Blade springs with a point bending moment at the tip, and lateral stiffness of blade.

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1 Background

This note records the derivation of a formula for the stiffness of a tapered blade spring when subject to a point moment at the tip, and for a lateral force at the tip. The formulae for loading with a vertical force at the tip are in various places including T030285. The reason for the derivation of the moment formula was to try to understand the effect of a longitudinal force acting off the neutral axis, however further thought has shown that that problem needs a different approach. The reason for the derivation of the lateral force equation was to confirm some FEA results because, as shown in T050255 (and other places) the lateral stiffness of the blades is significant.

2 Approach – moment on the blade tip

The approach was to copy the derivation given in the appendix of T030285 but modify the starting conditions for a moment rather than a force at the tip. The simple approach given in that paper for a triangular blade, does not work in the case of a point bending moment at the tip and so the full derivation for a trapezoidal blade is required.

See appendix 1 for the derivation.

3 Result – moment on the blade tip

Using the same symbols as in the previous paper and introducing *M* for the bending moment at the tip, the formula is

$$y = \gamma \frac{12Ml^2}{Eah^3}$$

Where

$$\gamma = \frac{1}{1-\beta}\left(\frac{1-\beta+\beta\,\mathrm{Logn}(\beta)}{1-\beta}\right)$$

γ varies between 1 for a triangular blade and 0.5 for a rectangular blade. For a rectangular blade the formula agree with standard textbook results (Roark).

This has been tested for a simple triangular blade using FEA.

4 Approach – lateral blade stiffness

Again the approach was similar but this time noting that the SMA of the blade varies with the cube of distance along the blade. See appendix 2 for derivation.

5 Result – lateral blade stiffness

At face value, the result is:

$$y = \varepsilon \frac{12Fl^3}{hEB^3}$$

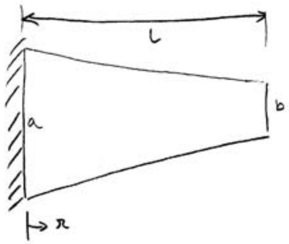
Where

$$\varepsilon = \left[-\ln(x) + \frac{x}{l} + \ln(l) - 1 \right]$$

where x is the distance along the blade measured from the tip. The problem with this is that when x is zero (at the blade tip), the value of ε is arbitrarily large because of the ln(0) term. This is because the second moment of area is tending to zero as you near the tip, faster than the bending moment. However, a real blade cannot come to an infinitely small point and be loaded there. I think the formula can be used at other locations along the blade despite the singularity at the tip, so one could imagine applying the formula at a fixed distance (say, 20mm) from the end. Further work would be needed to verify this.

6 APENDIX 1 – derivation of moment formula

①



$$I = \frac{wt^3}{12} = \frac{t^3}{12}\left(a - \frac{x}{l}(a-b)\right)$$

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = \frac{M}{C - Dx} \qquad \begin{cases} C = \dfrac{Et^3 a}{12} \\ D = \dfrac{Et^3}{12} \times \dfrac{a-b}{l} \end{cases} \qquad [1]$$

$$\frac{dy}{dx} = -\frac{M}{D}\ln(c - Dx) + E \qquad [2]$$

$$y = -\frac{M}{D}\left(-x - \frac{c\ln(c-Dx)}{D} + x\ln(c-Dx)\right) + Ex + F \qquad [3]$$

CHECK:

$$\frac{dy}{dx} = -\frac{M}{D}\left(-1 + \frac{c\cancel{D}}{\cancel{D}(c-Dx)} - \frac{Dx}{c-Dx} + \ln(c-Dx)\right) + E$$

$$= -\frac{M}{D}\left(\frac{-(c-Dx) + c - Dx}{c-Dx} + \ln(c-Dx)\right) + E$$

$$= -\frac{M}{D}\ln(c-Dx) + E$$

$$\frac{d^2y}{dx^2} = -\frac{M}{D} \times \frac{-D}{c-Dx}$$

$$= \frac{M}{c-Dx} \qquad \text{QED}$$

② Boundary conditions

$$x = 0 \Rightarrow y = \frac{dy}{dx} = 0$$

In [2]
$$0 = -\frac{M}{D}\ln(c) + E$$

$$\Rightarrow E = \frac{M}{D}\ln(c) \qquad [4]$$

In [3]
$$0 = -\frac{M}{D}\left(0 - \frac{c}{D}\ln c\right) + F$$

$$F = -\frac{Mc}{D^2}\ln c \qquad [5]$$

When $x = L$

$$y = -\frac{M}{D}\left(-L - \frac{c\ln(c-DL)}{D} + L\ln(c-DL)\right) + \frac{ML}{D}\ln c - \frac{Mc}{D^2}\ln c$$

