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The Optimal Filter Procedure

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THE OPTIMAL FILTER PROCEDURE

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1. A measure of the stochastic background

We start by expanding the metric perturbation $h_{\alpha\beta}(t, \vec{x})$ in plane waves [1]

$$h_{\alpha\beta}(t, \vec{x}) = \sum_A \int_{-\infty}^{\infty} df \int d\hat{\Omega} \tilde{h}_A(f, \hat{\Omega}) e^{2\pi i f(t - \hat{\Omega}\cdot\vec{x}/c)} e^A_{\alpha\beta}(\hat{\Omega}) \tag{1}$$

with $\alpha, \beta = 1, 2$ since we are in the TT gauge. For an isotropic, unpolarized and stationary background it holds that the ensemble average of the Fourier amplitudes is completely uncorrelated

$$< \tilde{h}^*_A(f, \hat{\Omega}) \tilde{h}_{A'}(f', \hat{\Omega}') >= \delta^2(\hat{\Omega}, \hat{\Omega}')\delta_{AA'}\delta(f - f')|h_0(f)|^2 \tag{2}$$

and

$$< \tilde{h}_A(f, \hat{\Omega}) >= 0 \tag{3}$$

The "power", at frequency, $|h_0(f)|^2$ is expressed in units of (strain)^2/Hz. In contrast $h_{\alpha\beta}(t, \vec{x})$ has dimension of (strain); and $\tilde{h}_A(f, \hat{\Omega})$ has dimensions of strain/Hz.

We are interested in the (frequency spectrum of the) energy density of the gravitational radiation [2]

$$\rho_G = \frac{c^2}{32\pi G} < \dot{h}_{\alpha\beta}(t, \vec{x}) \dot{h}^{\alpha\beta}(t, \vec{x}) > \tag{4}$$

To evaluate Eq.(4) we use the plane wave expansion given by Eq.(1), take the time derivative and carry out the sums over A, A' (factor of 4) and the integrations over $d\hat{\Omega}, d\hat{\Omega}'$ (factor of 4π) taking account of Eq.(2). We also integrate over df' and reduce the range of the df integration from $\infty \to +\infty$ to $0 \to +\infty$ (factor to 2, also note that $|h_0(f)|^2 = |h_0(-f)|^2$) to obtain

$$\rho_G = \frac{c^2}{G} 4\pi^2 \int_0^{\infty} df f^2 |h_0(f)|^2 \tag{5}$$

But by definition

$$\rho_G = \int_0^\infty df \frac{d\rho_G}{df} = \int_0^\infty df \frac{1}{|f|} \frac{d\rho_G}{d(lnf)} \tag{6}$$

Thus we can relate $|h_0(f)|^2$ to

$$\frac{1}{\rho_c} \frac{d\rho_G}{d(lnf)} = \Omega(f), \tag{7}$$

or

$$|h_0(f)|^2 = \rho_c \frac{G}{c^2 4\pi^2} \frac{1}{|f|^3} \Omega(f) \tag{8}$$

where ρ_c is the closure density of the universe defined through

$$\rho_c = \frac{3c^2 H_0^2}{8\pi G} \tag{9}$$

with H_0 the present-day value of the Hubble constant. Thus we can write

$$|h_0(f)|^2 = \frac{3}{32\pi^3} H_0^2 \frac{1}{|f|^3} \Omega(f) \tag{10}$$

Finally, we wish to connect the power spectrum $|h_0(f)|^2$ to the mean square fluctuations of the strain in the time domain. For a pair of Fourier conjugate variables Parseval's theorem states

$$\int_{-\infty}^{\infty} |h(t)|^2 dt^2 = \int_{-\infty}^{\infty} |\tilde{h}(f)|^2 df \tag{11}$$

The mean square fluctuations are related to the power spectrum since

$$< |h(t)|^2 >= \frac{1}{T} \int_{-T/2}^{T/2} |h(t)|^2 dt = \frac{1}{T} \int_{-\infty}^{\infty} |\tilde{h}(f)|^2 df = 2 \int_0^\infty |h_0(f)|^2 df \tag{12}$$

