LIGO Note : T040062-00-E
April 7, 2004
Hiro Yamamoto and Osamu Miyakawa / Caltech LIGO

Calculation of fields in a Dual Recycled Michelson Cavity using linear approximation

Hiro Yamamoto : hiro@ligo.caltech.edu
Osamu Miyakawa : miyakawa_o@ligo.caltech.edu
LIGO Project / California Institute of Technology

■ Revision history

■ Version 1.0 : April 7, 2004 : Initial release

Fabry-Perot is used to illustrate the basic idea. That part is copied from a separate note, and mathematics part for FP summation should be merged with the mathematics part at the end. Also, PRM calculation is appeneded without any consistency review.

Introduction

The time domain simulation of a compound optics system forming resonators uses a time step which is determined by cavity lengths of the system. Usually, the CPU time to simulate a given duration of physics process is proportional to the number of iterations, which, in turn, is inversely proportional to the simulation time step.

For a large scale interferometer, like LIGO and Virgo, the naive approach combining constituent optics becomes very slow because short cavities in the Michelson cavity determine the simulation time step. The time scale of interest is usually the trip time of laser through the long FP arm. The Michelson cavity is usually 100 to 1000 times shorter than the long arm, and the simulation becomes slower by this amount compared to the case that the simulation runs using the one trip time through the long arm as the simulation time step.

In order to overcome this problem, it was proposed to use an analytic calculation of the field evolution in the Michelson cavity over the longer time using an adiabatic approximation. This showed a good success to speed up the simulation of a system where different cavity sizes exist.

The original method used an adiabatic approximation, i.e., mirror positions and input fields do not change during the one time step. This revealed a limitation when the simulation was used to study a case when the frequency noise was important. Because the time dependence of the phase of a field is the frequency, this approach of using a constant input field during one time step can be used only for limited situations where derivatives of physics quantities can be neglected.

A solution to address this issue is to use a linear approximation of physics quantities, like input fields and mirror positions, during one time step. Then the lowest order effect due to the frequency change can be included in the calculation.

In this note, formulas are derived to calculate the field evolution in a Fabry Perot cavity and in a dual recycling Michelson cavity using a linear approximation. This result can be used to calculate the field evolution using a time step much longer than the time step determined by the cavity lengths in the Michelson cavity.

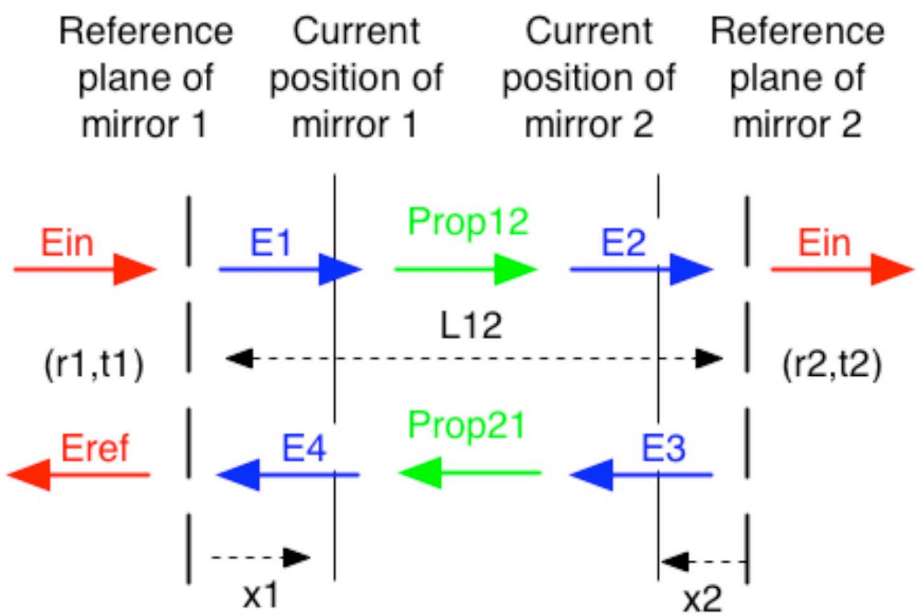
The full interferometer with long arms can be calculated by attaching propagators and end mirrors to this compound cavity. The bandwidth of length control systems of large interferometers is typically of the same order as the half spectral range of the long arm, and the calculation using the combination of compound Michelson cavity and end mirrors is an appropriate choice for most of the case.

