# StackSlide and Hough Search SNR and Statistics

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Abstract

The StackSlide Search and the Hough Search are two methods under development for finding periodic gravitational waves. This technical document derives the basic relationships that define the signal-to-noise ratio and statistics for these searches. It compares the sensitivities of these searches by finding and comparing the characteristic amplitude that can be detected for a given false alarm and false dismisal rate for each. We find that in theory, at least for one case, that the StackSlide search may be somewhat more sensitive that the Hough search, justifying the development of both of these search methods.

## I. Introduction

The StackSlide Search and the Hough Search are incoherent methods for finding periodic gravitational waves. In both methods the initial step is to divide the time-domain data into science-mode segments with a time-baseline of less than $40$ minutes ($30$ minute segments are typical) and each segment undergoes a Discrete Fourier Transform (DFT). The output of this step are called Short-time-baseline Fourier Transforms, or SFTs, and the time-baseline of each SFTs is denoted $T_{\rm SFT}$. For $T_{\rm SFT} < 40$ minutes the power of a periodic signal from an isolated source is expected to be found approximately at one frequency (though not necessarily centered on an SFT frequency bin of course). The searches are template based, were a template corresponds to the signal frequency at the Solar System Barycenter (SSB) for some reference time, and to one sky position and one set of spindown parameters. In the StackSlide Search the power in each SFT is stacked up, slid to correct for doppler shifts and spindown according to each template, and summed. The search looks for peaks in power above a threshold. In the Hough Search a cutoff on power is first applied to the SFTs; power above the cutoff is replaced by a 1, power below the cutoff is replaced by a 0. The Hough Search produces a number count (the number of SFTs with power above the cutoff) for each template, and looks for peaks in number count above a threshold. In the future these methods will be applied to the $\mathcal{F}$-statistic defined in Jaranowski, Królak and Schutz (JKS) [2]; the analysis will be basically the same as given here, except the number of degrees of freedom will double.

The overall goal of this document is to derive expressions for the characteristic amplitude, $h_0$, of a detectable signal for given false alarm and false dismisal rates, in terms of the power spectral density of the noise, $S_n$, the number of SFTs, $M$, and the observation time, $T_o$. For example, for a coherent search using the $\mathcal{F}$-statistic the characteristic amplitude for a $1\%$ false alarm rate and $10\%$ false dismisal rate is given by $h_0 = 11.4\sqrt{S_n/T_o}$, as given in reference [1]. Here we find similar expressions for the StackSlide and Hough searches. We find that in theory, at least for one case, that the StackSlide search may be somewhat more sensitive that the Hough search, justifying the development of both of these search methods.

## II. Optimal signal-to-noise ratio

In this section we work out expressions for what is called the optimal signal-to-noise ratio (optimal SNR).

First note that the strain of a periodic signal at the detector is given by

$$h(t) = F_{+}(t) A_{+} \cos\Phi(t) + F_{\times}(t) A_{\times} \sin\Phi(t), \tag{1}$$

where $F_{+}$ and $F_{\times}$ are the usual beam pattern response functions, $A_{+}$ and $A_{\times}$ are the amplitudes of the gravitational wave for the plus and cross polarizations, and $\Phi$ is the phase. The phase contains modulations from doppler shifts due to the relative motion between the source and the detector and the frequency evolution of the source.

During one SFT, if we Taylor expand the phase out to first order about the time at the midpoint of the SFT we can write

$$\Phi(t) \approx \Phi_{1/2} + 2\pi f_{1/2}(t - t_{1/2}). \tag{2}$$

Approximating $F_{+}$ and $F_{\times}$ as constants, the strain at discrete time $t_j$, where $j$ is the discrete time index, measured from the start of the SFT, can thus be approximated as

$$h_j \approx F_{+1/2} A_{+} \cos[\Phi_{1/2} + 2\pi f_{1/2}(t_0 + t_j - t_{1/2})] + F_{\times 1/2} A_{\times} \sin[\Phi_{1/2} + 2\pi f_{1/2}(t_0 + t_j - t_{1/2})], \tag{3}$$

where $t_0$ is the time as the start of the SFT, and $t_{1/2} - t_0 = T_{\rm sft}/2$. This simplifies to

$$h_j \approx F_{+1/2} A_{+} \cos(\Phi_0 + 2\pi f_{1/2} t_j) + F_{\times 1/2} A_{\times} \sin(\Phi_0 + 2\pi f_{1/2} t_j), \tag{4}$$

where in this equation $\Phi_0$ is the approximate phase at the start of the SFT (not the initial phase at the start of the observation), i.e.,

$$\Phi_0 = \Phi_{1/2} - 2\pi f_{1/2}(T_{\rm sft}/2). \tag{5}$$

Using these approximations, the signal can be treated as the sum of pure sinusoids during the time of one SFT.

