{m} \end{pmatrix}$
Eigenvalues	$\frac{g}{l_{eq}}, \frac{k}{m}$	$\frac{g}{l_{eq}}, \frac{k}{m}$
Eigenvectors	(1,0), (0,1)	(1,0), (0,1)

Eigenmodes	$x = x_0 \cos \sqrt{\frac{g}{l_{eq}}} t, z = z_0 \cos \sqrt{\frac{k}{m}} t$	$\theta = \theta_0 \cos \sqrt{\frac{g}{l_{eq}}} t,$ $r = r_0 \cos \sqrt{\frac{k}{m}} t$
Horizontal mode gravitational energy	$-mgl_{eq}$ $\approx O[1]$	$-mgl_{eq} \cos \theta$ $\approx -mgl_{eq} \left(1 - \frac{\theta^2}{2}\right)$ $\approx -mgl_{eq} \left(1 - \frac{x^2}{2l_{eq}^2}\right)$ $\approx O[1] + \frac{mgx^2}{2l_{eq}} + O[x^4]$
Horizontal mode elastic energy	$\frac{1}{2}k \left(\sqrt{x^2 + l_{eq}^2} - l_0\right)^2$ $\approx \frac{1}{2}k \left(l_{eq} \left(1 + \frac{x^2}{2l_{eq}^2}\right) - l_0\right)^2$ $\approx \frac{1}{2}k \left((l_{eq} - l_0)^2 + x^2 \left(1 - \frac{l_0}{l_{eq}}\right) + O[x^4]\right)$ $\approx O[1] + \frac{mgx^2}{2l_{eq}} + O[x^4]$	$\frac{1}{2}k(l_{eq} - l_0)^2$ $\approx O[1]$

5.2 A Solution

Because the normal mode formalism throws away important information, it's important to intercept the process and classify the energy flows before the production of the stiffness matrix. Looking at the details of the expansion in Eq (17) gives a clue as to how this can be done.

$$\begin{aligned}
E_T &= \frac{k_0}{2} \left(\sqrt{A_T^2 + l_{eq}^2} - l_0 \right)^2 - \frac{k_0}{2} (l_{eq} - l_0)^2 \\
&\approx \frac{k_0}{2} \left(l_{eq} \left(1 + \frac{A_T^2}{2l_{eq}^2} \right) - l_0 \right)^2 - \frac{k_0}{2} (l_{eq} - l_0)^2 \\
&= \frac{k_0}{2} \left(l_{eq} - l_0 + \frac{A_T^2}{2l_{eq}} \right)^2 - \frac{k_0}{2} (l_{eq} - l_0)^2 \\
&= \frac{k_0}{2} \left(l_{eq}^2 - 2l_{eq}l_0 + l_0^2 + A_T^2 - l_0 \frac{A_T^2}{l_{eq}} + \frac{A_T^4}{4l_{eq}^2} \right) - \frac{k_0}{2} (l_{eq} - l_0)^2 \\
&= \frac{k_0}{2} \left(A_T^2 - l_0 \frac{A_T^2}{l_{eq}} \right) + \frac{k_0 A_T^4}{8l_{eq}^2} \\
&= \frac{k_0 (l_{eq} - l_0) A_T^2}{2} + O[A_T^4]
\end{aligned} \tag{25}$$

The subexpression $l_{eq} - l_0 + A_T^2/2l_{eq}$ in the third line is the length of the spring, and it's already second-order in the transverse amplitude, A_T . The dominant terms in the result come from squaring this length, specifically the cross-terms between A_T^2 and l_{eq} or $-l_0$. That is, a relatively large (second-order) amount of energy can be stored in the spring despite this small (second-order) displacement, because it's acting against a large (zeroth-order) static force.

