LASER INTERFEROMETER GRAVITATIONAL WAVE OBSERVATORY

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Statistical properties of sky map measurements convolved with a finite-reponse antenna pattern		
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I. INTRODUCTION

The Kolomogorov-Smirnov (K-S) test provides a simple way to to assess the consistency of an observed cumulative distribution function (CDF) with a model CDF. However, if the CDF involves samples that may be correlated, its applicability needs to be examined. This note shows that the effect of pixel-to-pixel correlations is to modify the expected variance of the model and the effective number of degrees of freedom, but not the applicability of the model.

In order to apply the K-S test, it is necessary to know the underlying CDF of the model. This note derives the probability distribution function (PDF), and hence the CDF, of toy model consisting of an array of measurements which consist of an underlying uncorrelated noise process that is convolved with a 2-D kernel (the antenna pattern) that is responsible for introducing correlations among the measured pixels. Given the PDF and CDF of the underlying noise model and the correlation properties of the antenna pattern, the PDF and CDF of the resulting correlated pixels may be specified. Hence, the K-S test may be applied to the measured array of data.

II. THE MODEL

A. Underlying noise process

Consider a 2-D array or map of pixels specified by coordinates $\{x_i, y_j\}$. Let the underlying uncorrelated noise process be $n(x_i, y_j)$, with the following statistical properties that do not depend on position, $\{x_i, y_j\}$,

$$P(n(x_i, y_j)) \;=\; \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{n^2}{2\sigma_n^2}} \tag{1}$$

Figure 1 shows a 2-D array of Gaussian N[0,1] noise and the corresponding histogram of the PDF over the pixels in the map.

Figure 2 shows comparison of the CDF between data and a Gaussian N[0,1] model. The corresponding K-S statistic is 0.92.

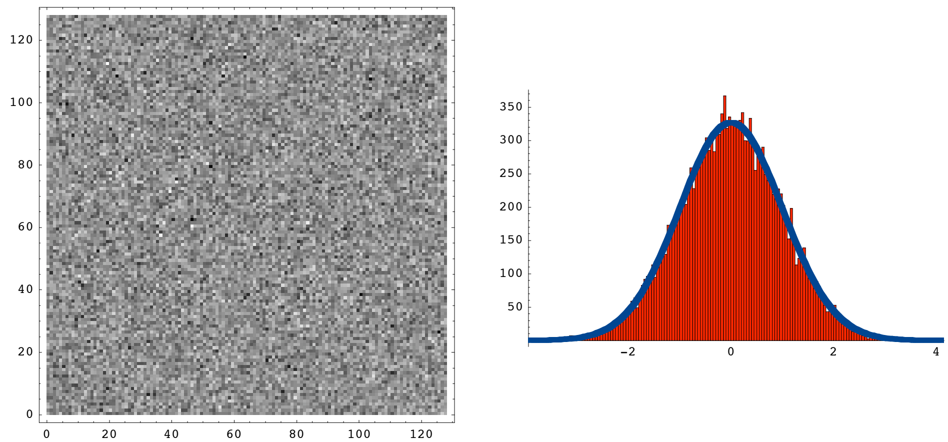

Figure 1: Left – 2-D map of N[0,1] noise process; Right – histogram of the PDF of pixel values. Blue solid curve corresponds to N[0,1] expected PDF.

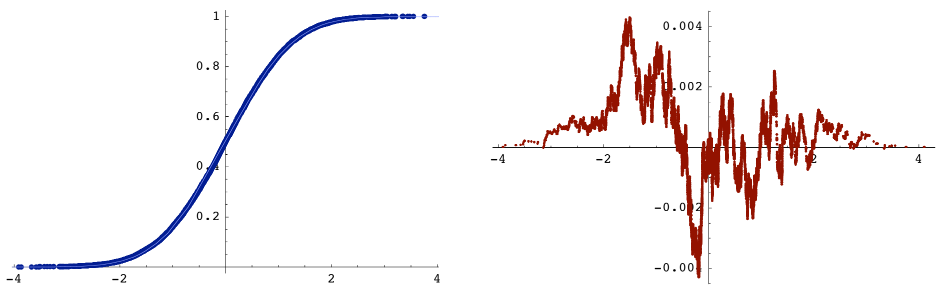

Figure 2: Left – CDF of data (blue dots) and model (Cyan line); Right – residuals between data and model. K-S statistic for this fit corresponds to KS = 0.92.

