LIGO Laboratory / LIGO Scientific Collaboration

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Suspension Blade Spring Internal Modes (Advanced LIGO)

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Change Record:

Revision –00: initial release.

Revision –01: The calculations (in version –00) for the internal modes based upon cantilevered, linearly tapered beams from referenced papers from Mabie & Rogers were in error by a factor of $\sqrt{g}$, where g is the gravitational acceleration. (This factor is only needed when using weight density instead of mass density.) In addition, the conclusion drawn from the wire transmissibility regarding the appropriate blade spring tip boundary condition was incorrect. For the typical blade spring, wire and mass parameters in LIGO suspensions, there is still enough coupled mass to enforce a simply supported blade tip boundary condition. Equations for the frequencies of a linearly tapered beam with clamped root and simply supported tip are incorporated and shown to be the asymptotic solution when the coupled tip mass is a large multiple of the blade spring mass.

1 Introduction

The advanced LIGO suspension design[1] achieves vertical isolation through the use of passive multiple stages of cantilevered blade springs which low pass filter the vibration. The frequencies of the rigid body modes of the coupled mass-spring system are designed to be set below the desired band of isolation, at about one to a few Hz. In addition to the rigid body modes, one must insure that the internal mode frequencies of the blade springs are high enough (and/or have enough damping) that they do not compromise isolation performance and so that thermal excitation does not exceed the displacement noise requirements[2;3]. The minimum internal mode frequency, calculated from one-dimensional considerations[4] is about 40 Hz; As a design goal the internal modes should be greater than 100 Hz, otherwise detailed modeling and/or measurement would be required to confirm acceptability.

Consider the quadruple pendulum suspension depicted in Figure 1. We wish to determine the bending mode frequencies of the cantilevered, blade springs. Much work has already been done to establish the internal blade spring modal frequencies. The finite element method has been used to establish[5,6] that:

- a simply supported blade tip boundary condition is a reasonable approximation for our typical parameters (i.e. that to a good approximation the first blade spring modal frequency is independent of the supported mass, the tip mass and the first rigid body bounce frequency)

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[1] N. Robertson, et. al., Advanced LIGO Suspension System Conceptual Design, 21 Oct 2003, LIGO-T010103-03

[2] N. Robertson, Seismic and Thermal Noise Peaks from Blade Internal Modes in an ETM/ITM Quadruple Pendulum, 24 Mar 2005, LIGO-T050046-01.

[3] N. Robertson, Evaluation of effect of blade internal modes on sensitivity of Advanced LIGO, T010174-00, 25 Oct 2001

[4] K. Strain, Estimate of the minimum frequency for resonances associated with blade-spring stages in quadruple pendulums for Advanced LIGO, 16 July 2003, ALUKGLA0007aJUL03

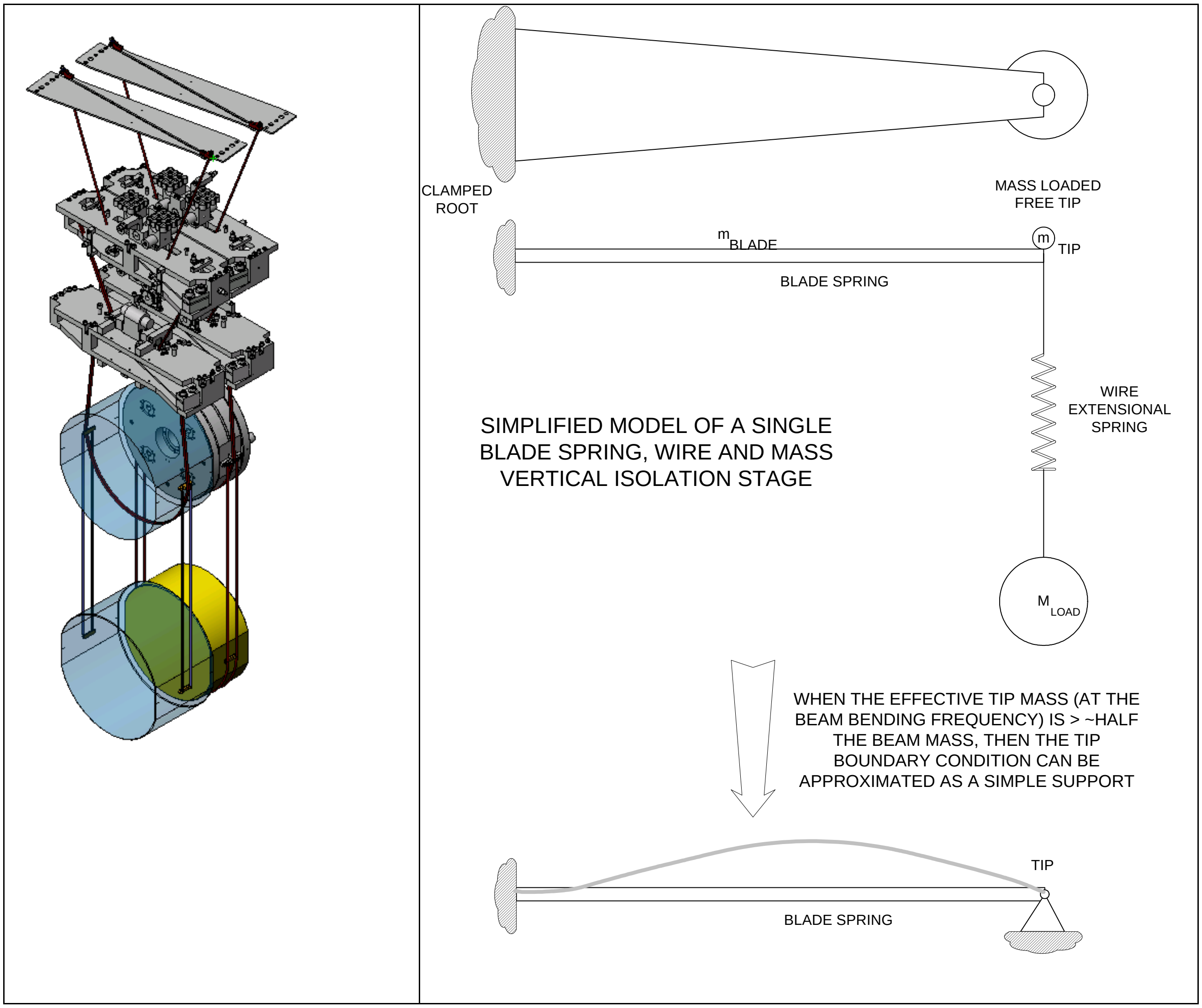
[5] J. Greenhalgh, Initial Exploration of Transmissibility by FEA of Blades, LIGO-T040024-00