For the purposes of this document the optimal SNR for this signal for one SFT will be define by

$$d^2_{\rm SFT} \equiv \frac{2}{S_n(f_{1/2})} \int_0^{T_{\rm SFT}} |h(t)|^2 dt \cong \frac{2}{S_n(f_k)} \sum_{j=0}^{N-1} |h_j|^2 \Delta t, \tag{6}$$

where $\Delta t$ is one over the sample rate, $N$ is the number of time samples in one SFT, $S_n$ is the one-side power spectral density of the noise, $f_{1/2}$ is the signal frequency at the midpoint of the SFT, assumed to be stationary to within $1/T_{\rm SFT}$ Hz during $T_{\rm SFT}$, $k$ is the index of the frequency bin with the signal power [in pseudo code $k = {\rm floor}(f_{1/2} * T_{\rm SFT} + 0.5)$], and $f_k = k/T_{\rm SFT}$.

Note that Parseval's Theorem can be used to relate Eq.(6) to the DFT of the signal

$$\sum_{j=0}^{N-1} |h_j|^2 \equiv \frac{1}{N} \sum_{k'=0}^{N-1} |\tilde{h}_{k'}|^2, \tag{7}$$

where the DFT is given by

$$\tilde{h}_{k'} = \sum_{j=0}^{N-1} h_j e^{-2\pi i j k'/N}. \tag{8}$$

Thus, the normalized time-integrated square amplitude of a signal can be found from the signal DFT by

$$d^2_{\rm SFT} \cong \frac{2\Delta t^2}{S_n(f_k)T_{\rm SFT}}\sum_{k'=0}^{N-1}|\tilde{h}_{k'}|^2, \tag{9}$$

where $N = T_{\rm SFT}/\Delta t$ has been used.

Ignoring leakage, which we deal with below, the power is confined to bin k and aliased to bin N-k (the Nyquist frequency is bin N/2). Thus, only these terms contribute to the sum and Eq. (9) reduces to

$$d^2_{\rm SFT} \cong \frac{4\Delta t^2}{T_{\rm SFT}}\frac{|\tilde{h}_k|^2}{S_n(f_k)}. \tag{10}$$

Comparing this with JKS Eqs. (39), (41), and (79) we see that $d^2_{\rm SFT}$ is the same as what JKS also call the optimal signal-to-noise ratio, for one SFT. Thus we can use the equations in JKS to relate $d^2_{\rm SFT}$ to the physical parameters that describe signal. For example, it then follows from JKS Eq. (84) that $d^2_{\rm SFT}$ averaged over SFTs is

$$< d^2_{\rm SFT} >\cong [A_+^2 < F^2_{+1/2}/S_n > +A_\times^2 < F^2_{\times 1/2}/S_n >]T_{\rm SFT}. \tag{11}$$

(Unless noted otherwise, $<>$ will indicate an average over SFTs throughout this document.)

However, it is easy to calculate Eq. (11) directly, including the effects of leakage by taking the DFT of $h_j$ given in Eq. (4) above. This gives:

$$\begin{aligned}\tilde{h}_k = & e^{i\Phi_0}\frac{(F_{+1/2}A_+ - iF_{\times 1/2}A_\times)}{2}\frac{1-e^{2\pi i(\kappa-k)}}{1-e^{2\pi i(\kappa-k)/N}} \\ & + e^{-i\Phi_0}\frac{(F_{+1/2}A_+ + iF_{\times 1/2}A_\times)}{2}\frac{1-e^{-2\pi i(\kappa+k)}}{1-e^{-2\pi i(\kappa+k)/N}},\end{aligned} \tag{12}$$

where $\kappa$ is defined by

$$\kappa \equiv f_{1/2}T_{\rm sft}, \tag{13}$$

and is usually not an integer. For $0 < \kappa < N/2$ and $|\kappa - k| << N$ the first term dominates and can be Taylor expanded to give:

$$\tilde{h}_k = Ne^{i\Phi_0}\frac{(F_{+1/2}A_+ - iF_{\times 1/2}A_\times)}{2}\left[\frac{\sin(2\pi\Delta\kappa)}{2\pi\Delta\kappa} + i\frac{1-\cos(2\pi\Delta\kappa)}{2\pi\Delta\kappa}\right], \tag{14}$$

where $\Delta\kappa \equiv \kappa - k$. Taking the absolute square of this equation, and substituting into Eq. (10) gives:

$$< d^2_{\rm SFT} >\cong \left[A_+^2\left\langle\frac{F^2_{+1/2}}{S_n}\frac{\sin^2(\pi\Delta\kappa)}{\pi^2\Delta\kappa^2}\right\rangle + A_\times^2\left\langle\frac{F^2_{\times 1/2}}{S_n}\frac{\sin^2(\pi\Delta\kappa)}{\pi^2\Delta\kappa^2}\right\rangle\right]T_{\rm SFT}, \tag{15}$$

where the relevant range for $\Delta\kappa$ is $0$ to $0.5$, corresponding to a frequency mismatch of $0$ to $1/2$ of an SFT bin.