In the present case of the stochastic background, the detection efficiency (antenna pattern) of the interferometer depends on $\hat{\Omega}$. Averaging over all angles and polarizations introduces a factor of $8\pi/5$ (see Eq.(3.17 of [1]). Thus the mean value of the detected strain squared is given by

$$< |h_d(t)|^2 >= \frac{16\pi}{5} \int_0^\infty |h_0(f)|^2 df \tag{13}$$

2. Detecting a stochastic background by cross-correlation

It is assumed that the gravitational strain $h(f)$ is much smaller than the noise $n_{1,2}(f)$ in the two detectors (assumed to be co-located and co-aligned) that are being correlated

$$\begin{aligned} s_1(t) &= n_1(t) + h(t) \\ s_2(t) &= n_2(t) + h(t) \end{aligned} \tag{14}$$

It then follows that the mean square gravitational strain

$$\frac{1}{T}\int_{-T/2}^{T/2} dt < s_1(t)s_2(t) > = < h^2(t) > \equiv < S > \tag{15}$$

We designate this mean square signal by $< S >$ and call it the "**statistic**". In obtaining Eq.(15) we set the terms $< n_1(t)n_2(t) >, < n_1(t)h(t) >$ and $< n_2(t)h(t) >$ equal to zero.

We can also calculate the variance of $< S >$

$$\sigma^2_{<S>} = < S^2 > - < S >^2 \simeq < S^2 > \tag{16}$$

where the last step follows because $< S >$ is much smaller than $< S^2 >$ which is dominated by the noise in the detectors.

$$\begin{aligned} \sigma^2_{<S>} &= \frac{1}{T^2}\int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt' < s_1(t)s_2(t)s_1(t')s_2(t')) > \\ &= \frac{1}{T^2}\int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt' < n_1(t)n_1(t') >< n_2(t)n_2(t') > \end{aligned} \tag{17}$$

but by definition

$$< n(t)n(t') > = \frac{1}{2}\int_{-\infty}^{\infty} df e^{2\pi i f(t-t')} P(f) \tag{18}$$

where $P(f)$ is the one-sided **power spectral density** (of the noise) in the detector. Namely $P(f)$ is defined through

$$< n^2(t) > = \int_0^{\infty} P(f) df \tag{19}$$

Inserting Eq.(18) into Eq.(17) and carrying out the integrations over dt yields $\delta^2(f - f')$. One of the δ-functions is integrated over df' and the other one is replaced by T, the length of the integration interval that generated it. Thus

$$\sigma^2_{<S>} = \frac{1}{4T} \int_0^\infty df P_1(f) P_2(f) \tag{20}$$

By using Eq.(13) in Eq.(15) the **expected** value of the statistic is

$$< S >= \frac{8\pi}{5} \int_{-\infty}^{\infty} df |h_0(f)|^2 \tag{21}$$

The statistic $< S >$ was defined by Eq.(15) in terms of strain in the time domain. We wish to obtain an equivalent expression for $< S >$ in terms of the measured amplitude spectral densities $h(f)$: (namely the measured strain per $\sqrt{\text{Hz}}$). We do this by using the equivalent of Eq.(1) without the dependence on angle and polarization

$$h(t) = \int_{-\infty}^{\infty} df \tilde{h}(f) e^{2\pi i f t} \equiv \sqrt{T} \int_{-\infty}^{\infty} df h(f) e^{2\pi i f t} \tag{22}$$

Here T is the time interval used to generate $h(f)$ from the time domain data. Recall that $|h(f)|^2 = P(f)$.

It then follows that

$$\begin{aligned} < S > &= \frac{1}{T} \int_{-T/2}^{T/2} dt < h_1(t) h_2(t) > \\ &= \int_{-T/2}^{T/2} dt \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df' < h_1^*(f) e^{-2\pi i f t} h_2(f') e^{2\pi i f' t} > \\ &= \int_{-\infty}^{\infty} df < h_1^*(f) h_2(f) > \end{aligned} \tag{23}$$

where we also used the reality of $h(t)$.

