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Detailed derivation of (2) in T060207

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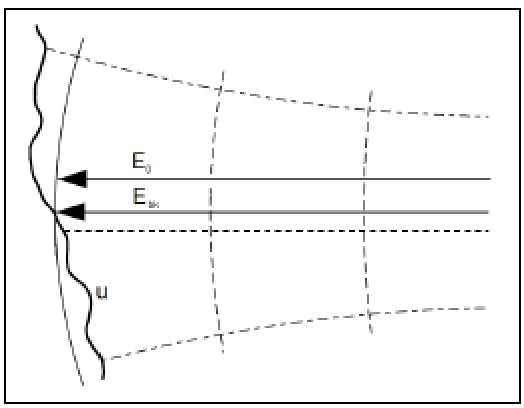
1. Physical configuration

Here the Parametric Instability (PI) "*R*" value is calculated for an arbitrary TM acoustic mode {m}vibration distorting the net electromagnetic field within a single (Adv LIGO) arm cavity. The arm cavity is held on exact {00} mode resonance, with no perturbations present other than the acoustic mode vibration and the finite TM mirror diameter. The cavity is pumped with a perfectly matched {00}field which maintains a steady state field strength E_0 (at frequency ω_0: a tacit "carrier" factor $e^{-i\omega_0 t}$ is understood to have been removed from all field amplitudes) of this mode within the cavity. The particular acoustic mode {m} (specific to only one TM) oscillates at *fixed* frequency ω_m. All other acoustic modes and the perturbation fields they generate have no influence on the dynamics of {m}. This is based on the assumption that for every other mode {m'}, $|\omega_{m'} - \omega_m| \gg \delta_m$. An incipient acoustic mode amplitude $\mathbf{u}_m(\mathbf{x})$ (**x** being the coordinate within the TM bulk) is excited (perhaps thermally, but in any case vanishingly small ……). Whether this amplitude will coherently grow or not depends on the balance of 1. the dissipation rate *D* of the internal energy of {m}, and 2. the rate of work, $\dot{W}$, done on the mode {m} via its [mirror surface] coupling to the *net* cavity field E_{tot}. This work is typically positive for a *Stokes* Doppler component of the cavity field, $E_{bk} e^{i\omega_m t}$, allowing the possibility of unstable growth of {m} if $R \equiv \langle \dot{W} \rangle_t / D > 1$ [4].



Calculations of the acoustic frequencies, wm, and modal shapes, $\mathbf{u}_m(\mathbf{x})$, and hence D_m are entirely the results of independent ($\forall$ {m}) FEA simulations of the Adv LIGO TM. Only the calculation of W_m involves an optical cavity (FFT) simulation.

An important distinction (and simplification) from the analysis in [1,2,3] is that we assume that the parametric feedback on {m} has no significant effect on ω_m. Therefore ω_m is merely an *ab initio* constant parameter. More exactly, the problem is one of coupled oscillators (the mode {m} and a sum over cavity modes which sufficiently approximates E_{tot}), so that ω_m will be shifted. However it may be shown that any relevant shift is $\leq \delta_m$ ($\sim 10^{-7}\ \omega_m$).

2. Derivation

First *D*, the dissipation of the acoustic mode {m}. We regard {m} as a classic SHO with dissipation parameterized as Q_m such that

$$D_m = \omega_m \mathrm{U}_{stored} / Q_m \qquad 1$$

We can also define a "width" $\delta_m = \omega_m / 2Q_m$. A FEA simulation of the TM mode {m} gives u(x), ω_m, and U_{stored} via

$$\mathrm{U}_{stored} = \frac{\mathrm{M}\omega_m^2 \int |\mathbf{u}(\mathbf{x})|^2 d\mathrm{v}}{2\int d\mathrm{v}} \qquad 2$$

Next, using the TM surface distortion $\mathrm{u}_z(\mathbf{x}_s)$ also obtained from the FEA, we calculate the field scattered ("reflected") from this TM:

$$E_{Scatt} = E_0 \exp\left\{ i4\pi \frac{\mathrm{u}_z(\mathbf{x}_s)}{\lambda} Cos\omega_m t \right\} \qquad 3$$