B. Correlation process

An instrumental response function, $\mathbf{K}$, will produce pixel-to-pixel correlations due to its finite response. As an example, consider a Gaussian correlation kernel,

$$\mathbf{K}(x_i, x_{i'}, y_j, y_{j'}) = \frac{1}{\mathcal{N}} e^{\frac{(x_i - x_{i'})^2 + (y_j - y_{j'})^2}{2\sigma_{xy}^2}}, \text{with} \tag{2}$$

$$\mathcal{N} \equiv \sum_{i,j} e^{\frac{(x_i - x_{i'})^2 + (y_j - y_{j'})^2}{2\sigma_{xy}^2}} \tag{3}$$

$$\text{so that} \tag{4}$$

$$\sum_{i,j} \mathbf{K}(x_i, x_{i'}, y_j, y_{j'}) = 1 \tag{5}$$

Figure 3 shows the correlation kernel for $\sigma_{xy} = 2.5$ pixels .

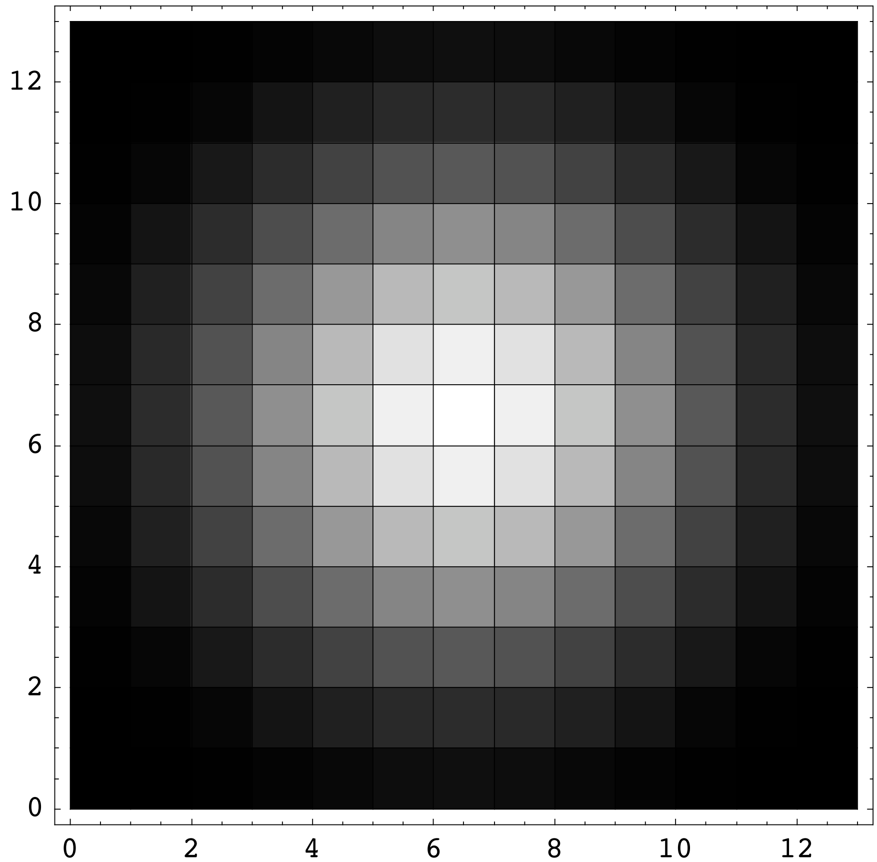

Figure 3: Correlation kernel corresponding to $\sigma_{xy} = 2.5$ pixels

Figure 4 shows a more "pathological" structured correlation kernel that is no longer simply a Gaussian.