Eqs.(23) and (20) allow us to calculate from the data the statistic and its error. In spite of the infinite range of the integrals, $< S >$ will remain finite because $|h_0(f)|^2$ is bounded (see Eq.(10)) and the experimental cross correlation

$$< h_1^*(f) h_2(f) > \tag{24}$$

is different from zero only for a finite frequency interval. In practice we limit the frequency integrals in both Eq.(23) and Eq.(20) to the physically relevant range by introducing an overlap function or an "optimal filter".

Nevertheless, Eqs.(23 and 20) show that as the range of the frequency integration increases, the statistic $< S >$ grows as Δf while its standard derivation grows only as $\sqrt{\Delta f}$, assuming that $< h_1(f)h_2(f) >$ and $P_1(f)P_2(f)$ are reasonably constant in that range. We also see that while $< S >$ is independent of the overall integration time, $\sigma_{<S>}$ decreases as $\sqrt{T}$. Thus the signal to noise ration is proportional to

$$\left(\frac{S}{N}\right)_{<S>} \propto \sqrt{T\Delta f} \tag{25}$$

Recall that the statistic $< S >$ is related to $|h_0(f)|^2$ and therefore

$$\left(\frac{S}{N}\right)_{<S>} \simeq \sqrt{T\Delta f}\left[\frac{|h_0|}{n_{\rm rms}}\right]^2 \tag{26}$$

where $n_{\rm rms}$ is the rms noise amplitude in the detectors.
We can call

$$\frac{|h_0|^2}{(n_{\rm rms})^2} \simeq \frac{|h_0(f)|^2}{[P_1(f)P_2(f)]^{1/2}}\Bigg|_{\text{averaged over the region of interest}} \tag{27}$$

the input $(S/N)_{in}$ ratio, and by our premises $(S/N)_{in} \ll 1$. In contrast, if one could determine $|h_0(f)|$ using a single interferometer, the corresponding input (S/N) ratio would be $\sqrt{(S/N)_{in}} \gg (S/N)_{in}$.

3. The Optimal filter

We can apply to Eq.(23) a "filter", $Q(f)$, in the frequency domain. The equation for the statistic then takes the form

$$< S >= \int_{-\infty}^{\infty} df < h_1^*(f)h_2(f) > Q(f) \tag{28}$$

Of course, the filter must be normalized

$$\int_{-\infty}^{\infty} df Q(f) = 1 \tag{29}$$

The shape (spectrum) of the filter is dictated by the properties of the detector and by the expected spectrum of the signal. A derivation of Eq.(28) and of Eq.(30) from first principles is given in the Appendix.

The two co-located and co-aligned interferometers $H1$ and $H2$ record amplitude spectral densities $h_1(f)$ and $h_2(f)$ respectively expressed in strain $/\sqrt{\rm Hz}$.

$Q(f)$ is the normalized optimal filter which we choose as

$$Q(f) = \frac{\mathcal{N}}{P_1(f)P_2(f)} \qquad \mathcal{N} = \frac{1}{\int_{-\infty}^{\infty} df/P_1(f)P_2(f)} \tag{30}$$

so that Eq.(29) is satisfied. $P_1(f), P_2(f)$ are the **calibrated**, noise dominated power spectral densities of $H1$ and $H2$.

It follows from Eq.(20) that the variance of S is

$$\sigma_S^2 =< (\mathcal{S}- < \mathcal{S} >)^2 > \simeq \frac{1}{4T}\int_{-\infty}^{\infty} df P_1(f)|Q(f)|^2 P_2(f) \tag{31}$$

with T is the length of the time record used to carry out the Fourier transform.