For gravitational waves emitted from a rotating triaxial ellipsoid, $A_+$ and $A_\times$ are given by

$$A_+ = \frac{1}{2}h_0(1+\cos^2\iota), \qquad A_\times = h_0\cos\iota, \tag{16}$$

where $\iota$ is the inclination angle between the source spin axis and the direction from the source to the SSB. This equation thus serves as a definition of $h_0$ [though see also JKS Eqs. (20)–(24) and the surrounding text for a further physical interpretation of $h_0$].

As stated in the introduction, the overall goal of this document is to derive expressions for the characteristic amplitude, $h_0$, of a detectable signal. Thus, it will be useful to estimate, for an average case, the expected value of $h_0$ in terms of $< d^2_{\rm SFT} >$. The value of $F^2_+$ and the value of $F^2_\times$ averaged over all sky positions is $1/5$. If we take $\cos\iota$ to be uniformly distributed between $-1$ and $1$ then the average value of $A^2_+$ is $7h_0^2/15$ and the average value of $A^2_\times$ is $h_0^2/3$. Ignoring leakage, for these average values, and a typical for value for $S_n$, Eq. (11) reduces to

$$h_0 = 2.5\sqrt{< d^2_{\rm SFT} > S_n/T_{\rm SFT}}. \tag{17}$$

(A better estimate that factors in the nonstationarity of the noise would be to replace $S_n$ with $< 1/S_n >^{-1}$.) Later in this document the statistics for the StackSlide and Hough searches will be used to estimate an upper limit on $< d^2_{\rm SFT} >$ for given false alarm and false dismisal rates, thus giving an estimated upper limit on $h_0$, which is what we call the characteristic amplitude. Leakage is ignored in these results to make it easier to compare with those given elsewhere. It would be easy to include the effect of leakage using Eq (15), which shows, worst case, leakage increases the characteristic amplitude by a factor of $1.57$.

## III. StackSlide power and signal-to-noise ratio

Let the discrete time samples of the data from the detector consist of a signal plus noise: $h_j + n_j$. An unnormalized SFT of the data is found by applying Eq.(8) which results in $\tilde{h}_{k'} + \tilde{n}_{k'}$. The Sum StackSlide Power, $\varrho$, will be defined in this document as the sum of the power in each SFT at the signal frequency (adjusted for doppler shift and spindown according to a template) and normalized as follows

$$\varrho = \sum_{\rm SFTs} \frac{4\Delta t^2}{S_n T_{\rm SFT}} |\tilde{h} + \tilde{n}|^2. \tag{18}$$

(From here on we suppress the frequency index, and understand that $\tilde{h}$ and $\tilde{n}$ are the values from the appropriate bin of each SFT for a given template.) We will assume that the noise is gaussian in a narrow band around the signal in each SFT (but not that the noise is stationary from SFT to SFT). The cross term between $\tilde{h}$ and $\tilde{n}$ then goes like the cosine of the uniformly distributed random phase difference between the signal and the noise in each SFT. Thus this term averages to zero as long as the noise is not grossly nonstationary from one SFT to the next, or at least it should not dominate the sum. Furthermore, we can replace the sum of $|\tilde{h}|^2/S_n$ and $|\tilde{n}|^2/S_n$ as $M$ times their average values, where $M$ is the number of SFTs. The Sum StackSlide Power becomes:

$$\varrho = \frac{4\Delta t^2}{T_{\rm SFT}} \left[ M < \frac{|\tilde{h}|^2}{S_n} > + M < \frac{|\tilde{n}|^2}{S_n} > \right]. \tag{19}$$

Note that on average

$$< \frac{S_n}{|\tilde{n}|^2} > \cong \frac{2\Delta t^2}{T_{\rm SFT}}. \tag{20}$$

Using this equation and Eq. (10) the Sum StackSlide Power is expected to be:

$$\varrho = 2M + M < d^2_{\mathrm{SFT}} > . \tag{21}$$

Brady and Creighton [3] use a slightly different normalization. If we define $\rho_n \equiv \varrho/2$, which is what Brady and Creighton call $\rho/S_n$ in their Eq. (3.1), the expected value of $\rho_n$ is

$$\rho_n = M + \frac{1}{2} M < d^2_{\mathrm{SFT}} > . \tag{22}$$

(Note that Brady and Creighton use $\rho$ for the unnormalized sum of the power. This is not used anywhere in this document.) The StackSlide code outputs what we call the Mean StackSlide Power, defined as $P \equiv \varrho/2M$; thus the expected value of $P$ is

$$P = 1 + \frac{1}{2} < d^2_{\mathrm{SFT}} > . \tag{23}$$

The first term on the right of Eqs. (21)–(23) represents the mean power in the noise for each normalization. The standard deviation of the noise, for one SFT, will equal its mean value (the statistics discussed in the next section will verify this). The standard deviation of the noise will be reduced by a factor of $\sqrt{M}$ after averging, and independent of which normalization is adopted the ratio of the power to the standard deviation of the noise power in all cases gives an expected StackSlide Power Signal-To-Noise Ratio of