C. Measured data

The measured data, $d(x_i, y_j)$, are produced by a convolution process between the instrumental response function, $\mathbf{K}$ and the underlying noise,

$$d(x_i, y_j) \ = \sum_{i',j'} \mathbf{K}(x_i - x_{i'}, y_j - y_{j'}) \ n(x_{i'}, y_{j'}) \tag{6}$$

Given the PDF for n, it is desired to determine the PDF for d. This follows directly by considering the problem in the Fourier transform space of the PDFs, where it is possible to manipulate the characteristic functions, $\phi_i(\omega)$ in a straightforward manner. The characteristic function for the PDF of the noise n is given by [1],

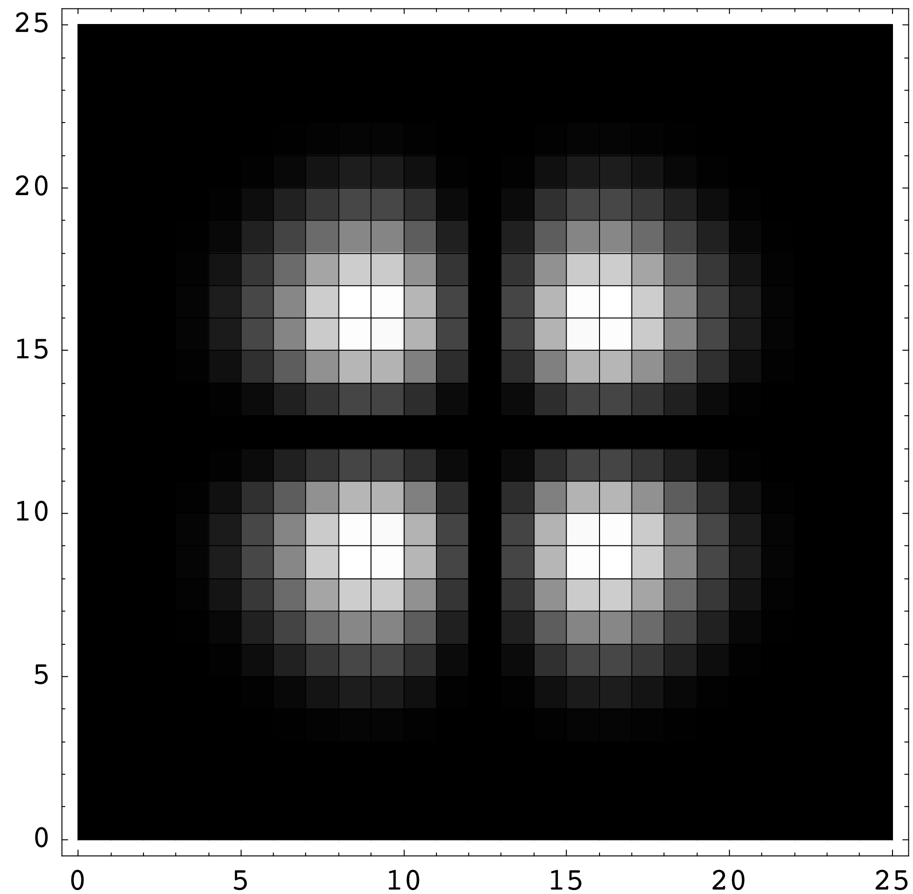

Figure 4: A more complicated correlation kernel having multiple lobes

$$\phi_n(\omega) = \int_{-\infty}^{+\infty} dn P(n) e^{i\omega n} \tag{7}$$

Using the PDF for n given in Eq. 1, we have,

$$\phi_n(\omega) = e^{-\sigma_n^2 \omega^2/2} \tag{8}$$

Now Eq. 6 shows that d corresponds to a weighted sum of independent RVs. Each RV n is an N[0,1] RV. Therefore the RV $q = \mathbf{K}_{ij} n_{ij}$ is an N[0,$\mathbf{K}_{ij}^2$] RV. Thus,

$$\phi_{\mathbf{K}_{ij} n_{ij}}(\omega) = e^{-\sigma_n^2 \mathbf{K}_{ij}^2 \omega^2/2} \tag{9}$$

Therefore, the sum in Eq. 6 results in the characteristic function for d given by,

$$
\begin{aligned}
\phi_d(\omega) &= \prod_{ij} \phi_{\mathbf{K}_{ij} n_{ij}}(\omega) && (10)\\
&= e^{-(\sum_{ij} \mathbf{K}_{ij}^2)\sigma_n^2 \omega^2/2} && (11)\\
\rightarrow \; P(d) &= \frac{1}{\sqrt{2\pi(\sum_{ij} \mathbf{K}_{ij}^2)\sigma_n^2}} e^{-d^2/(2(\sum_{ij} \mathbf{K}_{ij}^2)\sigma_n^2)} && (12)
\end{aligned}
$$