Preparation (many similar names are used)

```
Off[General::"spell"];
Off[General::"spell1"];
```

■ Field evolution in Fabry-Perot cavity

■ Fig. 1 : Fabry-Perot cavity schematic view



■ basic formulation

In Fig.1, vertical dashed lines are the reference positions of two mirrors. All fields are calculated at these reference positions which do not move. Vertical solid lines are the actual positions of mirrors. Their positions are specified by the distances from their respective reference position.

$E_2(t)$ is the field at the mirror 2 reference plane at time t, and $E_3(t)$ is the reflected field at the same place at the same time. The reflection occurs on the actual mirror position, where the incident field is $E_2(t) \cdot \mathrm{Exp}[i\, k\, x_2(t)]$ and the reflected field is $-r_2 \cdot E_2(t) \cdot \mathrm{Exp}[\mathrm{ik} \cdot x_2(t)]$. $E_3(t)$ is defined on the reference plane, so $E_3(t) = -r_2\, E_2(t) \cdot \mathrm{Exp}[\mathrm{i2k} \cdot x_2(t)]$. The approximation here is that $E_2(t)$ and $x_2(t)$ does not change in the time scale of v x / c $\simeq 10^{-6}\ 10^{-6}\ / 10^{8} = 10^{-20}$.

$$E_1(t) = t_1\, E_{\mathrm{in}}(t) - r_1 \cdot E_4(t) \cdot \mathrm{Exp}[\, i\, 2\, k \cdot x_1(t)\,]$$
$$E_2(t) = E_1(t - \tau) \cdot \mathrm{Exp}[-\, i\, k \cdot L_0\,]$$
$$E_3(t) = -r_2 \cdot E_2(t) \cdot \mathrm{Exp}[\, i\, 2\, k \cdot x_2(t)\,]$$
$$E_4(t) = E_3(t - \tau) \cdot \mathrm{Exp}[-\, i\, k \cdot L_0\,]$$

$$\tau = \frac{L_0}{c}$$

$E_1(t) = t_1 E_{in}(t) - r_1 \cdot E_4(t) \cdot \text{Exp}[i\, 2\, k \cdot x_1(t)]$
$= t_1 E_{in}(t) - r_1 \cdot E_3(t-\tau) \cdot \text{Exp}[-i\, k \cdot L_0] \cdot \text{Exp}[i\, 2\, k \cdot x_1(t)]$
$= t_1 E_{in}(t) + r_1 r_2 \cdot E_2(t-\tau) \cdot \text{Exp}[i\, 2\, k \cdot x_2(t-\tau)] \cdot \text{Exp}[-i\, k \cdot L_0] \cdot \text{Exp}[i\, 2\, k \cdot x_1(t)]$
$= t_1 E_{in}(t) + r_1 r_2 \cdot E_1(t-2\tau) \cdot \text{Exp}[-i\, k \cdot L_0] \cdot \text{Exp}[i\, 2\, k \cdot x_2(t-\tau)] \cdot \text{Exp}[-i\, k \cdot L_0] \cdot \text{Exp}[i\, 2\, k \cdot x_1(t)]$

$E_1(t)$
$= t_1 E_{in}(t) + r_1 r_2 \cdot E_1(t-2\tau) \cdot \text{Exp}[-i\, 2 \cdot k \cdot (L_0 - x_2(t-\tau) - x_1(t))]$
$= t_1 E_{in}(t) + R \cdot E_1(t-2\tau) \cdot \text{Exp}[i\, \phi(t)]$

$\phi(\text{t}) = 2 \cdot k \cdot (x_1(t) + x_2(t-\tau))$
$\text{R} = r_1 \cdot r_2 \cdot \text{Exp}[-i\, 2\, k\, L_0]$