[6] R. Jones, C. Torrie, Case Study: Using ANSYS to Predict the Lowest Flexural Internal Mode Frequency of the MC and RM Upper Blades, 3 Nov 2003, LIGO-T030273-04

- the first blade spring internal mode frequency is to a good approximation independent of the blade shape factor (or aspect ratio)

In the following, a theoretical formulation is presented which explicitly includes the shape factor and the effective tip mass as a function of frequency.

Figure 1: Quadruple Suspension



2 Boundary conditions

Each blade spring is rigidly clamped to its associated mass (or in the case of the top blade spring, to the upper structure). Each of the suspended masses (from 22 to 40 kg) far exceeds the mass of the blade springs (which have masses on the order of 0.5 kg) and so the clamped end (root) of the blade spring can be approximated as an idealized clamped boundary condition (zero vertical displacement and zero slope).

At the beam tip end it might seem at first that the large supported mass ensures a simply supported boundary condition (zero vertical displacement but no applied moment due to the flexibility of the wire). However, the first vertical-bounce mode of the suspension stages is set by design at about ~1 Hz. The ETM/ITM parameters are given in N. Robertson's paper[7], where the parameter definitions are given in M. Perreur-Lloyd's paper[8]. The value of the transmissibility (for motion at the blade root to motion at the supported mass) at ~100 Hz (the lowest intended frequency of the internal modes of the blade springs) is about 1%; See for example Figure 1 of T010174-00. The mass that the blade spring supports would seem then to be largely decoupled from the dynamics of the blade spring internal modes. However, from the formulation presented in the following section, we find that the first mode frequency is within ~10% of a simply supported end condition case if the effective tip mass is at least 30% of the blade mass. So the effective mass can be as little as ~0.1% of the supported mass before the simply supported boundary condition is no longer a good approximation.[9]

3 Theoretical modal frequencies

The blade spring designs used in GEO and adv. LIGO have constant thickness and a linearly tapered width. The geometry also generally includes a tip region with a deviation from a idealized trapezoidal geometry, as shown in Figure 2.

In the literature there are solutions for the elastic modal frequencies for a cantilevered beam with linearly tapered width, with[10] and without[11] a discrete of point tip mass (i.e. a mass with no rotary inertia). Results are also available for a linearly tapered beam width of constant thickness with a clamped root and simply supported end. In all three cases the frequency, f_i, for the i^{th} mode is calculated as follows:

$$f_i = \left(\frac{1}{2\pi}\right)\left(\frac{h}{l^2}\right)(lk_i)^2\sqrt{\frac{E}{12\rho}}$$

where

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[7] N. Robertson, Investigation of Wire Lengths in Advanced LIGO Quadruple Pendulum Design for ETM/ITM, 26 Jan 2005, LIGO-T040028-00.