$$SNR = (1 + \frac{1}{2} < d^2_{\mathrm{SFT}} >)\sqrt{M}. \tag{24}$$

# IV. StackSlide statistics

The Sum StackSlide Power given by Eq. (18) is the sum of the power from the appropriate bin (which depends on the template) from each SFT:

$$\varrho = \varrho_1 + \varrho_2 + \varrho_3 + ... + \varrho_M. \tag{25}$$

To illustrate the derivation of the statistics, in this section we will consider the case $M = 1$ and then use the above equation to generalize to arbitrary M. (The Hough statistics also makes use of the $M = 1$ case.) The cases of noise only and signal plus noise are considered respectively.

For $M = 1$ and the case of noise only,

$$\varrho = \frac{4\Delta t^2}{T_{\mathrm{SFT}}} \frac{|\tilde{n}|^2}{S_n}. \tag{26}$$

Define $x$ and $y$ as the real and imaginary parts of $\tilde{n}$, normalized such that:

$$x = \frac{2\Delta t}{\sqrt{S_n T_{\mathrm{SFT}}}} \mathrm{Re}(\tilde{n}). \tag{27}$$

$$y = \frac{2\Delta t}{\sqrt{S_n T_{\mathrm{SFT}}}} \mathrm{Im}(\tilde{n}). \tag{28}$$

Thus, $\varrho = x^2 + y^2$. For gaussian noise the mean values of $x$ and $y$ are zero, and the normalization is choosen so that the mean values of $x^2$ and $y^2$ are one. This can be seen from Eq. (21), which shows that the expected value of $\varrho = 2$ for $M = 1$ and $d^2_{\rm SFT} = 0$, and noting that $\bar{x^2} = \bar{y^2}$ when only noise is present since the phase of the noise is uniformly distributed. Thus the variances are: $\sigma^2_x = \bar{x^2} - (\bar{x})^2 = 1$; $\sigma^2_y = \bar{y^2} - (\bar{y})^2 = 1$.

Thus note that $\varrho$ is the sum of the squares of two guassian distributed variables with zero mean and unit variance, which is precisely the definition of a $\chi^2$ variable with two degrees of freedom. Thus $\varrho$ will follow the distribution of such a variable. However, for illustrative purposes, we will derive this. Note that the probability that $x$ and $y$ fall within $dx$ and $dy$ of their measured values is the product of the gaussian probability (with $\sigma_x = \sigma_y = 1$) for each:

$$\mathcal{P}(x,y)dxdy = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dxdy. \tag{29}$$

Making the substitutions $x = r\cos\theta$, $y = r\sin\theta$, $dxdy = rdrd\theta$,

$$\mathcal{P}(r,\theta)drd\theta = \frac{1}{2\pi} r e^{-r^2/2} drd\theta. \tag{30}$$

Thus $r^2 = x^2 + y^2 = \varrho$, or $r = \sqrt{\varrho}$ which is what is called the noise amplitude, for one SFT. To find the probability of getting a noise amplitude $r$ for any phase $\theta$, just integrate over all possible values of $\theta$ (i.e., from $0$ to $2\pi$). This results in the **Rayleigh Distribution** for the noise amplitude $r$:

$$\mathcal{P}(r)dr = re^{-r^2/2}dr. \tag{31}$$

Finally, substituting $\varrho = r^2$, $d\varrho = 2rdr$, gives

$$\mathcal{P}(\varrho)d\varrho = \frac{1}{2} e^{-\varrho/2} d\varrho. \tag{32}$$

This is the $\chi^2$ **Distribution for 2 Degrees of Freedom**.

From Eq. (25) it can be seen that in general $\varrho$ is a $\chi^2$ variable with $2M$ degrees of freedom. Thus, the derivation generalizes to give the $\boldsymbol{\chi^2}$ **Distribution for 2M Degrees of Freedom** for the distribution for $\varrho$:

$$\mathcal{P}(\varrho; M)d\varrho = \frac{1}{2^M \Gamma(M)} \varrho^{M-1} e^{-\varrho/2} d\varrho. \tag{33}$$

To compare with Brady and Creighton [3] make the substitution $\rho_n = \varrho/2$ and $d\rho_n = d\varrho/2$, and note that since $M$ is an integer that $\Gamma(M) = (M-1)!$; thus

$$\mathcal{P}(\rho_n; M)d\rho_n = \frac{1}{(M-1)!} \rho_n^{M-1} e^{-\rho_n} d\rho_n, \tag{34}$$

which is the integrand for the **Incomplete Gamma Function** as given in Brady and Creighton Eq (3.1). [Again note that $\rho_n$ is what Brady and Creighton call $\rho/S_n$ in their Eq. (3.1).]