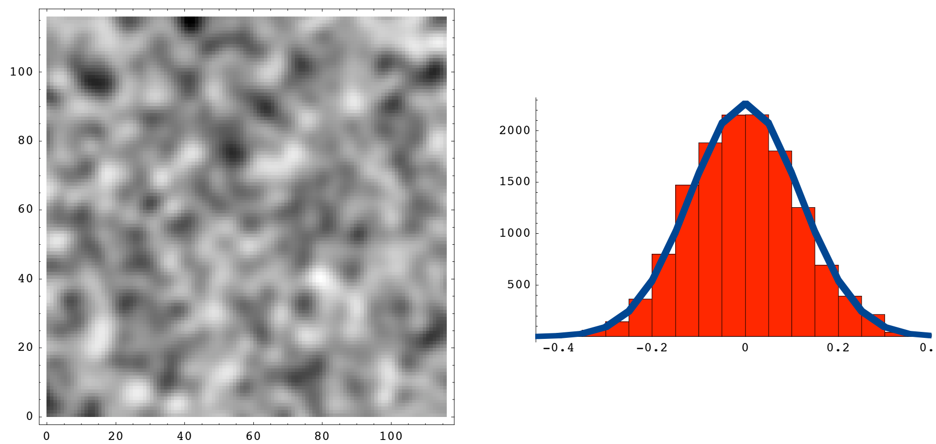

Figure 5: Left – 2-D map of the convolved signal d using a Gaussian correlation kernel; Right – histogram of the PDF of pixel values. Blue solid curve corresponds to N[0,$\sum_{ij} \mathbf{K}_{ij}^2$)] expected PDF when the underlying process n is N[0,1].

Figure 5 shows the 2-D map of measured data d and the corresponding PDF of the pixel values, along with the expected model.

Figure 6 shows the 2-D map of measured data d using the "pathological" correlation kernel of Figure 4 and the corresponding PDF of the pixel values, along with the expected model.

D. Number of effective pixels in the blurred map

Figure 7 shows comparison of the CDF between data and a Gaussian N[0,$\sum_{ij} \mathbf{K}_{ij}^2$)] model. As expected, the CDF for d is indeed consistent with a Guassian PDF with variance given by $\sigma_d^2 = \sigma_n^2 \sum_{ij} \mathbf{K}_{ij}^2$. In order to assess the significance of the residuals shown in Fig. 7 using the K-S test, the number of degrees of freedom or number of *effective* independent data points in the CDF. For the raw data n, this is simply the number of points in the array, $N = N_x N_y$. However, by convolution with the correlation kernel $\mathbf{K}$, the effective number

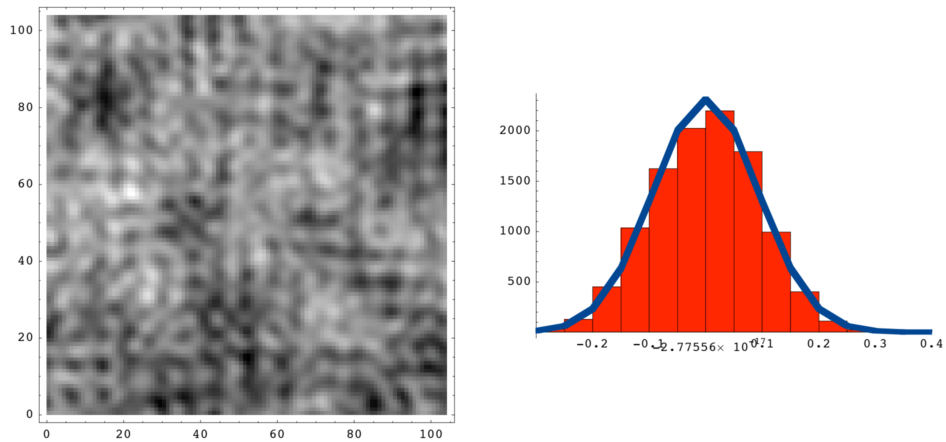

Figure 6: Left – 2-D map of the signal d produced by convolution with the multi-lobed kernel shown in Figure 4; Right – histogram of the PDF of pixel values. Blue solid curve corresponds to N[0,$\sum_{ij} \mathbf{K}_{ij}^2$)] expected PDF when the underlying process n is N[0,1].