By contrast, in the vertical case, we have

$$\begin{aligned}
E_L &= \frac{k_0}{2} (l_{eq} - l_0 + A_L)^2 - \frac{k_0}{2} (l_{eq} - l_0)^2 - mgA_L \\
&= \frac{k_0}{2} (l_{eq}^2 - 2l_{eq}l_0 + l_0^2 + 2l_{eq}A_L - 2l_0A_L + A_L^2) - \frac{k_0}{2} (l_{eq} - l_0)^2 - mgA_L \\
&= \frac{k_0 A_L^2}{2} + k_0 (l_{eq} - l_0) A_L - mgA_L \\
&= \frac{k_0 A_L^2}{2}
\end{aligned} \tag{26}$$

Here the displacement of the spring is the same as that of the mass, i.e., first-order (A_L), and the dominant second-order term in the result comes from the cross-term between this and itself. (The linear cross-terms cancel with the gravitational energy, and would not appear in the stiffness matrix in any case because zeroth- and first-order terms have zero second derivative.)

All this leads to a simple trick that can be used to identify the component of the elasticity that produces damping. If the equilibrium position is used, but all tensions and other static forces are arbitrarily set to zero (e.g., for a linear spring, by setting the unstretched length equal to the stretched length at equilibrium), then any restoring force that persists is necessarily the result of a first-order length change of a spring and thus not subject to dissipation dilution. For example, if we write the previous stiffness matrix (for either x , z or θ , r) in a more spring-centric way,

$$\mathbf{K} = \begin{pmatrix} \frac{k(l_{eq} - l_0)}{l_{eq}} & 0 \\ 0 & k \end{pmatrix}$$

then for $l_0 \rightarrow l_{eq}$, the upper left coefficient goes to zero, showing that all the x direction dissipation is diluted in this case.

6 Application to Violin Modes

In this section we develop the theory for the continuum case of a violin mode in a wire, both to demonstrate that dissipation dilution does apply and for comparison with finite-element simulations in ANSYS.

Therefore, consider a wire of unstretched length l_0 strung between two points L apart. The tension in the wire will be

$$T = YS \left(1 - \frac{l_0}{L} \right) \quad (27)$$

where Y is the Young's modulus and S is the cross-sectional area. The energy of the wire in the undisturbed state is

$$E_{v_0} = \frac{YS}{2} \int_0^L \left(1 - \frac{l_0}{L} \right)^2 dl = \frac{YSL}{2} \left(1 - \frac{l_0}{L} \right)^2 = \frac{T^2 L}{2YS} \quad (28)$$

If the wire is then pushed sideways, so that it has a profile $y(l)$ over the length $0 \leq l \leq L$, its energy increases in two ways.

6.1 Longitudinal Extension

First, even if it is perfectly flexible, pushing it out of the straight line between the endpoints stretches it longitudinally (this is the primary violin mode mechanism). An infinitesimal element

dl of the wire is stretched to $\sqrt{1 + \left(\frac{dy}{dl} \right)^2} dl$, so there is an increase of energy given by

$$\begin{aligned}
E_{VL} &= \frac{YS}{2} \int_0^L \left(\sqrt{1 + \left(\frac{dy}{dl} \right)^2} - \frac{l_0}{L} \right)^2 - \left(1 - \frac{l_0}{L} \right)^2 dl \\
&= \frac{YS}{2} \int_0^L \left(1 + \frac{1}{2} \left(\frac{dy}{dl} \right)^2 - \frac{l_0}{L} \right)^2 - \left(1 - \frac{l_0}{L} \right)^2 dl \\
&= \frac{YS}{2} \int_0^L \left(1 - \frac{l_0}{L} \right)^2 + \left(1 - \frac{l_0}{L} \right) \left(\frac{dy}{dl} \right)^2 + \frac{1}{4} \left(\frac{dy}{dl} \right)^4 - \left(1 - \frac{l_0}{L} \right)^2 dl \\
&\approx \frac{YS}{2} \left(1 - \frac{l_0}{L} \right) \int_0^L \left(\frac{dy}{dx} \right)^2 dx \\
&= \frac{T}{2} \int_0^L \left(\frac{dy}{dx} \right)^2 dx
\end{aligned} \tag{29}$$

This component of the energy is Type-I dissipation-diluted for the same reason as in Section 3.2: a first-order displacement in the direction we are interested in (transverse) is accomplished with only a second-order extension of the elastic element and thus a fourth-order energy loss.