[8] M. Perreur-Lloyd, Pendulum Parameter Descriptions and Naming Convention, 20 Jul 2004, LIGO-T040072-01

[9] The boundary conditions assumed in the suspension design reports to date are clamped (or approaching a clamp, or fixed support at the root) and a simple support at the tip, e.g. see:
J. Greenhalgh, Initial Exploration of Transmissibility by FEA of Blades, LIGO-T040024-00
R. Jones, C. Torrie, Case Study: Using ANSYS to predict the Lowest Flexural Internal Mode Frequency of the MC and RM Upper Blades, 3 Nov 2003, LIGO-T030273-04

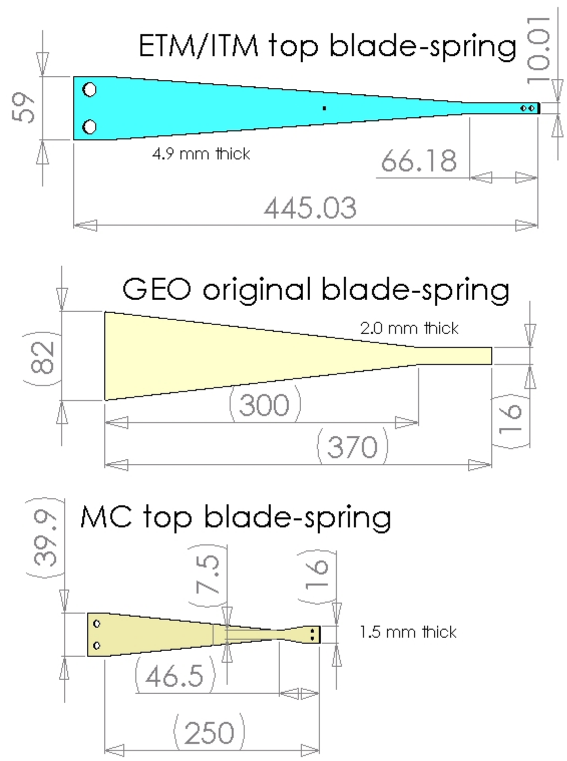
[10] H. Mabie, C. Rogers, Transverse Vibrations of Tapered Cantilever Beams with End Loads, J. Acoustical Soc. Am., v36, n3, Mar 1964, p 463-469.

[11] H. Mabie, C. Rogers, Transverse Vibrations of Double-Tapered Cantilever Beams, J. Acoustical Soc. Am., v51, n5, 1972.

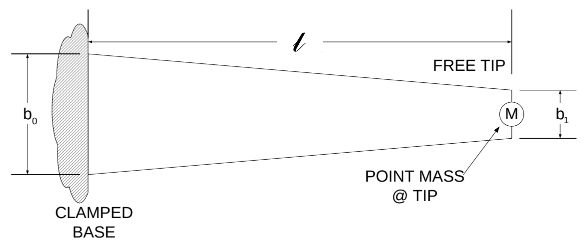
$(lk_i)^2$ is the numerical solution to the eigenvalue problem for the ith mode and is listed in tabular form as a function of the aspect ratio, $\beta = {}^{b_0}\!/\!_{b_1}$, thickness ratio $\alpha = {}^{h_0}\!/\!_{h_1}$, and the mass ratio, $R = M / m$, in the appendices.

Figure 2: Blade Spring Geometry (drawings have a common scale)

Blade Spring Example Designs



Model



M = mass of the tip point mass

m = mass of the blade spring

h = thickness of the blade-spring (h_0 is the thickness at the blade root, h_1 is the thickness at the blade tip; for LIGO suspension blades the thickness is constant, so $h_0 = h_1$)

ℓ = length of the blade-spring (taken as the distance from the clamp at the base to the point where the wire detaches from the clamp at the tip of the blade