Since, for the purposes of this threshold calculation, $\mathrm{u}_z(\mathbf{x}_s)/\lambda$ can be considered arbitrarily small it will be consistent to only retain and consider field components to first order in $\sim \mathrm{u}_z(\mathbf{x}_s)/\lambda$. In particular, the total cavity field, E_{tot}, into the TM surface can be reduced to three [Fourier] components:

$$E_{tot} \approx E_0 + E^s_{back} e^{i\omega_m t} + E^{as}_{back} e^{-i\omega_m t} \qquad 4$$

where $E^{s/as}_{bk} \propto iE_0$ via (3). E_{tot} may then be used to calculate the radiation pressure on the TM surface:

$$\mathrm{P}(\mathbf{x}_s) = 2\eta |E_{tot}(\mathbf{x}_s)|^2 / c \qquad 5$$

Where η is a constant (depending on units) which relates radiation pressure to the magnitude of the Poynting vector (which will cancel in the sequel). The factor two represents the doubling of pressure due to the nearly perfectly reflecting surface. In this situation we approximate "2"=1+R_{TM}. The only terms in (5) which will lead to time average work on {m} will be ones at frequency ω_m, and of quadrature ~$Sin\omega_m t$ (in phase with the *velocity* of {m}, consistent with (3)). With E_0 taken as pure real, such terms will then be $\propto \mathrm{Im}\left[E^{s/as}_{bk}\right]$:

$$|E_{tot}|^2 = E_0^2 + 2E_0 \,\mathrm{Im}\left[E^s_{bk} - E^{as}_{bk}\right] Sin\omega_m t + \text{ineffective terms} \qquad 6$$

Immediately at this stage the sign difference of work done by the Stokes component and by the anti-Stokes component emerges. Several special cases may make the factor $\mathrm{Im}\left[E_{bk}^{s}-E_{bk}^{as}\right]$ vanish. One would be if $u_z(\mathbf{x}_s)$ is spatially uniform, so that $E_{bk}^{s/as}$ are resonant and thus relatively real [3]. Similarly it may appear that a piece is missing from the total radiation pressure: that due to E_{scatt}- E_0 . However, aside from the usual static pressure already accounted for by the factor 2 in (5), this field gives no dynamic contribution to (6): the perturbation is pure phase modulation. Physically this says that light reflecting off a mirror (not part of a cavity) will *not* alter the damping of any of the mirror's mechanical modes. Therefore the active term in (6) exists only via circulating "back" fields due to the cavity.

Work is done by the radiation pressure on {m} at rate:

$$\dot{W}=\omega_m Sin\omega_m t\int \mathrm{P}(\mathbf{x}_s)\,\mathrm{u}_z(\mathbf{x}_s)ds = 4\frac{\eta}{c}\omega_m Sin^2\omega_m t\int E_0\,\mathrm{Im}\left[E_{bk}^{s}-E_{bk}^{as}\right]\mathrm{u}_z(\mathbf{x}_s)ds \qquad 7$$

Then combining (1,2,5,6,7) gives:

$$\left\langle \dot{W}\right\rangle_t / D = 4\frac{\eta Q_m}{\mathrm{M}c\omega_m^2}\frac{\int E_0\,\mathrm{Im}\left[E_{bk}^{s}-E_{bk}^{as}\right]\mathrm{u}_z(\mathbf{x}_s)ds\int d\mathrm{v}}{\int\left|\mathbf{u}(\mathbf{x})\right|^2 d\mathrm{v}} \qquad 8$$

Then noting that P_0 (cavity pump power) is $\eta\int\left|E_0\right|^2 ds$ gives

$$\left\langle \dot{W}\right\rangle_t / D = 4\frac{\mathrm{P}_0 Q_m}{\mathrm{M}c\omega_m^2}\frac{\mathrm{V}\int E_0\,\mathrm{Im}\left[E_{bk}^{s}-E_{bk}^{as}\right]\mathrm{u}_z(\mathbf{x}_s)ds}{\int\left|E_0\right|^2 ds\int\left|\mathbf{u}(\mathbf{x})\right|^2 d\mathrm{v}} \qquad 9$$

Which is essentially the expression presented in LIGO-T060207-00 [6]:

$$R=\frac{4PQ_m}{m\omega_m{}^2c}\left(\frac{V\int|E_0^s|Im(E_{bk}^s)u_z dA}{\int|E_0^s|^2dA\int|\vec{u}|^2dV}-\frac{V\int|E_0^{as}|Im(E_{bk}^{as})u_z dA}{\int|E_0^{as}|^2dA\int|\vec{u}|^2dV}\right) \qquad (2)$$