Specifically, the extension in an element dl is

$$d^2l = \left(\sqrt{1 + \left(\frac{dy}{dl} \right)^2} - 1 \right) dl \approx \left(1 + \frac{1}{2} \left(\frac{dy}{dl} \right)^2 - 1 \right) dl = \frac{1}{2} \left(\frac{dy}{dl} \right)^2 dl \tag{30}$$

If the profile is made to oscillate at angular frequency ω , i.e.,

$$y(l, t) = y(l) \cos \omega t \tag{31}$$

then the time behaviour is

$$d^2l(t) = \frac{1}{2} \left(\frac{dy}{dl} \cos \omega t \right)^2 = \frac{1}{2} \left(\frac{dy}{dl} \right)^2 (1 + \cos 2\omega t) dl \tag{32}$$

the oscillatory component as a phasor at 2ω is

$$d^2l = \frac{1}{2} \left(\frac{dy}{dl} \right)^2 dl \tag{33}$$

and that of the tension is

$$dT = \frac{Y_0(1+i\phi)S}{2} \left(\frac{dy}{dl} \right)^2 \tag{34}$$

Thus the power dissipation is

$$P_{VL} = \frac{Y_0 S \phi \omega}{2} \int_0^L \left(\frac{dy}{dl} \right)^4 dl \tag{35}$$

6.2 Bending

Second, if as inevitable in real life, the wire has a non-zero moment of area I , there will be an additional component of energy due to bending:

$$E_{VB} = \frac{YI}{2} \int_0^L \left(\frac{d^2 y}{dl^2} \right)^2 dl \quad (36)$$

Typically the integrand is only large near the ends of the wire, where there is a discontinuity between the sinusoidal profile of the violin mode and the fixed angle of the wire in the clamp. For typical wires/fibres of interest in gravity wave detector suspensions, the bending energy is less than 1% of the total violin mode energy. However it contributes essentially all of the damping because the extension and compression of the elastic material on opposite sides of the neutral plane within the wire is first order in the degree of bending, and thus not diluted:

$$P_{VB} = YI\phi\omega \int_0^L \left(\frac{d^2 y}{dl^2} \right)^2 dl = 2\phi\omega E_{VB} \quad (37)$$

6.3 Application to Violin Mode Profiles

Let the wire have linear mass density ρ_L . If the wire is ideally flexible then according to familiar theory, there is a series of modes of frequency

$$f_n^0 = \frac{nc}{2L} \quad (38)$$

where $n=1,2,\dots$ is the mode number, and $c = \sqrt{T/\rho_L}$ is the velocity of waves in the wire. The shape of each mode is

$$y^0(x,t) = A \sin\left(\frac{n\pi x}{L}\right) \cos(2\pi f_n^0 t), \quad 0 \leq x \leq L \quad (39)$$

If the wire has some lateral stiffness, then (Fletcher and Rossing, “The Physics of Musical Instruments”, p. 61, Eq 2.67)

$$f_n = nf_1^0 \left(1 + \beta + \beta^2 + \frac{n^2 \pi^2}{8} \beta^2 \right) \quad (40)$$

where $\beta = 2a/L$ and $a = 1/\lambda = \sqrt{YI/T}$ is the flexure correction as used in pendulum design. In this expression the terms $1 + \beta + \beta^2$ are from the Taylor expansion of $L(1 - \beta) = L(1 - 2a/L)$ in the denominator reflecting the fact that the primary effect of the stiffness is simply to reduce the effective length of the wire by a at each end. The final term is the effect of the stiffness of the wire in the middle section and is second order in β . Hereafter we neglect the second order effects.