- At this point we can make some simplifying assumptions:
    1. Because $P_1(f)$ is so much smaller in the fsr region than everywhere else, (i.e. the filter peaks) we restrict the integration to $\pm$ 200 Hz around the fsr.
    2. In this region $h(f_0)$ can be assumed constant. It follows then from Eqs.(21, 23 and 28) that

$$< \mathcal{S} >= |h(f_0)|^2 \int_{-\infty}^{\infty} Q(f) df = |h(f_0)|^2 \tag{32}$$

- Since in our code we use counts (uncalibrated data) we must **carefully distinguish** between calibrated data (in strain/$\sqrt{\text{Hz}}$) and uncalibrated data (in counts/$\sqrt{\text{Hz}}$. We designate the uncalibrated data by an overbar, where

$$h(f) = \overline{h}(f)R(f) \tag{33}$$

$R_1(f), R_2(f)$ are the complex response functions for $H1$ and $H2$. They can be expressed as

$$R(f) = \frac{1}{H(f)C(0)} \tag{34}$$

Here $H_1(f), H_2(f)$ are the dimensionless complex transfer functions for $H1$ and $H2$ normalized to unity at $f = 0$. Note that at 37.52 kHz (the fsr for $H1$ but not for $H2$) they differ by a factor of $\sim 140; C_1(0)$ and $C_2(0)$ are the sensing functions, evaluated at $f = 0$ and for $S4$ they are given by

$$
\begin{aligned}
C_1 &= 2.46 \times 10^{21} \\
C_2 &= 1.17 \times 10^{21}
\end{aligned}
\tag{35}
$$

- We can now express Eqs.(29,30) in uncalibrated data

$$
\begin{aligned}
1/\mathcal{N} &= \int df \frac{1}{\overline{psd1}\,\overline{psd2}} \frac{1}{|R_1|^2|R_2|^2} \\
&= \int df \frac{1}{\overline{psd1}\,\overline{psd2}} (C_1C_2)^2 |H_1(f)|^2 |H_2(f)|^2 \\
&= (C_1C_2)^2 \int df \frac{|\mathrm{respcc}(f)|^2}{\overline{psd1}\,\overline{psd2}}
\end{aligned}
\tag{36}
$$

We introduced the **expected** response of the cross correlation defined through

$$
\mathrm{respcc}(f) = H1^*(f)H2(f) \tag{37}
$$

Similarly it follows that

$$
Q(f) = (C_1C_2)^2 \mathcal{N} \frac{|\mathrm{respcc}(f)|^2}{\overline{psd1}\,\overline{psd2}} \tag{38}
$$

Next we express the statistic in uncalibrated data

$$
\begin{aligned}
< \mathcal{S} > &= \int df < \overline{h}_1^*(f) \overline{h}_2(f) > Q(f) R_1^*(f) R_2(f) \\
&= (C_1C_2)^2 \mathcal{N} \int df < \overline{cc}(f) > \frac{|\mathrm{respcc}(f)|^2}{\overline{psd1}\,\overline{psd2}} \frac{1}{C_1C_2H_1^*(f)H_2(f)} \\
&= C_1C_2\mathcal{N} \int df < \overline{cc}(f) > \frac{\mathrm{respcc}(f)^*}{\overline{psd1}\,\overline{psd2}}
\end{aligned}
\tag{39}
$$

where we used the notation

$$
< \overline{cc}(f) > = < \overline{h}_1^*(f) \overline{h}_2(f) > \tag{40}
$$

for the measured (uncalibrated) cross-correlation in $(\text{counts})^2/\text{Hz}$.

Finally introducing the normalization (Eq.36) into Eq.(39) we have

$$< \mathcal{S} >= \frac{\int df < \overline{cc} > (\text{respcc})^*/(\overline{psd1}\ \overline{psd2})}{\int df |\text{respcc}|^2/(\overline{psd1}\overline{psd2})} \cdot \frac{1}{C_1 C_2} \tag{41}$$

Only the real part of Eq.(41) can be different from zero, as can be seen from Eq.(32). However since the phase of $< \overline{cc} >$ depends on IFO tuning we evaluate both the real and imaginary part. Further

$$\sigma_{\mathcal{S}} = \frac{1}{\left[\int df |\text{respcc}|^2/\overline{psd1}\ \overline{psd2}\right]^{1/2}} \frac{1}{2\sqrt{T}} \cdot \frac{1}{C_1 C_2} \tag{42}$$

From Eqs.(41,42) we see that the output of our code must be multiplied by

$$1/C_1 C_2 = 3.47 \times 10^{-43} \tag{43}$$

to yield $< S >$ and $\sigma_{<S>}$ in units of $(\text{strain})^2/\text{Hz}$.