of independent points is reduced. By considering the effect of $\mathbf{K}$, it can be shown that the reduction in data points is given by $\sum_{ij} \mathbf{K}_{ij}^2 \leq 1$. In the limit $\mathbf{K}_{ij} \rightarrow \delta_{ij}$, $\sum_{ij} \mathbf{K}_{ij}^2 \equiv 1$ and $N_{eff} = N_x N_y$; in the other limit of $\mathbf{K}_{ij} = \text{const} = 1/(N_x N_y)$, the entire plane is averaged to produce a single average value for all pixels and then $\sum_{ij} \mathbf{K}_{ij}^2 \equiv 1/(N_x N_y)$, so that $N_{eff} = 1$. When applying the K-S test the appropriate number of data points to use is then given by $N_{eff} = N_x N_y \sum_{ij} \mathbf{K}_{ij}^2$. Using this value of N_{eff}, the K-S statistic is ~ 1.

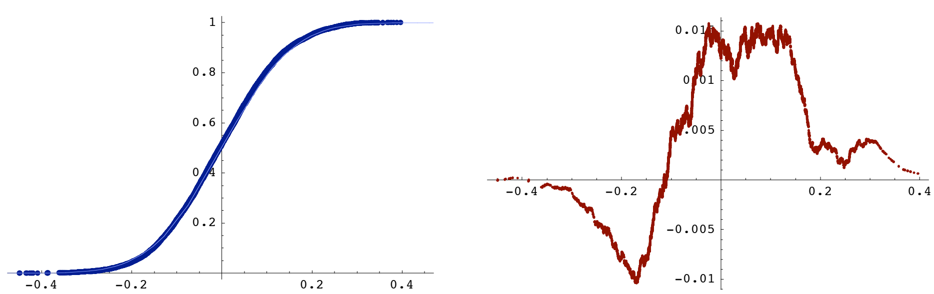

Figure 7: Left – CDF of data (blue dots) and model (Cyan line); Right – residuals between data and model. K-S statistic for this fit corresponds to KS ~ 1.

Last, Figure 8 shows a comparisons of CDFs and the residuals between data and Gaussian model for d even when the correlation kernel is highly structured and non-Gaussian.

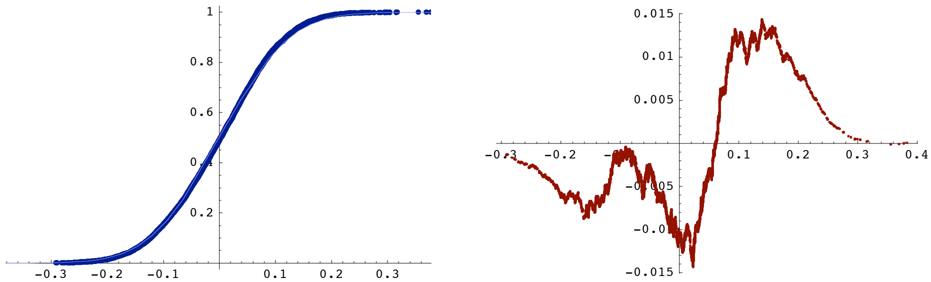

Figure 8: Left – CDF of data (blue dots) and model (Cyan line). In this case the mulit-lobed kernel of Figure 4 was applied to the raw data; Right – residuals between data and model. K-S statistic for this fit corresponds to KS ~ 1.

III. SPATIAL NONSTATIONARITY

In practice, for the application at hand, the antenna convolution kernel depends on sky position. This introduces a spatially nonstationary process during the convolution of the now position dependent kernel and the underlying noise. Here, $\mathbf{K}(x_i, x_{i'}, y_j, y_{j'})$ is no longer simplify a function of the differences, $\{x_i - x_{i'}, y_j - y_{j'}\}$:

$$d(x_i, y_j) \;=\; \sum_{i', j'} \mathbf{K}(x_i, x_{i'}, y_j, y_{j'})\; n(x_{i'}, y_{j'}) \tag{13}$$

Figure 9 presents a slightly more realistic example of a position-dependent correlation kernel.

Nonetheless, at each point $\{x_i, y_j\}$, the convolution process is linear and results in a weighted sum of independent noise RVs from the underlying noise. Now, though the variance of the weighted sum is position dependent:

$$\sigma_d^2(x_i, y_j) \;=\; \sigma_n^2 \sum_{i'j'} \mathbf{K}^2_{iji'j'} \tag{14}$$

Figure 10 shows the position-dependent standard deviation for this example.