E = elastic modulus of the blade spring material

ρ = mass density of the blade spring material

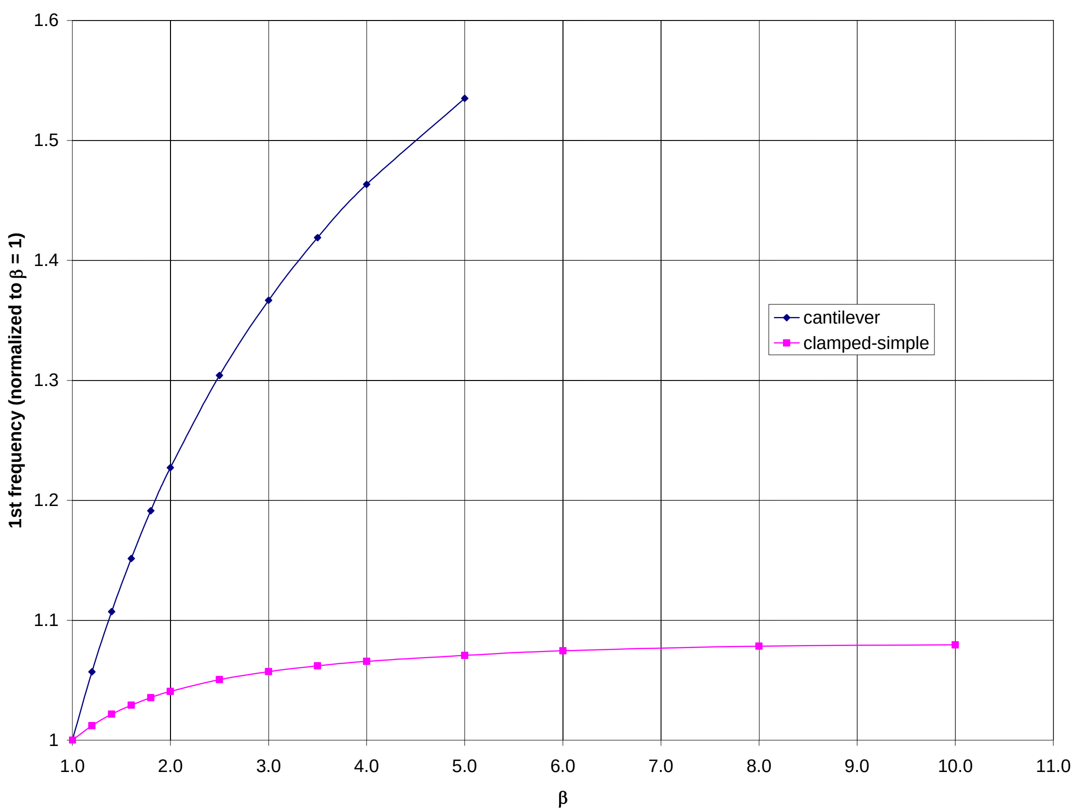
To date the suspension design has stuck to aspect ratios of the blade designs that are close to the ratio used in GEO (β = 5.125) and have scaled the first internal mode frequency[12] from the GEO blade by ℓ^2/h.

The effect on the first frequency of changing the aspect ratio of the blade spring is indicated in Figure 3. The two curves represent the limiting cases of:

- clamped root – free tip boundary condition: no effective tip mass (when the supported mass is very low compared to the blade mass or when the supported mass is decoupled due to a high internal mode frequency and a low vertical bounce mode), and
- clamped root – simply supported tip boundary condition: large effective tip mass (when the supported mass is high compared to the blade mass or when the internal blade frequency is low in comparison to the vertical bounce mode)

As noted by J. Greenhalgh (T040024) based on finite element analysis, there is very little variation in the first mode frequency with shape factor changes for the clamped root – simple support boundary condition.

Figure 3: Variation of 1st frequency with aspect ratio, β



The appendices A, B, and C give eigenvalues (coefficients of the frequency equation) for three boundary condition cases, all with clamped blade roots: (i) free tip, (ii) simply supported tip and

[12] For example, this is the formulation used in the current Matlab code for calculation of the suspension dynamics.

(iii) tip loaded with a discrete mass. When the tip mass is very small in comparison to the blade mass, the frequencies asymptote to the free tip boundary condition case. When the tip mass is very large in comparison to the blade mass, the frequencies asymptote to the simply supported boundary condition case.

4 Comparison with experimental results

The simply supported tip formulation above compares well with available experimental results[13,14] as indicated in Table 1. In each case the beam length used in the calculation is the length from the base clamp to the wire break-away from the wire clamp at the tip, and the tip width is taken as the minimum width of the blade.

For the upper blade of the Mode Cleaner triple suspension, the mass of the wire clamp at the tip was varied by adding mass, as indicated in Table 1. Note that the mass loaded tip formulation (appendix A) does not match these experimental results (for any reasonable assumed parameter values). It seems likely that the rotary inertia of the added tip mass is responsible for the observed decrease in first frequency, and not the added translational inertia.

Table 1: Comparison of theory to experiment

Blade Spring	Tip Mass (g)	Experiment (Hz)	Theory (Hz)	% Error
GEO Original design	0[⊗]	55	53.9	-2.0%
Mode Cleaner triple suspension upper blade (D020205)	0	90	88.4	-1.8%
	12	84	NA	
	17	82	NA	
	22	80	NA	
	32	77	NA	
ETM Top Blade	0		69.3	
ETM Middle Blade	0	95	96.6	1.7%
ETM Bottom Blade	0	111	113.7	2.4%

Unfortunately the eigenvalues listed in the literature (and the appendices) are limited to $\beta < 5$ and have rather coarse sampling for $R < 0.3$. To compare with these experimental results, the values in the literature were extrapolated and interpolated. A refined set of numerical solutions to extend the solution set to parameter ranges of interest to LIGO and to provide a more accurate estimation

[13] R. Jones, C. Torrie, Case Study: Using ANSYS to predict the Lowest Flexural Internal Mode Frequency of the MC and RM Upper Blades, 3 Nov 2003, LIGO-T030273-04

[14] SUS Team, Blade Characterization for the ETM Controls Prototype, LIGO-T040229-11

[⊗] The tip mass for the "original" GEO blade spring was not identified in the T030273-04 and is assumed here to be negligible.

would be of use. In the interim, the tabular results in the appendix or the eigenvalue coefficient fits in Table 2 can be used.