Therefore in the middle of the wire, several times distance a from the ends, the shape of the stiff wire is

$$y_n(x,t) = A \sin\left(\frac{n\pi(x-a)}{2L(1-2a/L)}\right) \cos(2\pi f_n t) \quad (41)$$

From Cagnoli et al. (Phys. Lett. A 272 2000 39 – 4, esp. Eq A.1) the shape of a wire near the endpoint when it has zero gradient at the endpoint and asymptotes to gradient dy/dx at large distances is

$$y(x) = \frac{dy}{dx} (x + ae^{x/a} - a) \quad (42)$$

Thus we can approximate the shape of the wire as

$$y_n(x,t) = A \left(\sin\left(\frac{2\pi n(x-a)}{2L(1-2a/L)}\right) + \frac{a2\pi n}{2L(1-2a/L)} \left(\exp\left(\frac{x}{a}\right) - (-1)^n \exp\left(\frac{L-x}{a}\right) \right) \right) \cos(2\pi f_n t) \quad (43)$$

For $L = A = n = 1$, $a = 0.1$, this has the shape shown in Figure 1:

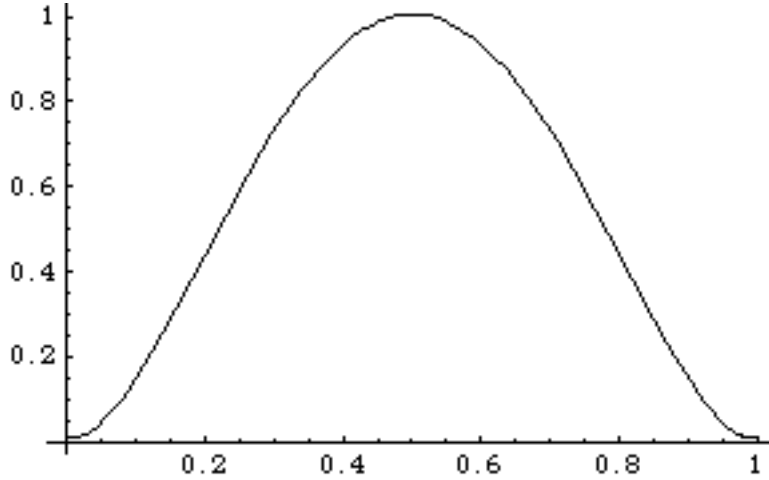


Figure 1: Shape of fundamental violin mode with stiff wire

To get a feeling for the relative sizes of the contributions, we integrate the two terms for the case of a test fibre of fused silica, 350 mm long, 450 μ m in diameter, tension 2.5 kgf, that has been extensively analyzed using ANSYS by Rahul Kumar, Calum Torrie and Liam Cunningham. (Other values for the test fibre are given in Table 2.)

For a 10 mm amplitude at the centre of the fibre for the fundamental ($n = 1$) mode, the energy from longitudinal extension is 17.3 mJ, whereas for bending it is 0.256 mJ, for a dissipation dilution factor of 67.6.

Table 2: Numerical values for test fibre

Note: symbol names as per text except that II is I , rhoL is linear mass density ρ_L , rho is volumetric mass density, ρ , b is relative elongation $(1-l_0/L)$.

$L \rightarrow 0.35$
 $r \rightarrow 0.000225$
 $Y \rightarrow 7.2 \times 10^{10}$
 $\text{II} \rightarrow 2.01289 \times 10^{-15}$
 $S \rightarrow 1.59043 \times 10^{-7}$
 $g \rightarrow 9.81$
 $m \rightarrow 2.5$
 $T \rightarrow 24.525$
 $a \rightarrow 0.00243093$
 $A \rightarrow 0.01$
 $n \rightarrow 1$
 $\text{rho} \rightarrow 2202$
 $\text{rhoL} \rightarrow 0.000350213$
 $b \rightarrow 0.0007496$
 $l_0 \rightarrow 0.349738$

7 Application to Pendulum with Bending Stiffness

In this section, I review the conclusions of Cagnoli et al. (2000) who analyzed a pendulum suspended by a fibre with bending stiffness and attempted to sort out the confusion in the literature over a factor of 2 in the expression for the net damping. They treated the wire as a beam under tension and showed that when bending stiffness is included, two new effects arise, both tending to increase the frequency of the pendulum, and of the same magnitude. This was a useful contribution because it reduced the issue to whether both or just one of these mechanisms contribute to damping. However I will argue that they came down incorrectly on the side of both doing so.