$E_1(t)$
$= t_1 E_{in}(t) + R \cdot \text{Exp}[i\, \phi(t)] \cdot E_1(t-2\tau)$
$= t_1 E_{in}(t) + R \cdot \text{Exp}[i\, \phi(t)] \cdot (t_1 E_{in}(t-2\tau) + R \cdot \text{Exp}[i\, \phi(t-2\tau)] \cdot E_1(t-4\tau))$
$= t_1 E_{in}(t) + R \cdot \text{Exp}[i\phi(t)] \cdot t_1 E_{in}(t-2\tau) + R^2\, \text{Exp}[i(\phi(t) + \phi(t-2\tau))] \cdot E_1(t-4\tau)$
$= t_1 E_{in}(t) + R \cdot \text{Exp}[i\phi(t)] \cdot t_1 E_{in}(t-2\tau) + R^2\, \text{Exp}[i(\phi(t) + \phi(t-2\tau))] \cdot (t_1 E_{in}(t-4\tau) + R \cdot \text{Exp}[i\, \phi(t-4\tau)] \cdot E_1(t-6\tau))$
$= t_1 E_{in}(t) + R \cdot \text{Exp}[i\phi(t)] \cdot t_1 E_{in}(t-2\tau) +$
$R^2 \cdot \text{Exp}[i(\phi(t) + \phi(t-2\tau))] \cdot t_1 E_{in}(t-4\tau) + R^3\, \text{Exp}[i(\phi(t) + \phi(t-2\tau) + \phi(t-4\tau))] \cdot E_1(t-6\tau)$

$$E_1(t) = t_1 E_{in}(t) + t_1 \cdot \sum_{n=1}^{N-1} R^n E_{in}(t - 2\,\text{n}\tau) \cdot \text{Exp}[i \sum_{m=0}^{n-1} \phi(t - 2\,\text{m}\tau)] + R^N\, \text{Exp}[i \sum_{m=0}^{N-1} \phi(t - 2\,\text{m}\tau)] \cdot E_1(t - 2\,\text{N}\tau)$$

$E_1(t + 2\,\text{N}\tau)$
$= t_1 E_{in}(t + 2\,\text{N}\tau) +$
$t_1 \cdot \sum_{n=1}^{N-1} R^n E_{in}(t + 2\,\text{N}\tau - 2\,\text{n}\tau) \cdot \text{Exp}[i \sum_{m=0}^{n-1} \phi(t + 2\,\text{N}\tau - 2\,\text{m}\tau)] + R^N\, \text{Exp}[i \sum_{m=0}^{N-1} \phi(t + 2\,\text{N}\tau - 2\,\text{m}\tau)] \cdot E_1(t)$

■ Summary of formulation

$E_1(t + 2\,\text{N}\tau)$
$= t_1 E_{in}(t + 2\,\text{N}\tau) +$
$t_1 \cdot \sum_{n=1}^{N-1} \{R^n E_{in}(t + 2\,\text{N}\tau - 2\,\text{n}\tau) \cdot \text{Exp}[i \sum_{m=0}^{n-1} \phi(t + 2\,\text{N}\tau - 2\,\text{m}\tau)]\} + R^N\, \text{Exp}[i \sum_{m=0}^{N-1} \phi(t + 2\,\text{N}\tau - 2\,\text{m}\tau)] \cdot E_1(t)$

$\phi(\text{T}) = 2 \cdot k \cdot (x_1(T) + x_2(T-\tau) - x_1(t) - x_2(t))$
$\text{R} = r_1 \cdot r_2 \cdot \text{Exp}[-i\, 2\, k\, (L_0 - x_1(t) - x_2(t))]$

■ Linear approximation

$x_j(t + \Delta) = x_j(t) + v_j \cdot \Delta$
$E_{in}(t + m\,\tau) = E_{in}(t)\, \text{Exp}[\beta \cdot m \cdot \tau]$

■ single summation

■ Calculation

$$\frac{1}{2k}\sum_{m=0}^{n-1}\phi(t+2\,\mathrm{N}\tau-2\,\mathrm{m}\tau) = \sum_{m=0}^{n-1}\{x_1(\ t+2\,\mathrm{N}\tau-2\,\mathrm{m}\tau)+x_2(\ t+2\,\mathrm{N}\tau-2\,\mathrm{m}\tau-\tau)\} =$$
$$\sum_{m=0}^{n-1}\left\{x_1(t)+\tau\cdot 2\,(N-m)\cdot v_1+x_2\,(t)+\tau\cdot(2\,N-2\,m-1)\cdot v_2\right\} =$$
$$n\,\{x_1(t)+x_2(t)+\tau\cdot(2\,N\cdot v_1+(2\,N-1)\cdot v_2)\}-2\,\tau(v_1+v_2)\,n(n-1)/2 =$$
$$n\,\{x_1(t)+x_2(t)+\tau\cdot(2\,N\cdot v_1+(2\,N-1)\cdot v_2)-\tau(v_1+v_2)\,(n-1)\}$$

```
FSUM1[n_] := n (x1 + x2 + τ (2 N v1 + (2 N - 1) v2) - τ (v1 + v2) (n - 1))
Simplify[FSUM1[N]]
```