Table 2: Frequency Coefficient Fits

Tip Boundary Condition	1st Frequency Coefficient Fit, $c_1 = (\ell k_1)^2$
Simply Supported *(best fit with experiment)*	$\log\left[(lk_1)^2\right] = -0.0328\log(\beta)^2 + 0.0655\log(\beta) + 1.1883$ $\log\left[(lk_2)^2\right] = -0.009\log(\beta)^2 + 0.0195\log(\beta) + 1.6987$ $\log\left[(lk_3)^2\right] = -0.0029\log(\beta)^2 + 0.009\log(\beta) + 2.0181$
Tip Mass	for β = 5 and 0 < R < ~4: $(lk_2)^2 = 9.15e^{\left(-R^{0.67}/0.27\right)} + 16.507$
Free	for R = 0 and 1 < β < ~6: $(lk_1)^2 = 2.7215\log(\beta) + 3.5014$ $(lk_2)^2 = 5.1833\log(\beta) + 21.987$ $(lk_3)^2 = 6.7725\log(\beta) + 60.996$

Appendix A: Eigenvalues for Varying Tip Mass and Beam Aspect Ratio

TABLE II. Factor $[lK]^2$.

β	R	Fundamental frequency	Second harmonic	Third harmonic
1.0	0	3.5160	22.035	61.70
	0.2	2.6127	18.208	53.55
	0.4	2.1680	17.176	52.06
	0.6	1.8926	16.701	51.44
	0.8	1.7007	16.428	51.11
	1.0	1.5573	16.250	50.89
	2.0	1.1582	15.861	50.45
	3.0	0.9628	15.720	50.30
	4.0	0.8416	15.647	50.21
1.2	0	3.7168	22.415	62.06
	0.2	2.7202	18.348	53.58
	0.4	2.2440	17.312	52.13
	0.6	1.9527	16.844	51.54
	0.8	1.7517	16.576	51.21
	1.0	1.6017	16.403	51.02
	2.0	1.1879	16.026	50.59
	3.0	0.9862	15.891	50.45
	4.0	0.8616	15.822	50.37
1.4	0	3.8923	22.743	62.39
	0.2	2.8100	18.451	53.58
	0.4	2.3061	17.414	52.17
	0.6	2.0017	16.951	51.60
	0.8	1.7924	16.690	51.29
	1.0	1.6374	16.522	51.11
	2.0	1.2113	16.157	50.71
	3.0	1.0048	16.026	50.57
	4.0	0.8772	15.961	50.50
1.6	0	4.0485	23.030	62.68
	0.2	2.8863	18.530	53.57
	0.4	2.3581	17.492	52.20
	0.6	2.0418	17.035	51.64
	0.8	1.8260	16.779	51.35
	1.0	1.6664	16.614	51.17
	2.0	1.2301	16.259	50.79
	3.0	1.0197	16.134	50.67
	4.0	0.8900	16.070	50.59
1.8	0	4.1873	23.286	62.95
	0.2	2.9519	18.590	53.55
	0.4	2.4019	17.552	52.22
	0.6	2.0756	17.102	51.68
	0.8	1.8540	16.849	51.41
	1.0	1.6908	16.693	51.22
	2.0	1.2457	16.344	50.87
	3.0	1.0318	16.222	50.74
	4.0	0.9002	16.159	50.68
2.0	0	4.3152	23.520	63.20
	0.2	3.0088	18.636	53.54
	0.4	2.4395	17.601	52.23
	0.6	2.1045	17.155	51.71
	0.8	1.8777	16.908	51.44
	1.0	1.7111	16.750	51.28
	2.0	1.2589	16.414	50.92
	3.0	1.0422	16.294	50.81
	4.0	0.9090	16.233	50.74
2.5	0	4.5852	24.021	63.74
	0.2	3.1244	18.715	53.48
	0.4	2.5141	17.685	52.24
	0.6	2.1609	17.252	51.77
	0.8	1.9238	17.012	51.52
	1.0	1.7506	16.862	51.35
	2.0	1.2837	16.542	51.04
	3.0	1.0615	16.429	50.92
	4.0	0.9252	16.372	50.87
3.0	0	4.8057	24.441	64.24
	0.2	3.2120	18.760	53.44
	0.4	2.5690	17.735	52.24
	0.6	2.2020	17.312	51.80
	0.8	1.9569	17.082	51.57
	1.0	1.7790	16.937	51.42
	2.0	1.3014	16.630	51.12
	3.0	1.0752	16.522	51.02
	4.0	0.9367	16.467	50.97
3.5	0	4.9894	24.802	64.66
	0.2	3.2808	18.786	53.38
	0.4	2.6115	17.768	52.24
	0.6	2.2329	17.354	51.81
	0.8	1.9822	17.128	51.60
	1.0	1.8002	16.988	51.45
	2.0	1.3145	16.691	51.18
	3.0	1.0787	16.589	51.08
	4.0	0.9452	16.536	51.04
4.0	0	5.1456	25.119	65.06
	0.2	3.3365	18.800	53.33
	0.4	2.6452	17.788	52.23
	0.6	2.2575	17.381	51.83
	0.8	2.0017	17.162	51.62
	1.0	1.8168	17.025	51.48
	2.0	1.3243	16.737	51.22
	3.0	1.0929	16.637	51.12
	4.0	0.9516	16.586	51.08
5.0	0	5.3977	25.655	65.74
	0.2	3.4210	18.810	53.25
	0.4	2.6948	17.810	52.22
	0.6	2.2934	17.415	51.84
	0.8	2.0301	17.203	51.64
	1.0	1.8406	17.072	51.52
	2.0	1.3386	16.798	51.28
	3.0	1.1038	16.703	51.19
	4.0	0.9606	16.655	51.15