3. FFT implementation

The expression (9) for *R* needs to be interpreted in terms of well defined arm cavity FFT (static) simulations. At first this appears paradoxical since the physical problem is dynamic, since E_{scatt} consists of time dependent side bands of different frequency light in the cavity, which feed back on the TM in a time modulated way. However we see already in (9) that there are no explicitly time dependent quantities. This results from recognizing and formulating this problem of interest (*threshold* feedback) as a purely linear one which allows the time dependence of each Fourier component to be factored out.

Aside from those quantities in (9) obtainable from TM FEA simulations, the only unknown ones are $E_{bk}^{s/as}$. Each of $E_{bk}^{s/as}$ is the steady state field which would result from the excitation of a perfect, {00} resonant arm cavity with a field equal to the $\mp\omega_m$ component of E_{scatt} (3). But this is exactly what our static FFT algorithms are designed to calculate. The implementation proceeds as follows:

1. An FFT simulation is configured for an ideal (*no* distortions) Adv LIGO arm cavity. The input excitation field is $\propto E_0$, the ideal Adv LIGO Gaussian beam which matches into this cavity. This "baseline" simulation is performed to establish the precise numerical cavity resonant length L_0. By FFT convention the relaxed field E_{bk}^0, of this simulation at the TM reflection surfaces are purely *real*.

2. Next we perturb this ideal input excitation via a phase distortion upon *transmission* into the cavity. This is performed by a standard FFT transverse phase map:

$$\varphi(x_s) = 2\pi \frac{\mathrm{u}_z(\mathbf{x}_s)}{\lambda_0} \square \; \pi$$

   which causes the cavity to be excited by a wavefront of exactly the same distortion amplitude as the physical acoustic scattering,(3), would for each *individual* frequency component

3. Now consider an FFT simulation for one of these frequency components, $\omega_0 \mp \omega_m$. One way to do this would be to specify a new wavelength $\lambda_0\left(1 \pm \omega_m / \omega_0\right)$ for the simulation (FFT knows *only* spatial quantities, e.g. λ) while maintaining the cavity length L_0. Instead we choose to fix $\lambda = \lambda_0$ for all simulations while *microscopically* changing the cavity length $\mathrm{L}_0 \to \mathrm{L}_0\left(1 \pm \omega_m / \omega_0\right)$.

4. The result of the FFT simulation (with distorted input and L *fixed* at $\mathrm{L}_0\left(1 \pm \omega_m / \omega_0\right)$) is a transverse map of the steady state cavity field at the longitudinal position of and into the distorting (input) TM. It is the sum $\left(E_{bk}^{s/as} / E_{bk}^{0} + E_0^{'}\right)$ where $E^{'}$ and $E_{bk}^0 \propto E_0$. The proportionality constants are precisely, analytically related (tracing the undistorted cavity resonance curve, at least to first order in perturbation) to the cavity length change $\pm \mathrm{L}_0 \omega_m / \omega_0$. Therefore the extraneous term $E_0^{'}$ can be exactly removed from the map, and the properly normalized, for use in (9), term $E_{bk}^{s/as}$ extracted.

5. The previous step is repeated to separately generate E_{bk}^{s} and E_{bk}^{as}, which are then inserted into Eqn. (9).

4. Cavity mode reduction

Once again, since the problem at hand is essentially linear, we can always decompose the excitation field E_{scatt} (3) into a sum of cavity eigenmodes (of normalized transverse form ψ_j) and calculate a partial R_j which takes into account only the work done by each single modal component. We proceed to calculate $E_{bk}^{s/as}$ and thence R_j via (9) for such a single component.