To understand a pendulum on a fibre with bending stiffness, three effects need to be considered. First there is the basic pendulum effect as analyzed in Section 3.2 above. For reference, the basic pendulum energy is

$$E_{PG0} = \frac{TL}{2} \left(\frac{x}{L} \right)^2 \quad (44)$$

and this is lossless.

Second there is the bending of the wire near the pivot point, which is governed by the same theory as bending in the context of violin modes in Section 6.2 above.

Specifically, as shown by Cagnoli et al., from solving the beam equation, the shape of the wire with a horizontal force F at the end is (their Eq (1) in the notation of this document):

$$x(z) = \frac{F}{T} (z + ae^{z/a} - a) \quad (45)$$

Applying (this document's) Eq (36), with $y \rightarrow x$ and $l \rightarrow z$, the bending energy is

$$E_{PB} = \frac{YI}{4a} \left(\frac{F}{T} \right)^2 \approx \frac{YI}{4a} \left(\frac{x}{L} \right)^2 = \frac{Ta}{4} \left(\frac{x}{L} \right)^2 \quad (46)$$

and the power loss is

$$P_{PB} = 2\phi\omega E_{PB} \quad (47)$$

Third, there is a modification to the basic pendulum effect related to the longitudinal stretching analyzed in the context of violin modes in Section 6.1. This additional longitudinal effect comes from the fact that the suspension fibre is bent out of a straight line near the pivot and thus either the tension must increase, or the straight-line distance between the endpoints must decrease. For a standard pendulum as analyzed by Cagnoli et al., the wire is effectively shortened. As they show, if the length of the wire is L and the projection on the z -axis is l , then

$$L = \int_0^l \sqrt{1 + \left(\frac{dx}{dz} \right)^2} dz \approx \int_0^l 1 + \frac{1}{2} \left(\frac{dx}{dz} \right)^2 dz \approx l \left(1 + \frac{1}{2} \left(\frac{x}{l} \right)^2 \left(1 + \frac{a}{2l} \right) \right) \quad (48)$$

Inverting gives, to second order,

$$l = L \left(1 - \frac{1}{2} \left(\frac{x}{L} \right)^2 \left(1 + \frac{a}{2L} \right) \right) \quad (49)$$

Multiplying through by $T = mg$ and throwing away the constant term gives Eq (44) plus an extra term

$$E_{PG} = \frac{TL}{2} \left(\frac{x}{L} \right)^2 \left(1 + \frac{a}{2L} \right) \quad (50)$$

The extra term represents the fact that as far as the curvature of the locus of the mass is concerned, the pendulum has effectively been shortened by $a/2$. (This is despite the fact that the straight line defined by the main section of the wire always passes through a point a below the break-off point, as illustrated in Fig 1. of Cagnoli et al. (2000).) Adding in Eq (46), which has the same magnitude as the extra term gives the total energy

$$E_P = \frac{TL}{2} \left(\frac{x}{L} \right)^2 \left(1 + \frac{a}{L} \right) \quad (51)$$

This says that as far as the energy and thus frequency is concerned, the pendulum has effectively been shortened by a . However as seen just above, it's not a pendulum pivoting about a point a down from the break-off point, it's a pendulum pivoting about a point $a/2$ down from the break-off via a surprisingly complex mechanism, involving winding and unwinding of wire around a curve, that just happens to require an amount of elastic energy cycling equivalent to the difference between a length of $a/2$ and a to operate.