- We can also "fit" the uncalibrated data either without or with an optimal filter.

  For the cross-correlations, and using the optimal filter of Eq.(30) or Eq.(38), we proceed as follows

$$|h(f_0)|^2 Q(f) = \mathcal{N} \frac{< \overline{h}_1^*(f) \overline{h}_2(f) >}{\overline{psd1}\ \overline{pds2}} \frac{|\text{respcc}(f)|^2}{H_1^*(f) H_2(f)} C_1 C_2$$

or

$$|h(f_0)|^2 (C_1 C_2) \mathcal{N} \frac{|\text{respcc}(f)|^2}{\overline{psd1}\ \overline{psd2}} = \mathcal{N} \frac{< \overline{cc}(f) >}{\overline{psd1}\ \overline{psd2}} (\text{respcc})^* C_1 C_2$$

or

$$|h(f_0)|^2 |\text{respcc}(f)|^2 =< \overline{cc}(f) > (\text{respcc})^* \frac{1}{C_1 C_2} \tag{44}$$

That is, if we fit

$$< \overline{cc}(f) > (\text{respcc})^* \tag{45}$$

for the component that behaves spectrally as $|\text{respcc}(f)|^2$, we obtain $|h(f_0)|^2 C_1 C_2$. Again only the real part of Eq.(44) should return a good fit if the phases have been properly adjusted.

The above result should be equivalent to that obtained from calculating the statistic $< \mathcal{S} >$ by integration. As discussed below, numerically the two methods agree.

4. Injections

We injected signals in the frequency domain by adding a random signal to actual data (or to simulated data) as follows and redefining

$$\begin{aligned} \overline{h}_1(f) &= \overline{n}_1(f) + \alpha \cdot rn(f) \cdot H1(f) \cdot \\ \overline{h}_2(f) &= \overline{n}_2(f) + \alpha \cdot rn(f) \cdot H2(f) \cdot \end{aligned} \tag{46}$$

Here $\overline{n}_{1,2}(f)$ is the (uncalibrated) amplitude spectral density for detectors 1,2 in strain/$\sqrt{\text{Hz}}$. Namely $\overline{n}_{1,2}(f)$ are the properly normalized Fourier transforms of the time series, such that

$$|\overline{n}_{1,2}(f)|^2 = \overline{P}_{1,2}(f) \equiv \overline{psd1,2} \tag{47}$$

The Fourier transforms were carried out over segments of length $\Delta t = 32$ s or with BW $= 1/32$ Hz.

$$rn(f) = r_1(f) + ir_2(f) \tag{48}$$

where $r_1(f), r_2(f)$ are vectors containing real random numbers, Gaussian distributed with zero mean and unit standard deviation. The real parameter α defines the injection strength. $H1(f)$ and $H2(f)$ are the complex transfer functions for the two detectors normalized to unity at zero frequency as already discussed in connection with Eq.(34).

The program calculates

$$\overline{h}_1^*(f)\overline{h}_2(f)$$

for each 32 s segment. It then averages these values over three frames, or 24 segments outputting

$$< \overline{h}_1^*(f)\overline{h}_2(f) >=< \overline{cc}(f) > \tag{49}$$

If the injected signals are much smaller than $\overline{n}_{1,2}(f)$ and if the averaging is adequate, we expect as shown in Eq.(15) that

$$< \overline{cc}(f) >= \alpha^2 \langle |rn(f)|^2 \rangle H1^*(f)H2(f) \tag{50}$$

Since

$$\langle |rn(f)|^2 \rangle = \langle |r_1(f)|^2 + |r_2(f)|^2 \rangle = 2$$

and using the notation of Eq.(37) we expect

$$< \overline{cc}(f) >= 2\alpha^2 \mathrm{respcc}(f) \tag{51}$$

When this result is introduced into the analysis program at Eq.(41) we find for the statistic (setting $C_1 = C_2 = 1$)

$$< S > = 2\alpha^2 \tag{52}$$

In this limit the standard deviation is independent of the injection as is obvious from Eq.(42). This is not anymore true when the injections modify the values of the pds's.