Table from *H. Mabie, C. Rogers, Transverse Vibrations of Tapered Cantilever Beams with End Loads, J. Acoustical Soc. Am., v36, n3, Mar 1964, p 463-469.*

Appendix B: Eigenvalues for Cantilevered Beam Aspect Ratio and Thickness Ratio Variation

Table from *H. Mabie, C. Rogers, Transverse Vibrations of Double-Tapered Cantilever Beams, J. Acoustical Soc. Am., v51, n5, 1972.*

N.B.: For LIGO SUS, α = 1 (constant thickness)

TABLE I. Factor $(lk)^2$ $\alpha = h_0/h_1$, $\beta = b_0/b_1$.

	β	Fundamental frequency	Second harmonic	Third harmonic	Fourth harmonic	Fifth harmonic
$\alpha=1.0$	1.0	3.5160	22.035	61.70	120.9	199.8
	1.2	3.7168	22.415	62.06	121.2	200.2
	1.4	3.8927	22.742	62.39	121.6	200.5
	1.6	4.0485	23.029	62.68	121.9	200.8
	1.8	4.1886	23.286	62.95	122.1	201.1
	2.0	4.3152	23.519	63.20	122.4	201.4
	2.5	4.5852	24.021	63.74	123.0	202.0
	3.0	4.8057	24.440	64.24	123.5	202.5
	3.5	4.9894	24.802	64.67	124.0	203.0
	4.0	5.1456	25.123	65.06	124.4	203.5
	5.0	5.3977	25.656	67.54	125.2	204.4
$\alpha=1.2$	1.0	4.3089	25.032	68.56	133.5	220.2
	1.2	4.5501	25.466	68.99	133.9	220.6
	1.4	4.7607	25.837	69.36	134.3	221.0
	1.6	4.9484	26.163	69.71	134.7	221.4
	1.8	5.1162	26.455	70.01	135.0	221.7
	2.0	5.2684	26.719	70.29	135.3	222.0
	2.5	5.5928	27.289	70.91	136.0	222.7
	3.0	5.8574	27.767	71.47	136.5	223.3
	3.5	6.0787	28.179	71.96	137.1	223.9
	4.0	6.2665	28.540	72.40	137.6	224.4
	5.0	6.5695	29.155	73.19	138.5	225.4
$\alpha=1.4$	1.0	5.1207	27.982	75.22	145.7	239.8
	1.2	5.4019	28.463	75.72	146.2	240.3
	1.4	5.6482	28.878	76.14	146.7	240.8
	1.6	5.8666	29.241	76.53	147.1	241.2
	1.8	6.0629	29.566	76.88	147.4	241.6
	2.0	6.2400	29.862	77.19	147.8	241.9
	2.5	6.6193	30.500	77.90	148.5	242.7
	3.0	6.9285	31.036	78.52	149.2	243.4
	3.5	7.1867	31.498	79.07	149.8	244.0
	4.0	7.4066	31.904	79.57	150.3	244.6
	5.0	7.7612	32.594	80.42	151.3	245.7
$\alpha=1.6$	1.0	5.9492	30.891	81.76	157.7	259.0
	1.2	6.2705	31.422	82.32	159.2	259.5
	1.4	6.5521	31.880	82.79	158.7	260.0
	1.6	6.8022	32.281	83.21	159.2	260.5
	1.8	7.0262	32.641	83.59	159.6	260.9
	2.0	7.2291	32.966	83.96	159.9	261.3
	2.5	7.6624	33.672	84.73	160.8	262.1
	3.0	8.0163	34.265	85.41	161.5	262.9
	3.5	8.3123	34.776	86.03	162.2	263.6
	4.0	8.5632	35.227	86.56	162.8	264.2
	5.0	8.9694	35.992	87.52	163.9	265.4
$\alpha=1.8$	1.0	6.7912	33.772	88.19	169.4	277.6
	1.2	7.1535	34.352	88.79	170.0	278.3
	1.4	7.4704	34.851	89.32	170.5	278.8
	1.6	7.7518	35.291	89.79	171.0	279.3
	1.8	8.0044	35.684	90.21	171.5	279.8
	2.0	8.2329	36.041	90.59	171.9	280.2
	2.5	8.7208	36.813	91.45	172.8	281.1
	3.0	9.1192	37.462	92.20	173.6	282.0
	3.5	9.4526	38.023	92.85	174.3	282.7
	4.0	9.7356	38.517	93.45	174.9	283.4
	5.0	10.1927	39.358	94.50	176.1	284.7