Given a false alarm rate, $f_a$, a cutoff in Sum Stack Slide Power, $\varrho_c$, can be defined such that $f_a$ is the chance of finding power in a bin above $\varrho_c$ due to noise alone:

$$f_a = \int_{\varrho_c}^{\infty} \mathcal{P}(\varrho; M)d\varrho. \tag{35}$$

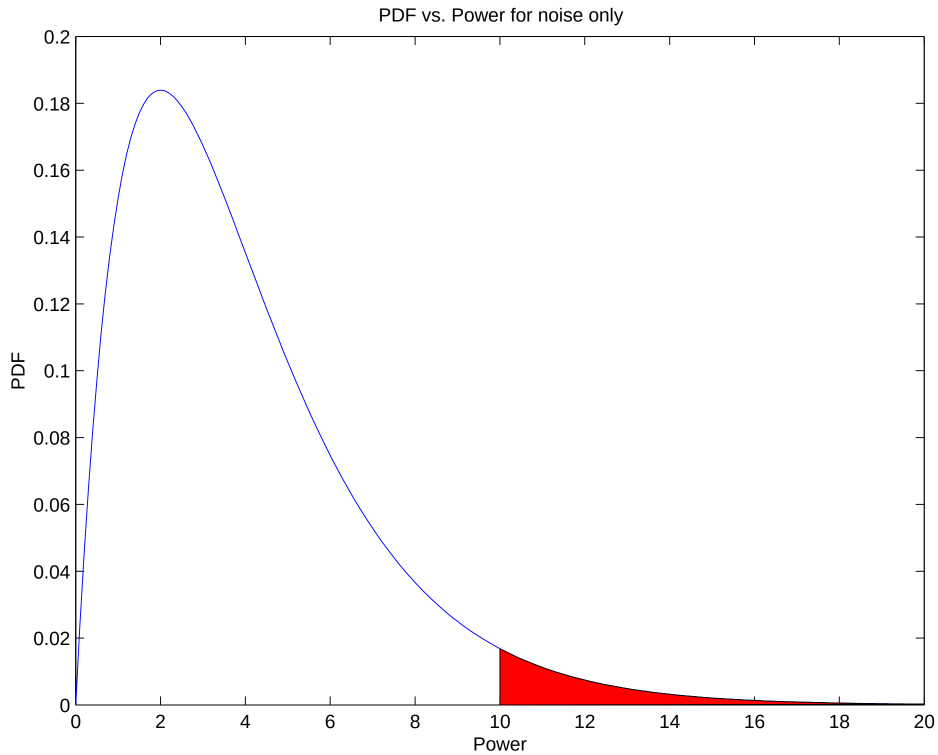


Figure 1: An example Probability Density Function vs. StackSlide Power for noise only. A cutoff of $\varrho_c = 10$ is shown. The area in red is the false alarm rate, $f_a$, for this cutoff.

Note that if $f_a$ is one over the number of templates and bins searched (i.e., the number of trials), then only one event above $\varrho_c$ would be expected in all those trials. In other words, in this case the loudest event would be expected to be probably not less than $\varrho_c$, though probably not much more than this.

Now consider the second main case consider in this section: $M = 1$ and a signal plus noise. The Sum StackSlide Power is

$$\varrho = \frac{4\Delta t^2}{T_{\rm SFT}} \frac{|\tilde{h} + \tilde{n}|^2}{S_n}. \tag{36}$$

Define $x$ and $y$ as the real and imaginary parts of $\tilde{h} + \tilde{n}$, normalized such that:

$$x = \frac{2\Delta t}{\sqrt{S_n T_{\rm SFT}}} {\rm Re}(\tilde{h} + \tilde{n}). \tag{37}$$

$$y = \frac{2\Delta t}{\sqrt{S_n T_{\rm SFT}}} {\rm Im}(\tilde{h} + \tilde{n}). \tag{38}$$

Thus, $\varrho = x^2 + y^2$, as before. Due to the presense of a signal, their mean values are now $\bar{x} = d_{\rm SFT} \cos\phi$ and $\bar{y} = d_{\rm SFT} \sin\phi$ respectively [see Eq. (10)], where $\phi$ is a phase associated with the signal. The variance in $x$ and $y$ still just comes from the noise, and thus we still have $\sigma_x = \sigma_y = 1$. (This can also be seen from Eq. (21) again, which shows that the expected value of $\varrho = 2 + d^2_{\rm SFT}$ for $M = 1$. Thus the mean values of $x^2$ and $y^2$ must be $1 + d^2_{\rm SFT} \cos^2\phi$ and $1 + d^2_{\rm SFT} \sin^2\phi$ respectively. Thus the variances are: $\sigma_x^2 = \bar{x^2} - (\bar{x})^2 = 1$; $\sigma_y^2 = \bar{y^2} - (\bar{y})^2 = 1$.)