The corresponding convolved image corresponding to Equation 13 is shown in Figure 11. However, since the raw image $d(x_i, y_j)$ exhibits a spatially varying standard deviation, the normalized quantity,

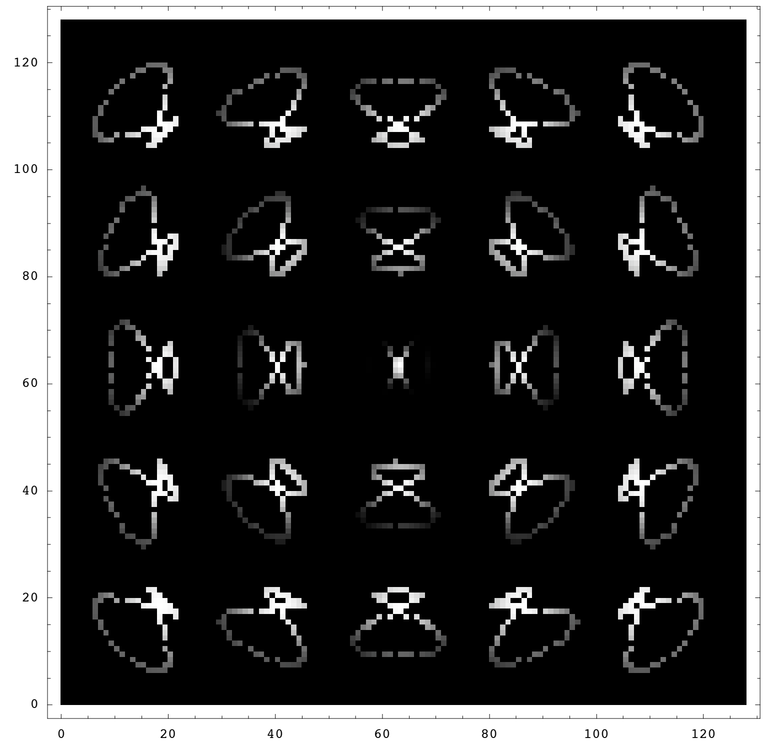

Figure 9: Example of a position dependent correlation kernel reminiscent of the gravradiometer antenna pattern. The "X" of each point spread function (PSF) is centered on the pixel to the PSF corresponds.

$$\xi(x_i, y_j) \;=\; \frac{d(x_i, y_j)}{\sigma(x_i, y_j)} \tag{15}$$

is shown. It can be seen that a histogram of the distribution of values ξ is consistent with the expected N[0,1] PDF.

Finally, the Kolmogorov-Smirnov test for this case is shown in Figure 12. As before, the effective number of degrees of freedom for this test is smaller than $N = N_x N_y$ due to the correlation introduced by the kernel. Generalizing on the discussion of Section II D, the effect of a position dependent kernel is to introduce a spatially averaged version of the expression for N_{eff} when the sum $\sum_{ij} \mathbf{K}_{ij}^2$ depends on position: $\sum_{ij} \mathbf{K}_{ij}^2 \to 1/(N_x N_y) \sum_{iji'j'} \mathbf{K}_{iji'j'}^2$, thus, $N_{eff} = \sum_{iji'j'} \mathbf{K}_{iji'j'}^2$.