$\alpha=2.0$	1.0	7.6469	36.632	94.52	180.8	295.9
	1.2	8.0497	37.260	95.18	181.5	296.6
	1.4	8.4019	37.803	95.75	182.1	297.2
	1.6	8.7149	38.278	96.26	182.6	297.8
	1.8	8.9958	38.705	96.73	183.1	298.3
	2.0	9.2495	39.092	97.14	183.6	298.7
$\alpha=2.0$	2.5	9.7925	39.931	98.09	184.6	299.7
	3.0	10.2355	40.637	98.88	185.4	300.6
	3.5	10.6061	41.247	99.60	186.2	301.5
	4.0	10.9210	41.785	100.24	186.9	302.2
	5.0	11.4291	42.700	101.36	188.2	303.6
$\alpha=2.5$	1.0	9.8345	43.713	110.04	208.8	340.4
	1.2	10.3388	44.462	110.84	209.7	341.3
	1.4	10.7807	45.109	111.53	210.4	342.0
	1.6	11.1723	45.679	112.15	211.0	342.7
	1.8	11.5240	46.188	112.70	211.6	343.3
	2.0	11.8419	46.652	113.21	212.1	343.8
	2.5	12.5217	47.656	114.34	213.3	345.0
	3.0	13.0762	48.501	115.28	214.3	346.1
	3.5	13.5402	49.234	116.14	215.3	347.1
	4.0	13.9345	49.880	116.90	216.1	348.0
	5.0	14.5710	50.982	118.24	217.6	349.5
$\alpha=3.0$	1.0	12.0798	50.740	125.26	236.1	383.6
	1.2	12.6871	51.608	126.18	237.0	384.6
	1.4	13.2183	52.358	126.99	237.9	385.4
	1.6	13.6893	53.019	127.71	238.6	386.2
	1.8	14.1120	53.612	128.37	239.3	386.9
	2.0	14.4940	54.150	128.96	239.9	387.5
	2.5	15.3100	55.319	130.26	241.3	388.9
	3.0	15.9768	56.306	131.35	242.5	390.2
	3.5	16.5340	57.160	132.34	243.5	391.3
	4.0	17.0074	57.914	133.24	244.5	392.3
	5.0	17.7721	59.201	134.79	246.2	394.1
$\alpha=3.5$	1.0	14.3724	57.739	140.23	262.7	425.7
	1.2	15.0823	58.725	141.30	263.8	426.8
	1.4	15.7030	59.577	142.23	264.8	427.8
	1.6	16.2538	60.329	143.04	265.6	428.7
	1.8	16.7477	61.004	143.78	266.4	429.5
	2.0	17.1943	61.618	144.46	267.1	430.2
	2.5	18.1476	62.952	145.93	268.7	431.8
	3.0	18.9251	64.075	147.19	270.0	433.2
	3.5	19.5753	65.052	148.30	271.2	434.5
	4.0	20.1287	65.913	149.33	272.3	435.6
	5.0	21.0204	67.388	151.07	274.3	437.7
$\alpha=4.0$	1.0	16.7028	64.717	155.05	289.0	467.0
	1.2	17.5167	65.826	156.25	290.2	468.3
	1.4	18.2278	66.782	157.28	291.3	469.4
	1.6	18.8582	67.626	158.21	292.3	470.4
	1.8	19.4234	68.383	159.04	293.1	471.3
	2.0	19.9344	69.071	159.79	293.9	472.1
	2.5	21.0250	70.568	161.44	295.7	473.9
	3.0	21.9146	71.832	162.87	297.2	475.4
	3.5	22.6576	72.074	164.12	298.5	476.8
	4.0	23.2893	73.900	165.23	299.7	478.1
	5.0	24.3079	75.560	167.21	301.9	480.4
$\alpha=5.0$	1.0	21.4573	78.682	184.33	340.5	547.9
	1.2	22.4790	80.026	185.80	342.1	549.5
	1.4	23.3714	81.186	187.06	343.4	550.8
	1.6	24.1621	82.212	188.18	344.5	552.0
	1.8	24.8702	83.132	189.17	345.6	553.1
	2.0	25.5096	83.973	190.11	346.5	554.1
	2.5	26.8749	85.794	192.10	348.7	556.2
	3.0	27.9873	87.336	193.82	350.5	558.1
	3.5	28.9164	88.678	195.33	352.1	559.8
	4.0	29.7047	89.865	196.70	353.6	561.3
	5.0	30.9770	91.895	199.09	356.1	564.1