For calibration purposes we note that

$$1/C_1C_2 = 3.5 \times 10^{-43}$$

and therefore the injected power, $(\mathrm{strain})^2/\mathrm{Hz}$ is

$$|h(f_0)|^2 = 7 \times 10^{-43} \alpha^2/\mathrm{Hz}$$

or

$$\Omega(f_0) = 6.8 \times 10^{-11} \alpha^2 |f_0|^3$$

The results that we have obtained are listed in the Table and shown in Fig.1. Both the statistic $< S >$ and the fitted values to the spectrum are given, as a function of the parameter α^2. The expected value (in the appropriate limit) is $2\alpha^2$ as in Eq.(52). The values of $T, \Delta f$ are

$$T = 768 \text{ s} \qquad \Delta f \simeq 130 \text{ Hz}$$

so that $\sqrt{T\Delta f} \simeq 30$.

Table 1 Results of injections using three simulated data frames

α^2	1	0.1	0.01	10^{-3}	10^{-4}
$< S >=$ rstat	1.832	0.237	0.019	2.7×10^{-3}	2.7×10^{-4}
σ	0.010	0.003	0.001	0.4×10^{-3}	2.7×10^{-4}
fit	1.893	0.252	0.020	4.2×10^{-3}	0.6×10^{-4}
$\sqrt{\Delta\chi^2}$	150	56	15	7.6	0.2
$(S/N)^*_{in}$	100	10	1.0	0.1	0.01
$(S/N)_{out}$	180	79	19	6.8	1
istat	0.014	-0.001	3×10^{-4}	4×10^{-4}	-0.4×10^{-4}
σ	0.010	0.003	0.001	4×10^{-4}	2.7×10^{-4}
fit	8×10^{-4}	-0.01	0.001	0.001	-1.5×10^{-4}
$\sqrt{\delta\chi^2}$	0.06	2.5	0.8	2.0	0.5

Notes:

1. $(S/N)_{in}$ is calculated from Eq.(27) taking $(n_{\rm rms})^2 = 0.02$. $(S/N)_{out} = < S >/\sigma$
2. Only the last three entries satisfy the limit of small signal to noise input.
3. There may be a bug in the code (the injection or analysis part), but the results seem convincing. Need to make longer runs with small $(S/N)_{in}$.

Appendix

Introducing the optimal filter in the evaluation of the statistic $< S >$

The filter is optimal in the sense that it maximizes the (S/N) ratio for the statistic. We start from Eq.(3.52) of ref. [1].

$$< S > = \frac{1}{T} \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df' \delta_T(f - f') < \tilde{h}_1^*(f) \tilde{h}_2(f') > Q(f') \qquad (A1)$$

where $\tilde{h}_1(f), \tilde{h}_2(f')$ and $Q(f')$ are the Fourier transforms of $h_1(t), h_2(t)$ and $Q(t - t')$ and $\delta_T(f - f')$ is a finite approximation to the δ-function. Replacing $\tilde{h}_{1,2}(f) = \sqrt{T} h_{1,2}(f)$ and carrying out the integration over df' immediately leads to Eq.(28).

Following the same steps as in section 2 leading to Eq.(31) we find

$$< S > = \frac{8\pi}{5} \int_{-\infty}^{\infty} df |h_0(f)|^2 Q(f) \qquad (A2)$$

Similarly from the steps leading to Eq.(20) we find

$$\sigma^2_{<S>} = \frac{1}{4T} \int_{-\infty}^{\infty} df P_1(f) P_2(f) Q(f) \qquad (A3)$$

As shown in [1] the form of $Q(f)$ that maximizes $< S > / \sigma_{<S>}$ is simply

$$Q(f) = \mathcal{N} \frac{1}{P_1(f) P_2(f)} \qquad (A4)$$

This form is used in Eq.(30) of section 3.