As before, $\varrho$ is the sum of the squares of two guassian distributed variables with unit variance, but with nonzero mean, which is precisely the definition of a noncentral $\chi^2$ variable with two degrees of freedom. Thus $\varrho$ will follow the distribution of such a variable. However, for illustrative purposes again, we will derive this. Note that the probability that $x$ and $y$ fall within $dx$ and $dy$ of their measured values is the product of the gaussian probability (with $\sigma_x = \sigma_y = 1$) for each:

$$\mathcal{P}(x, y)dxdy = \frac{1}{\sqrt{2\pi}} e^{-(x-\bar{x})^2/2} \frac{1}{\sqrt{2\pi}} e^{-(y-\bar{y})^2/2} dxdy. \tag{39}$$

Making the substitutions $x = r\cos\theta$, $y = r\sin\theta$, $dxdy = rdrd\theta$,

$$\mathcal{P}(r, \theta)drd\theta = r\left[\frac{1}{2\pi} e^{-rd_{\mathrm{SFT}}\cos(\theta-\phi)} d\theta\right] e^{-(r^2+d^2_{\mathrm{SFT}})/2} dr. \tag{40}$$

The variable $r = \sqrt{\varrho}$ is the amplitude of the signal in the presence of noise, for one SFT. To find the probability of getting the amplitude $r$ for any phase $\theta$, just integrate the factor in square brackets over all possible values of $\theta$, i.e., from $0$ to $2\pi$. By symmetry this is also twice the integral from $0$ to $\pi$ independent of the signal phase $\phi$, and thus this integral results in the Modified Bessel Function $I_0(rd_{\mathrm{SFT}})$, e.g., see Abramowitz and Stegun Eq. (9.6.16) [4]. The distribution for the signal amplitude $r$ in the presense of noise is the **Rice Distribution**

$$\mathcal{P}(r)dr = rI_0(rd_{\mathrm{SFT}})e^{-(r^2+d^2_{\mathrm{SFT}})/2} dr. \tag{41}$$

Finally, substituting $\varrho = r^2$, $d\varrho = 2rdr$, gives

$$\mathcal{P}(\varrho)d\varrho = \frac{1}{2} I_0\left(\sqrt{\varrho d^2_{\mathrm{SFT}}}\right) e^{-(\varrho+d^2_{\mathrm{SFT}})/2} d\varrho. \tag{42}$$

This is the $\chi^2$ distribution for $2$ degrees of freedom with noncentrality parameter $d^2_{\mathrm{SFT}}$.

To generalization to $2M$ degrees of freedom, note that the noncentrality parameter after summing the power from $M$ SFTs, will on average be M $< d^2_{\mathrm{SFT}} >$. The distribution for $\varrho$ will be a **Noncentral $\chi^2$ Distribution for 2M Degrees of Freedom**:

$$\mathcal{P}(\varrho; M, < d^2_{\mathrm{SFT}} >)d\varrho \propto \left(\frac{\varrho}{M < d^2_{\mathrm{SFT}} >}\right)^{\frac{M-1}{2}} I_{M-1}\left(\sqrt{\varrho M < d^2_{\mathrm{SFT}} >}\right) e^{-(\varrho+M<d^2_{\mathrm{SFT}}>)/2} d\varrho. \tag{43}$$

Note that the right side of this equation has not been normalized.

Given a false dismissal rate, $f_d$, and the cutoff in Sum Stack Slide Power, $\varrho_c$, below which signals are rejected as likely due to noise [which determines the false alarm rate, see Eq. (35) above] there is a value $< d^2_{\mathrm{SFT}} >$ such that there is $f_d$ chance that the signal will be dismissed. This can be found by solving:

$$f_d = \int_0^{\varrho_c} \mathcal{P}(\varrho; M, < d^2_{\mathrm{SFT}} >)d\varrho. \tag{44}$$

Thus, Eqs. (35) and (44) can be used to find the minimum SNR that can be detected using the StackSlide search for fixed false alarm and false dismissal rates.