Appendix C: Eigenvalues for Cantilevered Beam with a Simply Supported End and Varying Aspect Ratio

Table from *H. Mabie, C. Rogers, Transverse Vibrations of Tapered Cantilever Beams with End Support, J. Acoustical Soc. Am., v44, n6, 1968.*

TABLE II. Factor $(lK)^2$.

β	Fundamental frequency	Second harmonic	Third harmonic
1.0	15.417	49.964	104.24
1.2	15.604	50.139	104.43
1.4	15.751	50.275	104.55
1.6	15.867	50.386	104.67
1.8	15.963	50.477	104.78
2.0	16.044	50.555	104.86
2.5	16.195	50.702	105.00
3.0	16.299	50.807	105.12
3.5	16.374	50.884	105.23
4.0	16.431	50.944	105.29
5.0	16.507	51.024	105.39

Appendix D: Parameters for Blade Spring Internal Mode Frequency Calculations

Blade Spring Properties:

N/m^2	E	modulus of elasticity	1.86E+11
--	nu	Poisson's ratio	0.3
kg/m^3	rho	density	7800
--	Q	damping	1.00E+04

Wire Properties:

N/m^2	Ew	modulus of elasticity	2.00E+11
--	nuw	Poisson's ratio	0.25
kg/m^3	rhow	density	7800

									ETM/ITM Controls Prototype		
			GEO "original" blade	MC Upper Blade					Top	Middle	Bottom
Units	**Symbol**	**Description**	aka "Reference Blade"	D020205-A					D040298	D040297	D040296
m	h	blade uniform thickness	0.002	0.0015					0.00429	0.0046	0.0042
m	lb	blade length	0.37	0.25					0.475	0.415	0.365
m	lt	wire clamp length	0.001	0.001					0.0058	0.0057	0.005
m	l	total blade length	0.371	0.251					0.4808	0.4207	0.37
m	a	blade root width	0.082	0.039878					0.095	0.059	0.049
m	b	blade tip width	0.016	0.007509					0.01	0.01	0.01
m		blade tip length	0.07	0.046455					0.0455	0.066	0.0705
kg	M	tip mass	0	0	0.012	0.017	0.022	0.032	0	0	0
--	beta	ratio of tip to root widths	0.195121951	0.18829931					0.105263158	0.169491525	0.204081633
--	alpha	shape factor	1.318011395	1.32268722					1.386483723	1.335966788	1.311978582
m	l^2/h	length^2/thickness	68.8205	42.0006667					53.88546387	38.4757587	32.5952381
kg	m	blade mass	0.2835924	0.0695807					0.844647804	0.520767702	0.3575754
	beta	ratio of root to tip widths	5.13	5.31					9.50	5.90	4.90
	alpha	ratio of root to tip thickness	1	1					1	1	1
Experimental results for frequency measurements:											
hz	f1	1st internal frequency	**55**	**90**	**84**	**82**	**80**	**77**		**95**	**111**
				86							
Case 1: Beam with linear taper in width, constant thickness, clamped root, simply supported tip											
	(lk1)^2	1st mode eigenvalue	16.5297	16.5405					16.6336	16.5692	16.5152
	(lk2)^2	2nd mode eigenvalue	51.0512	51.0632					51.1873	51.0956	51.0354
	(lk3)^2	3rd mode eigenvalue	105.4452	105.4633					105.7127	105.5151	105.4218
hz	f1	1st mode frequency	**53.9**	**88.4**					**69.3**	**96.6**	**113.7**
hz	f2	2nd mode frequency	166.4	272.8					213.1	297.9	351.3
hz	f3	3rd mode frequency	343.8	563.4					440.1	615.3	725.6
		1st mode frequency error	-2.0%	-1.8%						1.7%	2.4%