$N\,((1+N)\,\tau\,v_1+N\,\tau\,v_2+x_1+x_2)$

$$\mathrm{N}(\ \tau\,v_1+x_1(t)+\mathrm{N}\ \tau\,v_1+x_2(t)+\mathrm{N}\ \tau\,v_2) = \mathrm{N}(\ \tau\,v_1+\frac{x_1(t)+x_1(t+2\,\mathrm{N}\tau)+x_2(t)+x_2(t+2\,\mathrm{N}\tau)}{2}\ ) \simeq N^2\,\tau(v_1+v_2)$$

```
FullSimplify[Series[FSUM1[n], {n, 0, 2}]]
```

$((\tau+2\,N\,\tau)\,v_1+2\,N\,\tau\,v_2+x_1+x_2)\,n-\tau\,(v_1+v_2)\,n^2+O[n]^3$

$$\sum_{m=0}^{n-1}\phi\,(t+2\,N\tau-2\,m\tau) = 2\,k\,(FS1\cdot n-FS2\cdot n^2) = 2\,k\cdot FS2\cdot n\,(2\,N-n)$$
$$FS1 = (\tau+2\,N\,\tau)\,v_1+2\,N\,\tau\,v_2 \simeq 2\,N\,(v_1+v_2)\,\tau = 2\,N\cdot FS2$$
$$FS2 = \tau\,(v_1+v_2)$$

■ Summary

$$\sum_{m=0}^{n-1}\phi\,(t+2\,N\tau-2\,m\tau) = 2\,k\cdot FS2\cdot n\,(2\,N-n)$$
$$FS2 = \tau\,(v_1+v_2)$$

■ double summation

■ calculation

$$DS[N] = \sum_{n=1}^{N-1} \left\{R^n E_{in}(t + 2N\tau - 2n\tau) \cdot Exp\left[i \sum_{m=0}^{n-1} \phi(t + 2N\tau - 2m\tau)\right]\right\}$$

$$= \sum_{n=1}^{N-1} \{R^n E_{in}(t) Exp[2(N-n)\beta\tau] \cdot Exp[i\, 2k \cdot FS2 \cdot n(2N-n)]\}$$

$$= E_{in}[t + 2N\tau] \sum_{n=1}^{N-1} \{R^n Exp[-n2\beta\tau] \cdot Exp[i\, 2k \cdot FS2 \cdot n(2N-n)]\}$$

$$= E_{in}[t + 2N\tau] \sum_{n=1}^{N-1} \{(R \cdot Exp[-2\beta\tau + i\, 2k \cdot FS2 \cdot 2N])^n \cdot Exp[-i\, 2k \cdot FS2 \cdot n^2]\}$$

$$= E_{in}[t + 2N\tau] \sum_{n=1}^{N-1} \left\{\tilde{R}^n \cdot Exp[-i\, 2k \cdot FS2 \cdot n^2]\right\}$$

$$\tilde{R} = R \cdot Exp[-2\beta\tau + i\, 4kN \cdot FS2]$$

$$2k\, FS2 = 4 \times \pi / \lambda \times \tau (v_1 + v_2) = 12 / 10^{-6} \times 10^{-7}\, 10^{-6} = 10^{-6} \sim 10 \cdot \tau$$

```
SP[a_, N_, M_] := FullSimplify[Sum[(a^n n^M), {n, 1, N-1}]]
```

$$SP[a\_, N\_, M\_] := FullSimplify\left[\sum_{n=1}^{N-1} (a^n n^M)\right]$$

```
SP[a, N, 0]
SP[a, N, 2]
SP[a, N, 4]
```

$$\frac{-a + a^N}{-1 + a}$$

$$\frac{-a(1+a) + a^N (a(1+a) - 2(-1+a)\, a N + (-1+a)^2 N^2)}{(-1+a)^3}$$

$$\frac{1}{(-1+a)^5} (-a(1+a)(1+a(10+a)) + a^N (a(1+a)(1+a(10+a)) - 4(-1+a)\, a(1+a(4+a)) N + 6(-1+a)^2 a(1+a) N^2 - 4(-1+a)^3 a N^3 + (-1+a)^4 N^4))$$

numerical values

```
SP[0.9, ∞, 0]
SP[0.9, ∞, 2]
SP[0.9, ∞, 4]
SP[0.9, ∞, 6]
SP[0.9, 100, 0]
SP[0.9, 100, 2]
SP[0.9, 100, 4]
SP[0.9, 100, 6]
SP[0.9, 50, 0]
SP[0.9, 50, 2]
SP[0.9, 50, 4]
SP[0.9, 50, 6]
```

9.