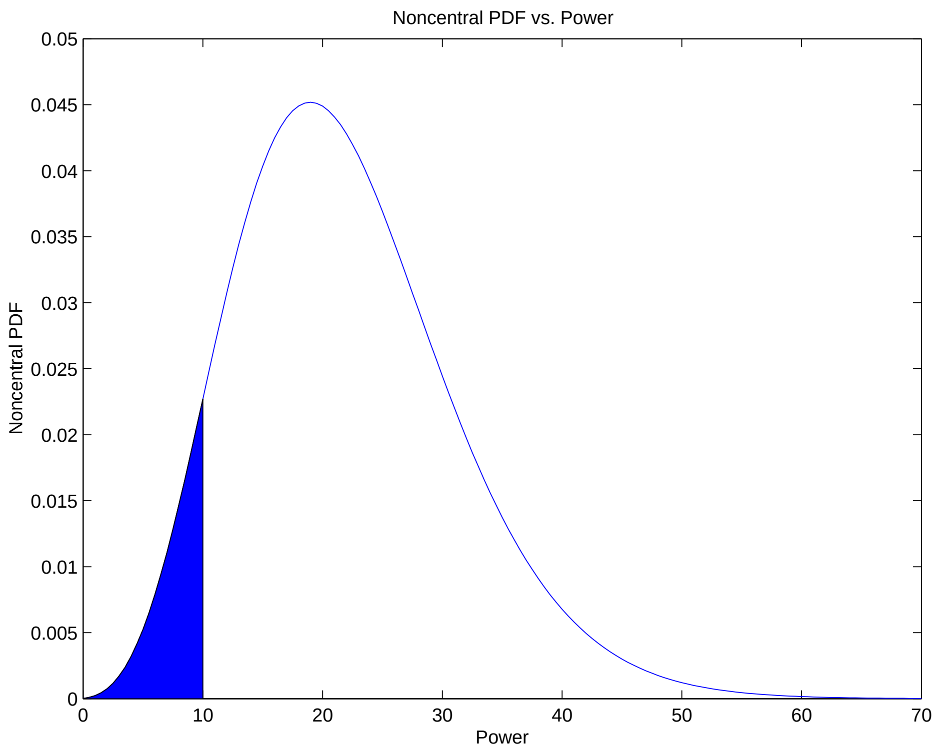


Figure 2: An example Noncentral Probability Density Function vs. StackSlide Power. A cutoff of $\varrho_c = 10$ is shown. The area in blue is the false dismissal rate, $f_d$, for this cutoff and a noncentrality parameter $M < d^2_{\mathrm{SFT}} >$.

# V. Hough number count and statistics

To understand the Hough number count and statistics, first consider setting an arbitrary cutoff $\varrho_c$ on the power in one SFT, and setting the power to $1$ in each frequency bin of each SFT if it is above this cutoff or to $0$ if it is below this cutoff. The number of $1$'s that occur in the $M$ SFTs from the bins corresponding to a template gives the number count for that template. Thus, the number count is an integer between $0$ and $M$.

Later we will adjust $\rho_c$ to minimize the SNR that can be detected for fixed false alarm and false dismissal rates (i.e. maximizing the sensitivity of the search). The SNR for a given $\rho_c$ is found in the four steps.

First define $\eta$ to be the probability that an SFT frequency bin will have a 1 after applying the $\varrho_c$ cutoff procedure. This is given by integrating the PDF in Eq. (32)

$$\eta = \int_{\varrho_c}^{\infty} \frac{1}{2} e^{-\varrho/2} d\varrho = e^{-\varrho_c/2}. \tag{45}$$

The situation is analogous to that shown in Fig. 1, except the area in red is $\eta$.

Second, since $\eta$ is the probability of getting a $1$ and $1-\eta$ is the probability of getting a $0$, the probability of getting a number count $n$ for $M$ SFTs is the **Binomial Distribution**

$$\mathcal{P}(n; M) = \frac{M!}{n!(M-n)!} \eta^n (1-\eta)^{M-n}. \tag{46}$$

If we set a false alarm rate, $f_a$, and define a cutoff on number count, $n_c$, such that there is an $f_a$ chance of finding a number count greater than or equal to $n_c$ due to noise alone, then $n_c$ is found by solving

$$f_a = \sum_{n=n_c}^{M} \frac{M!}{n!(M-n)!} \eta^n (1-\eta)^{M-n}. \tag{47}$$

The situation is analogous to that shown in Fig. 1, the area in red is $f_a$, but this is a discrete distribution and the horizontal axis would the number count from $0$ to $M$ with a cutoff at $n_c$.

Third, when the signal is present we define $\zeta$ as the probability of getting a $1$ and $1-\zeta$ as the probability of getting a $0$. We set the false dismissal rate, $f_d$, such that when the signal is present there is an $f_d$ chance the number count will be less than $n_c$ and thus dismissed. This condition determines $\zeta$ for fixed $f_d$:

$$f_d = \sum_{n=0}^{n_c-1} \frac{M!}{n!(M-n)!} \zeta^n (1-\zeta)^{M-n}. \tag{48}$$

The situation is analogous to that shown in Fig. 2, the area in blue is $f_d$, but this is a discrete distribution and the horizontal axis would be the number count from $0$ to $M$ with a cutoff at $n_c - 1$.

Fourth, given $\zeta$ from the third step, the average value of $< d^2_{\mathrm{SFT}} >$ that would be necesary to result in probability $\zeta$ is found by solving the integral of Eq. (42) such that

$$\zeta = \int_{\varrho_c}^{\infty} \frac{1}{2} I_0 \left( \sqrt{\varrho d^2_{\mathrm{SFT}}} \right) e^{-(r^2 + <d^2_{\mathrm{SFT}})>/2} d\varrho. \tag{49}$$

Note that $\varrho_c$ is the same cutoff as in Eq. (45). The situation is analogous to that shown in Fig. 1, except the area in red is $\zeta$.