Thus even though the bending energy and the extra gravitational energy have the same magnitude and involve the Young's modulus in the same way (through the flexure correction, $a = \sqrt{YI/T}$), they are associated with quite different damping behaviour. So, while the method of complex spring constants used as one alternative in Cagnoli et al. (2000) happens to give the right total damping, this is not an indication that it can be trusted in general, because it does so at the cost of a spurious back-story as to where the damping is actually occurring. That is, it is not the case that, as claimed in the second last section of the main body of the paper, that, "if it is applied to each spring constant, it gives the energy lost by the relevant force." Rather, it gives the right answer only because we "know" in general terms what the right answer is. As we have seen several times now, it doesn't follow at all that where an elasticity appears there is damping. After all, Eq (44) could equally well be written

$$E_{PG0} = \frac{YS(L-l_0)L}{2} \left(\frac{x}{L} \right)^2 \quad (52)$$

If the method of complex spring constants were applied to this version of the energy, it would yield the erroneous conclusion that the gravitational component was not diluted. Rather, one has to consider the change in strain in the elastic material. By explicit assumption in the derivation of Eq (48), the length of the wire does not change. Thus the extra term has Type-II diluted damping, just as for the rest of the gravitational energy it is associated with.

Even if there *were* a length change of $ax^2/4L^3$, as there would be in a horizontally constrained version of a pendulum with stiff wire, the dissipation would still be Type-I diluted because, by inspection (of Eq (49), not (50)), the length change of the wire considered as a spring is second-order in the amplitude (x/L).

8 Conclusions

- To assess whether there will be dissipation dilution there is no substitute for evaluating the stress and strain in the elastic material itself.
- Whenever there is any sort of mechanism, no matter how trivial, such as a pendulum, which converts the longitudinal tension in a wire into a transverse restoring force, there is scope for the strain in the elastic material to be other than first-order in the amplitude at the nominal output.
- Non-diluted damping is associated with a change in strain that is first-order in the amplitude at the output.
- "Type-I" dissipation dilution is associated with a change in strain that is second-order in the amplitude. Examples include the violin modes of a wire, pendulums with vertical mode frequencies much less than the pendulum mode frequency, and a pendulum where the mass is constrained to move in a horizontal plane rather than an arc. The damping is fourth-order in the amplitude or second-order in the total stored energy. Thus the decay rate is second-order in amplitude and thus negligible for small amplitudes.
- "Type-II" dissipation is associated with a change in strain that is higher than second order in the amplitude. The primary example is the standard pendulum without bending stiffness, where the

energy is offloaded to the gravitational field and the longitudinal strain is approximately constant (fourth-order in amplitude?). The decay rate is even slower than for Type-I dilution.

- Other indicators of dilution include large static stresses and stress/strain that varies at twice the frequency at the output.
- Normal mode analysis throws away the information required to decide whether a particular source of elasticity has diluted damping because it implicitly substitutes a new set of springs (described by the stiffness matrix) that are connected directly between the coordinates of the system and are first-order in those coordinates.
- Some of this information can be recovered by generating the stiffness matrix a second time with all static stresses relieved. Any stiffness that persists with the static stress off is necessarily involves first-order displacements of the elastic elements and is thus non-diluted.
- Normal mode analysis also throws away the information need to tell apart Type-I and Type-II problems. Worse, it will freely substitute one type of system for the other in the output eigenmodes depending on the coordinate system used. Thus a standard pendulum (Type-II, with gravitational potential energy and movement of the mass along an arc) will be converted to a horizontally constrained pendulum (Type-I with elastic restoring force and motion in a line) if solved in x, z coordinates. This needs to be kept in mind when interpreting results from ANSYS and similar programs.
- Cagnoli et al. (2000) is right in respect of the flexure correction and the total damping, but the location of the damping implied by the method of complex spring constants that it seeks to rehabilitate is spurious. The method simply should not be used.