1710.

1.84851×10^{6}

4.9956×10^{9}

8.99973

1706.82

1.80909×10^{6}

4.48366×10^{9}

8.94846

1525.96

1.10257×10^{6}

1.34843×10^{9}

Summation expression of the field

calculation

$FS2 = \tau\,(v_1 + v_2)$

$R = r_1 \cdot r_2 \cdot \mathrm{Exp}[\,i\,2\,k\,(L_0 - x_1(t) - x_2(t))\,]$

$\tilde{R} = R \cdot \mathrm{Exp}[-2\,\beta\,\tau + i\,4\,kN \cdot FS2]$

$E_1\,(t + 2\,N\tau)$

$$= t_1\,E_{in}\,(t + 2\,N\tau) + t_1 \cdot \sum_{n=1}^{N-1} \left\{R^n\,E_{in}\,(t + 2\,N\tau - 2\,n\tau) \cdot \mathrm{Exp}\left[i \sum_{m=0}^{n-1} \phi\,(t + 2\,N\tau - 2\,m\tau)\right]\right\} +$$

$$R^N \operatorname{Exp}\left[i \sum_{m=0}^{N-1} \phi(t + 2N\tau - 2m\tau)\right] \cdot E_1(t)$$

$$= t_1 E_{in}(t + 2N\tau) + t_1 E_{in}[t + 2N\tau] \sum_{n=1}^{N-1} \left\{ \tilde{R}^n \cdot \operatorname{Exp}[-i\ 2k \cdot FS2 \cdot n^2] \right\} + R^N \operatorname{Exp}[i\, 2k \cdot FS2 \cdot N^2] \cdot E_1(t)$$

$$\simeq t_1 E_{in}(t + 2N\tau) \left\{ 1 + \frac{-\tilde{R} + \tilde{R}^N}{-1 + \tilde{R}} + SP2 \right\} + R^N \operatorname{Exp}[i\, 2k \cdot FS2 \cdot N^2] \cdot E_1(t)$$

$$= t_1 E_{in}(t + 2N\tau) \left\{ \frac{1 - \tilde{R}^N}{1 - \tilde{R}} - i\, 2k \cdot FS2 \cdot SP2 \right\} + R^N \operatorname{Exp}[i\, 2k \cdot FS2 \cdot N^2] \cdot E_1(t)$$

$$SP2 = \frac{\tilde{R}(1 + \tilde{R}) - \tilde{R}^N \left(\tilde{R}(1 + \tilde{R}) + 2(1 - \tilde{R})\tilde{R} N + (1 - \tilde{R})^2 N^2\right)}{(1 - \tilde{R})^3}$$

■ Summary

$$E_1(t + 2N\tau) = t_1 E_{in}(t + 2N\tau) \left\{ \frac{1 - \tilde{R}^N}{1 - R} - i\, 2k \cdot FS2 \cdot SP2 \right\} + R^N \operatorname{Exp}[i\, 2k \cdot FS2 \cdot N^2] \cdot E_1(t)$$

$$FS2 = \tau (v_1 + v_2)$$

$$R = r_1 \cdot r_2 \cdot \operatorname{Exp}[-i\, 2k (L_0 - x_1(t) - x_2(t))]$$

$$\tilde{R} = R \cdot \operatorname{Exp}[-2\beta\tau + i\, 4\, kN \cdot FS2]$$

$$SP2 = \frac{\tilde{R}(1 + \tilde{R}) - \tilde{R}^N \left(\tilde{R}(1 + \tilde{R}) + 2(1 - \tilde{R})\tilde{R} N + (1 - \tilde{R})^2 N^2\right)}{(1 - \tilde{R})^3}$$