These for steps gives the value of $< d^2_{\rm SFT} >$ such that the signal will be found, for given values of $f_a$, $f_d$, and $\varrho_c$. The value of $\varrho_c$ can be adjusted to minimize $< d^2_{\rm SFT} >$ for fixed $f_a$ and $f_d$. This give the minimum SNR that can be detected for fixed false alarm and false dismissal rates.

## VI. Comparison between StackSlide and Hough

Equations (35) and (44) can be used to find the minimum SNR that can be detected using the StackSlide search for fixed false alarm and false dismissal rates.

Equations (45)–(49) can be used to find the minimum SNR that can be detected using the Hough search for fixed false alarm and false dismissal rates, the SNR is further minimized by adjusting the value of $\varrho_c$ keeping $f_a$ and $f_d$ fixed.

For a $1\%$ false alarm rate, $10\%$ false dismissal rate, and $M = 1887$ we find that $< d^2_{\rm SFT} >= .17\sqrt{1887/M}$ for the StackSlide Search, and that $< d^2_{\rm SFT} >= .21\sqrt{1887/M}$ for the Hough Search. (The scaling with $1/\sqrt{M}$ needs to be confirmed.) Thus, using Eq. (17) we find for the StackSlide Search

$$h_0 = 6.8 M^{1/4}\sqrt{S_n/T_o}, \tag{50}$$

and for the Hough Search

$$h_0 = 7.6 M^{1/4}\sqrt{S_n/T_o}. \tag{51}$$

Note that $T_o = M T_{\rm SFT}$ is the science mode observation time corresponding to $M$ SFTs.

This shows the estimated StackSlide result to be $\sim 12\%$ better than the estimated Hough result. Furthermore, we find that if $f_a$ is set to one over the total number of templates (including the number of frequencies searched), so that $\varrho_c$ is an estimate of the loudest event, that the estimated StackSlide result remains $\sim 10 - 15\%$ better than the estimated Hough result.

Note that a $10 - 15\%$ increase in sensitivity could increase the number of potential sources. However, the absolute sensitives of each search method and a detailed model of the distribution of sources (i.e., in the Milky Way and Halo) would have to be considered. Also, the Hough code is currently several times faster than the StackSlide code (though a better comparison is needed using the same hardware, the same optimization, and thresholds that trigger the event handling loop in each code the same number of times). Thus, for fixed computational power, there could be an interesting trade-off the StackSlide and Hough methods. Each method may be optimal under different circumstances. We are only beginning to test this with fake and real data.

## APPENDIX: Matlab code

The following Matlab code was used to find the numerical results:

```
function d2 = FindStackSlideULonSNR(M,falseArate,falseDrate)
% Usage: d2 = FindStackSlideULonSNR(M,falseArate,falseDrate)
% M = num of SFTs; falseArate = false alarm rate, falseDrate = false dismisa

rhozero=chi2inv(1-falseArate,2*M);
disp('Stack slide power cutoff =')
disp(rhozero/(2.0*M))
```

```
empty = [];
delta = fzero('diffcdfncx2',rhozero,empty,rhozero,2*M,falseDrate);
d2 = delta/M;
return;

function d2 = FindHoughULonSNR(rhozero,M,falseArate,falseDrate)
% Usage: d2 = FindHoughULonSNR(rhozero,M,falseArate,falseDrate)
% rhozero = cutoff on Power that turns SFT power into a one or a zero
% M = num of SFTs; falseArate = false alarm rate, falseDrate = false dismisa
% d2 = "d-squared" = optimial JKS signal to noise ratio in one SFT
eta = exp(-rhozero/2.0);
nzero = binoinv(1-falseArate,M,eta)+1;
empty = [];
zeta = fzero('diffcdfbino',nzero/M,empty,nzero-1,M,falseDrate);
d2 = fzero('diffcdfncx2',rhozero,empty,rhozero,2,1-zeta);
return;


function y = diffcdfncx2(delta,rhozero,nu,falseDrate)
% Usage: y = diffcdfncx2(delta,rhozero,nu,falseDrate)
% delta is non-centrality parameter
% rhozero is power cutoff
% nu = number of degrees of freedom
% falseDrate desired false dismisal rate (e.g., 5%)
if (delta < 0.0)
   y = 1 + abs(delta);
else
   y = ncx2cdf(rhozero,nu,delta) - falseDrate;
end
return;

function y = diffcdfbino(zeta,nzero,M,falseDrate)
% Usage: y = diffcdfbino(zeta,nzero,M,falseDrate)
% zeta = probability of getting one on any given trial when signal present
% nzero = cutoff in number count
% M = number of SFTs
% falseDrate desired false dismisal rate (e.g., 5%)
if (zeta < 0.0)
   y = 1 + abs(zeta);
else
   y = binocdf(nzero,M,zeta) - falseDrate;
end
return;
```

# References

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