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# Quantifying the error on spectral amplitude estimates

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# 1 Analysis

Suppose we want to estimate the (complex) amplitude $\tilde{A}$ of the component at a frequency $\omega$ of a time series $x(t)$. We evaluate this integral:

$$\tilde{A} = \frac{1}{T}\int_0^T x(t)e^{-i\omega t}dt \tag{1}$$

We may write the time series $x(t)$ as a sum of the component of interest $Ae^{i\omega t}$ and everything else $n(t)$.

$$x(t) = Ae^{i\omega t} + n(t) \tag{2}$$

The integral becomes:

$$\tilde{A} = A + \left(\frac{1}{T}\int_0^T n(t)e^{-i\omega t}dt\right) \tag{3}$$

Now suppose that the noise term $n(t)$ is a random variable with Gaussian distribution with zero mean. We can consider $\tilde{A}$ to be a random variable estimating the value of $A$. Because the integral over the noise has an expectation value of zero, the mean value of $\tilde{A}$ corresponds with the true value of $A$:

$$\langle\tilde{A}\rangle = A \tag{4}$$

To quantify the variance of $\tilde{A}$ I first transform to the discrete case, which is what we will be implementing:

$$\tilde{A} = A + \frac{1}{N}\sum_{j=0}^{N} n_j e^{-i\omega t_j} \tag{5}$$

where $n_j = n(t_j)$. The variance of $\tilde{A}$, which I will denote by $D\{\tilde{A}\}$, may now be written in terms of the variance of $n$:

$$D\{\tilde{A}\} = \frac{1}{N^2}D\left\{\sum_{j=0}^{N} n_j e^{-i\omega t_j}\right\} \tag{6}$$

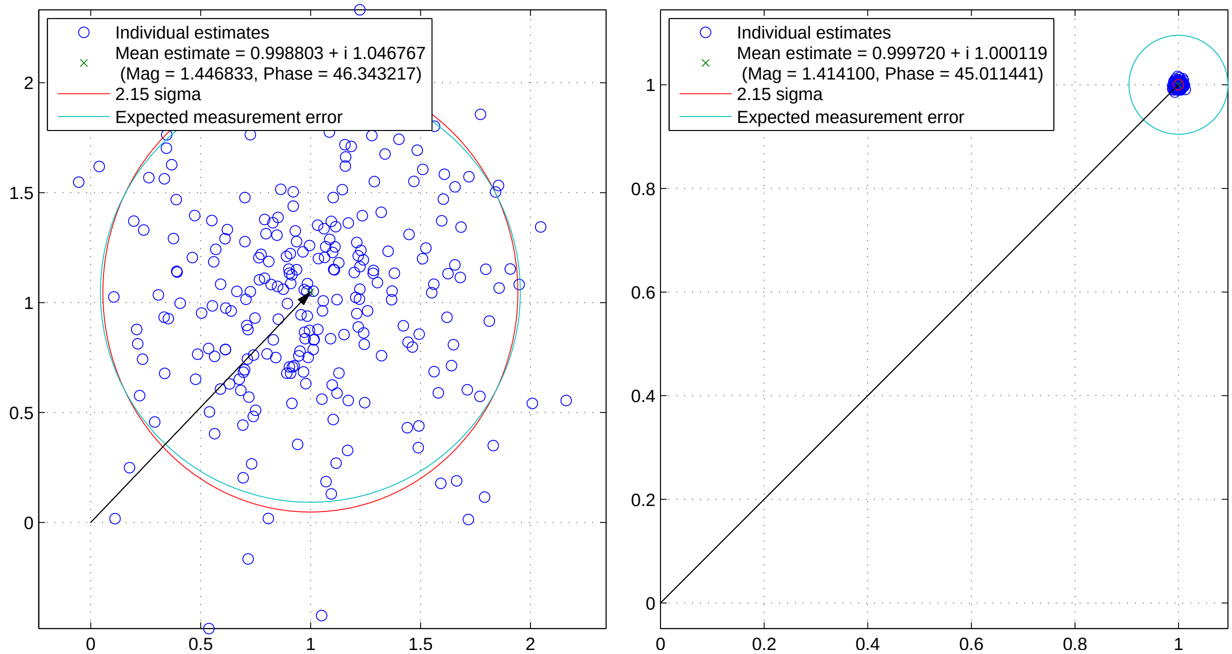
If $n_j$ is a gaussian random variable characterized by (stationary) mean $\mu = 0$ and variance $\sigma^2 = D\{n\}$, then this may be simplified:

$$D\{\tilde{A}\} = \frac{1}{N^2}N\sigma^2 = \frac{\sigma^2}{N} \tag{7}$$

For sufficiently small signal to noise ratios ($\sqrt{|A|^2/\langle n^2\rangle}$), we may simply estimate $D\{n\} \approx D\{x\}$, i.e. use the approximation that the variance of the *noise* is equal to the variance of the entire time series $x$.

# 2 Numerical Simulation

Two simulations were performed, each estimating the amplitude of a signal at a frequency of 1144.3 Hz at an amplitude of $\sqrt{2}$ and phase of 45° in 64 seconds of complex time series




(a) Low SNR. Error is estimated accurately, as seen by the coincidence of the predicted error (blue circle) with the measured error (red ellipse). The amplitude of the simulated signal ($\sqrt{2}$ at a phase of 45°, or, in cartesian coordinates, real and imaginary components of unity) is also correctly recovered.

(b) High SNR simulation. Error is overestimated due to breakdown of the small signal approximation.

Figure 1: Results of two numerical simulations of the described algorithm

sampled at 2048 Hz. The 64 second series was broken down into 256 segments and the spectral component was estimated for each segment. The small signal approximation was used: the variance of the noise was estimated from the variance of the entire time series.

In the first test the signal-to-noise ratio was 0.1 and both the amplitude and the error on the amplitude estimate were accurately estimated. In the second test the signal-to-noise ratio was 10. In this case, while the amplitude was accurately estimated, the error on the estimates were not, owing to the breakdown of the small signal approximation. The estimates for both simulations are displayed in the complex plane in Fig. 1.