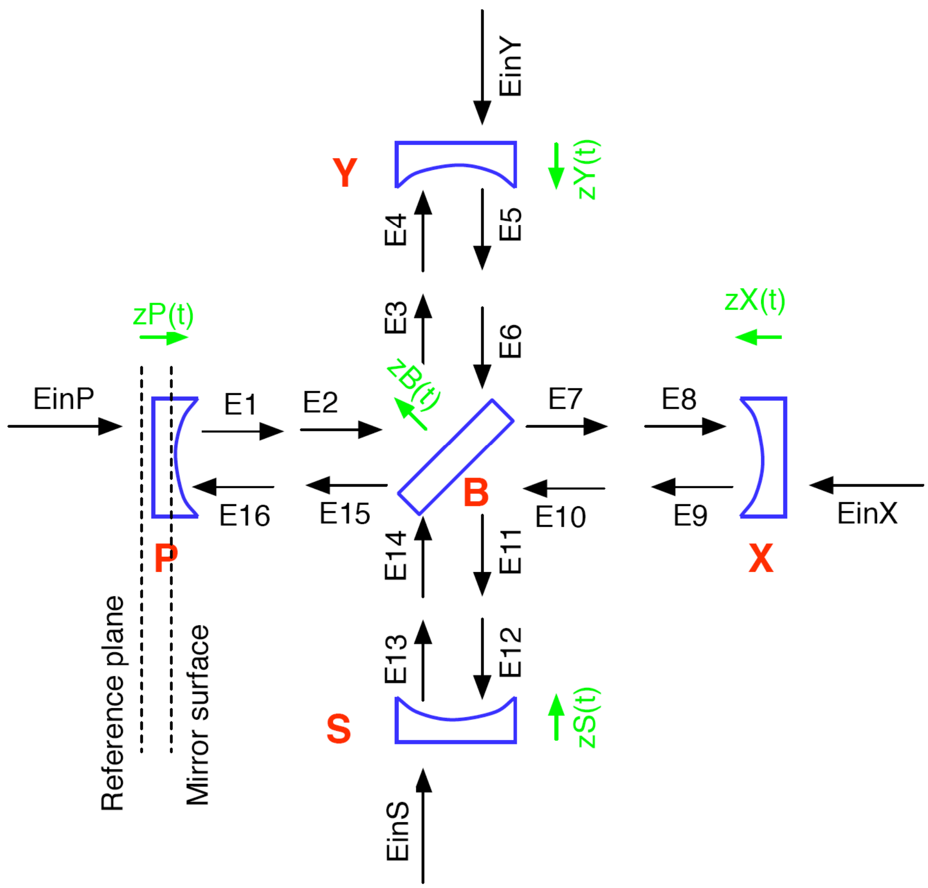
■ Reflected amd transmitted field

$$E_1(t) = t_1 E_{in}(t) - r_1 \cdot E_4(t) \cdot \operatorname{Exp}[i\, 2k \cdot x_1(t)]$$
$$E_2(t) = E_1(t - \tau) \cdot \operatorname{Exp}[-i\, k \cdot L_0]$$
$$E_3(t) = -r_2 \cdot E_2(t) \cdot \operatorname{Exp}[i\, 2k \cdot x_2(t)]$$
$$E_4(t) = E_3(t - \tau) \cdot \operatorname{Exp}[-i\, k \cdot L_0]$$

$$E_{tra}(t) = t_2 \cdot E_2(t) = t_2 \cdot E_1(t - \tau) \cdot \operatorname{Exp}[-ik \cdot L_0]$$
$$E_{ref}(t) = r_1 \cdot \operatorname{Exp}[-i\, 2k \cdot x_1(t)] \cdot E_{in}(t) + t_1 \cdot E_4(t)$$
$$= r_1 \cdot \operatorname{Exp}[-i\, 2k \cdot x_1(t)] \cdot E_{in}(t) + t_1 \cdot (t_1 \cdot E_{in}(t) - E_1(t)) / (r_1 \cdot \operatorname{Exp}[i\, 2k \cdot x_1(t)])$$
$$= E_{in}(t) \cdot \frac{r_1^2 + t_1^2}{r_1 \cdot \operatorname{Exp}[i2k \cdot x_1(t)]} - E_1(t) \cdot \frac{t_1}{r_1 \cdot \operatorname{Exp}[i\, 2k \cdot x_1(t)]}$$

■ Dual recycling Michelso cavity : Formulation

■ Fig.2 Schematic view of dual recycling Michelson cavity



■ Definitions

All fields are assumed to have a common constant angular frequency, ω, e.g., the carrier frequency. When incoming fields have different frequencies, like carrier and sideband components, fields with different frequencies are calculated separately. The fluctuation of frequency of fields can be simulated as the time dependence of phase of fields.