$$y = \frac{M}{D}\left(L + \frac{c}{D}\ln(c-DL) - L\ln(c-DL)\right) + \frac{M}{D}L\ln c - \frac{Mc}{D^2}\ln c$$

$$= \frac{M}{D}\left(L + \left(\frac{c}{D} - L\right)\ln(c-DL) + L\ln c - \frac{c}{D}\ln c\right)$$

$$= \frac{M}{D}\left(L + \left(\frac{c}{D} - L\right)\left(\ln(c-DL) - \ln c\right)\right)$$

$$= \frac{M}{D}\left(L + \left(\frac{c}{D} - L\right)\ln\left(\frac{c-DL}{c}\right)\right) \qquad [6]$$

$$\frac{c-DL}{c} = \frac{a-(a-b)}{a} = \beta \left(= \frac{b}{a}\right)$$

$$\frac{c}{D} - L = \frac{aL}{a-b} - L = L\left(\frac{a}{a-b} - 1\right) = L\left(\frac{a-(a-b)}{a-b}\right)$$

$$= L\frac{b}{a-b}$$

$$= L\left(\beta/1-\beta\right)$$

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From [6]

$$y = \frac{M}{D}\left(L + L\left(\frac{\beta}{1-\beta}\right)\ln\beta\right)$$

$$= \frac{12\,M\,l_1}{Et^3\,(a-b)} \times l\left(1 + \frac{\beta\ln\beta}{1-\beta}\right)$$

as $\beta \to 0$ (triangular blade)

$$\left.\begin{aligned}\frac{\beta\ln\beta}{1-\beta} &\to 0\\ b &\to 0\end{aligned}\right\}\quad y \to \frac{12\,M\,l^2}{E\,t^3\,a}$$

as $\beta \to 1$ (~~the~~ rectangular blade)

$$\frac{\beta\ln\beta}{1-\beta} \to -1 \text{ (numerically)} \quad \text{but } a-b \to 0$$

Recast as

$$y = \frac{12\,M\,l^2}{E\,t^3 a} \times \left[\frac{1}{1-\beta}\left(\frac{1-\beta+\beta\ln\beta}{1-\beta}\right)\right]$$

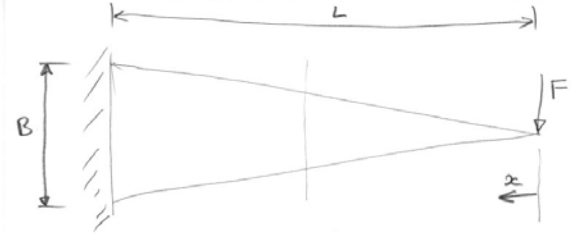
as $\beta \to 0$, $[\;] \to 1$ (as ~~before~~ above)

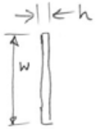
$\beta \to 1$, $[\;] \to 0.5$

$$y \to \frac{6\,M\,l^2}{E\,t^3 a}$$ (check with standard result

$$= \frac{Ml^2}{2EI}$$

Appendix 2 – derivation of lateral force formula





$w = kx$

$k = \frac{B}{L}$

$$\frac{d^2y}{dx^2} = \frac{M}{EI} \qquad I = \frac{hw^3}{12}$$

$$= \frac{M}{E} \times \frac{12}{hk^3x^3}$$

But $M = Fx$, so $\frac{d^2y}{dx^2} = \underbrace{\frac{F}{E} \times \frac{12}{hk^3}}_{J} \times x^{-2} = Jx^{-2}$

$$\left.\begin{aligned} \frac{dy}{dx} &= -Jx^{-1} + A \\ y &= -J\ln x + Ax + B \end{aligned}\right\} \text{when } x = L,\ y = \frac{dy}{dx} = 0$$

$$A = 0 + \frac{J}{L} = \frac{J}{L}$$

$$B + AL - J\ln L = 0 \quad \Rightarrow B = J\ln L - J$$

So

$$y = -J\ln x + \frac{J}{L}x + J\ln L - J$$

$$= J\left(-\ln x + \frac{x}{L} + \ln L - 1\right)$$

$$= \frac{12FL^3}{hEB^3}\,\varepsilon \quad \text{where } \varepsilon = \left[-\ln x + \frac{x}{L} + \ln L - 1\right]$$