For each Fourier component of E_{scatt}:

$$E_{scatt}^{s/as} \approx 2\pi i \frac{\mathrm{u}_z(\mathbf{x}_s)}{\lambda_0} E_0 \qquad 10$$

For brevity we follow only one modal component of this field (e.g. $\mathrm{u}_z(\mathbf{x}_s)E_0 \propto \psi_j$) giving

$$E_{scatt}^{s/as}\{j\} \approx 2\pi i \frac{\psi_j(x)}{\lambda_0} \mathsf{E} \int \psi_j \psi_0 \mathrm{u}_z(\mathbf{x}_s) ds \qquad 11$$

where $E_0 \equiv \mathsf{E}\psi_0$. Since this is a pure eigenmode the steady state expression for $E_{bk}^{s/as}\{j\}$ is simply,

$$E_{bk}^{s/as}\{j\} = E_{scatt}^{s/as}\{j\} / [1 - \tilde{r}\exp\{i2k^{s/as}\mathrm{L}_0 + i2\vartheta_G^j\}] \qquad 12$$

which now does explicitly differ, s/as, through $k^{s/as}$. In this expression ϑ_G^j is the cavity Gouy phase for mode {j}, and $\tilde{r}$ is the effective RT reflectivity within the cavity for mode {j}. Recall now that E_0 and thus E are pure real, so that $E_{scatt}^{s/as}\{j\}$ is pure imaginary, giving

$$\mathrm{Im}[E_{bk}^{s/as}\{j\}] = 2\pi \frac{\psi_j(x)}{\lambda_0} \mathsf{E} \int \psi_j \psi_0 \mathrm{u}_z(\mathbf{x}_s) ds \, \mathrm{Re}\left[\frac{1}{1 - \tilde{r}\exp\{i2k^{s/as}\mathrm{L}_0 + i2\vartheta_G^j\}}\right] \qquad 13$$

For small RT phase deviations from resonant, $2\left(k^{s/as}\mathrm{L}_0 + \vartheta_G^j\right) < \pi$, to first order

$$\mathrm{Re}\left[\frac{1}{1 - \tilde{r}\exp\{i2k^{s/as}\mathrm{L}_0 + i2\vartheta_G^j\}}\right] \approx \left(\frac{c}{2\mathrm{L}_0\delta_j}\right) \frac{1}{1 - \left(\Delta\omega^{s/as}/\delta_j\right)^2} \qquad 14$$

where we define $\frac{c}{2\mathrm{L}_0\delta_j} \equiv \frac{\tilde{r}}{1-\tilde{r}} \approx \frac{1}{1-\tilde{r}}$, and $\left(k^{s/as}\mathrm{L}_0 + \vartheta_G^j\right) \equiv \mathrm{L}_0\Delta\omega^{s/as}/c$.

Using (13) and (14) in (5,6,7) we arrive at the rate of work done on {m}by this individual modal component:

$$\dot{W}\{j\} = 4\frac{\eta\pi}{c\lambda_0 L_0 \delta_j} \omega_m Sin^2\omega_m t \left(\int \mathsf{E}\psi_0\psi_j \mathrm{u}_z(\mathbf{x}_s) ds\right)^2 \left(\frac{1}{1 - \left(\Delta\omega^s/\delta_j\right)^2} - \frac{1}{1 - \left(\Delta\omega^{as}/\delta_j\right)^2}\right)$$

Which may be simplified using $\delta_j \equiv \omega_j / 2Qj$, $P_0 = \eta E^2 \int |\psi_0|^2 ds$, and dividing by D_m

$$R_j = \langle \dot{W}\{j\} \rangle_t / D = 4 \frac{P_0 Q_m Q_j}{L_0 M c \omega_m^2} \Lambda_m^j \left(\frac{1}{1 - (\Delta\omega^s / \delta_j)^2} - \frac{1}{1 - (\Delta\omega^{as} / \delta_j)^2} \right) \qquad 15$$

Where $\Lambda_m^j \equiv \frac{V\left(\int \psi_0 \psi_j u_z(\mathbf{x}_s) ds\right)^2}{\int |\mathbf{u}(\mathbf{x})|^2 dv}$ is the geometrical coupling factor of [1,2]. This is exactly the contribution of {j} in the R expression of LIGO-T060207 and [1,2]:

$$R = \frac{4PQ_m}{mL\omega_m{}^2 c} \left(\sum_i \frac{Q_i^s \Lambda_i^s}{1 + (\Delta\omega_i^s / \delta_i^s)^2} - \sum_j \frac{Q_j^{as} \Lambda_j^{as}}{1 + (\Delta\omega_j^{as} / \delta_j^{as})^2} \right) \qquad (1)$$

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