Fields at various locations in the dual recycling Michelson cavity (DRM) have following relations. En[t] is the amplitude of the field N at time t without the fast oscillating part, i.e., the full amplitude is Exp[$i\,\omega$ t] En[t].

E1lhs[t] and E13lhs[t] are there to stop recursion. rfM and rbM are amplitude reflectance from the front and back side of mirror M, and tfM and tbM are transmittance from respective sides. $\Pi_{A\leftarrow B}$ is the propagation phase from B to A. For a scalar field case, it is Exp[-$i\,\omega\,L_{\mathrm{MN}}$] with L_{MN} being the distance between mirror M and N. When the phase of the field going through a cavity is changing, the time derivative of the phase is properly included in the form effectively changing the frequency. τM is the trip time between mirror M and the beam splitter, i.e., τM = $L_{\mathrm{BS}-M}$ / speed of light.

Relations among fields at different locations

```
E1lhs[t_] := tbP[t] EinP[t] + rfP[t] E16[t];
E2[t_] := ΠB←P E1[t - τP];
E3[t_] := rfB[t] E2[t] + tbB[t] E14[t];
E4[t_] := ΠY←B E3[t - τY];
E5[t_] := rfY[t] E4[t] + tbY[t] EinY[t];
E6[t_] := ΠB←Y E5[t - τY];
E7[t_] := rbB[t] E14[t] + tfB[t] E2[t];
E8[t_] := ΠX←B E7[t - τX];
E9[t_] := rfX[t] E8[t] + tbX[t] EinX[t];
E10[t_] := ΠB←X E9[t - τX];
E11[t_] := rbB[t] E10[t] + tfB[t] E6[t];
E12[t_] := ΠS←B E11[t - τS];
E13lhs[t_] := rfS[t] E12[t] + tbS[t] EinS[t];
E14[t_] := ΠB←S E13[t - τS];
E15[t_] := rfB[t] E6[t] + tbB[t] E10[t];
E16[t_] := ΠP←B E15[t - τP];
```

Independent fields

When above equations are solved, they come to the following two independent equations.

```
E1lhs[t]
```

```
EinP[t] tbP[t] +
 rfP[t] ΠP←B (rfB[t - τP] ΠB←Y (rfY[t - τP - τY] ΠY←B (E1[t - 2 τP - 2 τY] rfB[t - τP - 2 τY] ΠB←P +
          E13[t - τP - τS - 2 τY] ΠB←S tbB[t - τP - 2 τY]) + EinY[t - τP - τY] tbY[t - τP - τY]) +
    ΠB←X tbB[t - τP] (EinX[t - τP - τX] tbX[t - τP - τX] + rfX[t - τP - τX] ΠX←B (E13[t -
             τP - τS - 2 τX] rbB[t - τP - 2 τX] ΠB←S + E1[t - 2 τP - 2 τX] ΠB←P tfB[t - τP - 2 τX])))
```

```
E13lhs[t]
```

```
EinS[t] tbS[t] +
 rfS[t] ΠS←B (ΠB←Y (rfY[t - τS - τY] ΠY←B (E1[t - τP - τS - 2 τY] rfB[t - τS - 2 τY] ΠB←P +
          E13[t - 2 τS - 2 τY] ΠB←S tbB[t - τS - 2 τY]) + EinY[t - τS - τY] tbY[t - τS - τY])
    tfB[t - τS] + rbB[t - τS] ΠB←X (EinX[t - τS - τX] tbX[t - τS - τX] +
      rfX[t - τS - τX] ΠX←B (E13[t - 2 τS - 2 τX] rbB[t - τS - 2 τX] ΠB←S +
          E1[t - τP - τS - 2 τX] ΠB←P tfB[t - τS - 2 τX])))
```

■ Basic relations

■ Basic idea of linear appoximation

When a field evolution over a time step of τ is given as

E(t0) = M(t0) E(t0-τ),

a longer jump can be given as

E(t0) = M(t0) M(t0-τ) M(t0-2 τ) ... M(t0-(N-1)τ) E(t0-N τ)

In the expression of M(t-n τ), use a linearlinzed formula - time dependence is linear -, like

r(t) = r0*exp($i\,\omega$ Δt), Δt=t-t0,

which is valid between t and t-Nτ. In this formula, r0 and ω are evaluated using given values at time t and t-Nτ.
By using this linear approximastion, the product of M(t)s shown above can be simplified as is shown below.