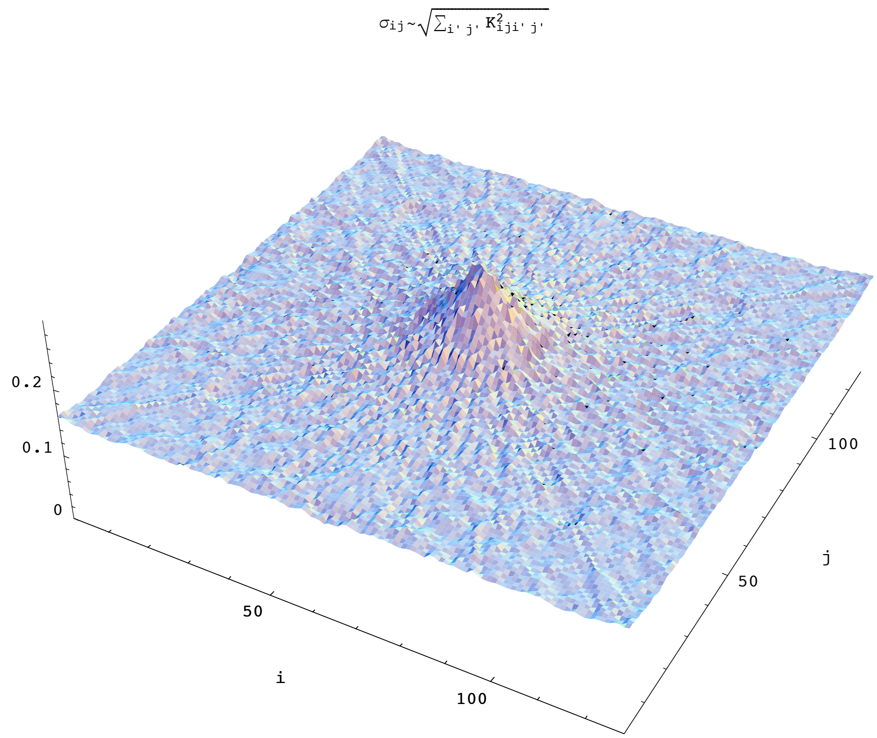

Figure 10: Spatial dependence of signal standard deviation for the PSFs shown in Figure 9.

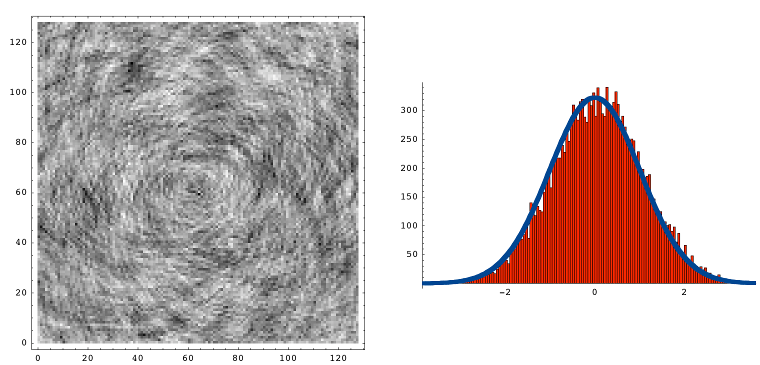

Figure 11: Left – 2-D map of the signal $\xi(x, y) \equiv d(x, y)/\sigma(x, y)$ produced by convolution with the position-dependent kernel shown in Figure 9 and normalizing to the PSF standard deviation of Figure 10; Right – histogram of the PDF of pixel values. Blue solid curve corresponds to N[0,1] expected PDF from the normalized values $\xi(x, y)$

IV. SUMMARY

The Kolmogorov-Smirnov test may be applied to the spatially resolved sky maps produced by targeted stochastic GW searches to determine whether the residuals of the sky map are consistent with an underlying Gaussian noise process or whether there is structure

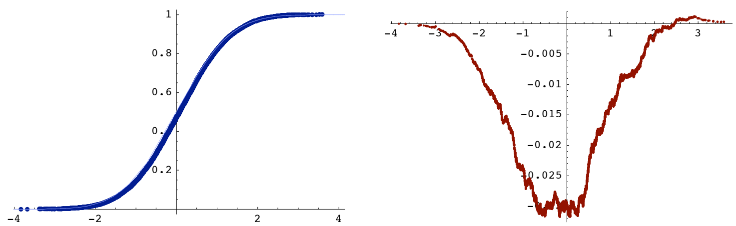

Figure 12: Left – CDF of data (blue dots) and model (Cyan line); Right – residuals between data and model. While there appears to be a small systematic deviation between the N[0,1] model and the data, with the corresponding N_{eff} reduced by the correlations introduced by the kernel, the K-S statistic for this fit corresponds to KS ~ 1.

in the map. The blurring introduced by the finite angular-resolution antenna pattern introduces correlations among neighboring pixels that reduces the effective number of degrees of freedom or independent data points in the map. However, by considering the details of the convolution process, it is possible to take this into account quantitiatively. In addition, the effect of a position dependent antenna-resolution function is still tractable. The analysis may be generalized to underlying CDF models that themselves contain correlations among pixels. Reference [1] discusses this in detail.

V. REFERENCES

[1] B.R. Frieden, "Probability, Statistical Optics, and Data Testing," Springer Series in Information Sciences, No. 10, Springer-Verlag, 2^{nd} Ed., New York, 1991.