Another approximation used is to keep only the first term proportional to time, e.g., exp[$i\,\omega$ t] $\simeq$ 1 + $i\,\omega$ t. Typical value of this expansion parameter is estimated as follows.

$$\omega_{\rm xxx}\, N\,\tau = N \times O(k\,v\,\tau) = N \times O(\frac{2\pi}{\lambda}\,v\,\frac{L}{c}) = N \times O\Big(\frac{2\pi}{10^{-6}\,(m)}\,10^{-6}\,(m/s)\,\frac{2\times 10\,(m)}{3\,10^{8}\,(m/s)}\Big) = N \times O(10^{-6})$$

This is a case of 10 m cavity, with a mirror moving at 10^{-6} m/s. For this case, the accuracy. or the second order correction which is neglected, is O(10^{-6}) with N = 1000.

■ Calculation of fields

Right hand sides of E1lhs and E13lhs are split into two parts, one depending on the past values of E1 and E13 and the part depending on the external fields.

```
E1Internal[t_] := E1lhs[t] /. {EinP[_] → 0, EinX[_] → 0, EinY[_] → 0, EinS[_] → 0}
E13Internal[t_] := E13lhs[t] /. {EinP[_] → 0, EinX[_] → 0, EinY[_] → 0, EinS[_] → 0}
E1External[t_] := Simplify[E1lhs[t] - E1Internal[t]]
E13External[t_] := Simplify[E13lhs[t] - E13Internal[t]]
```

```
E1Internal[t]
```

```
rfP[t] ΠP←B (rfB[t - τP] rfY[t - τP - τY] ΠB←Y ΠY←B
     (E1[t - 2 τP - 2 τY] rfB[t - τP - 2 τY] ΠB←P + E13[t - τP - τS - 2 τY] ΠB←S tbB[t - τP - 2 τY]) +
   rfX[t - τP - τX] ΠB←X ΠX←B tbB[t - τP]
     (E13[t - τP - τS - 2 τX] rbB[t - τP - 2 τX] ΠB←S + E1[t - 2 τP - 2 τX] ΠB←P tfB[t - τP - 2 τX]))
```

```
E1External[t]
```

```
EinP[t] tbP[t] + rfP[t] ΠP←B (EinX[t - τP - τX] ΠB←X tbB[t - τP] tbX[t - τP - τX] +
     EinY[t - τP - τY] rfB[t - τP] ΠB←Y tbY[t - τP - τY])
```

```
E13Internal[t]
```

```
rfS[t] ΠS←B (rfY[t - τS - τY] ΠB←Y ΠY←B
     (E1[t - τP - τS - 2 τY] rfB[t - τS - 2 τY] ΠB←P + E13[t - 2 τS - 2 τY] ΠB←S tbB[t - τS - 2 τY])
     tfB[t - τS] + rbB[t - τS] rfX[t - τS - τX] ΠB←X ΠX←B
     (E13[t - 2 τS - 2 τX] rbB[t - τS - 2 τX] ΠB←S + E1[t - τP - τS - 2 τX] ΠB←P tfB[t - τS - 2 τX]))
```

```
E13External[t]
```

```
EinS[t] tbS[t] + rfS[t] ΠS←B (EinX[t - τS - τX] rbB[t - τS] ΠB←X tbX[t - τS - τX] +
     EinY[t - τS - τY] ΠB←Y tbY[t - τS - τY] tfB[t - τS])
```

■ Linear approximation

Mirrors are moving at constant velocities, and they are reflected as the change of reflectivity of each mirror. The change of the external input fields are explicitly assumed to be of the form
$E_0 \times \mathrm{Exp}[i\,\beta\,t]$
where E_0 and β are constant during the time to calculate the field evolution.

v_B is $\frac{1}{\sqrt{2}}$ of the beam splitter mirror velocity due to the mirror tilt.
t should be interpreted as t-t0, or the current time is set to 0.

```
LinearMotion := {rfP[t_] → rfP0 Exp[ⅈ 2 k vP t ], rfB[t_] -> rfB0 Exp[ⅈ 2 k vB t],
   rbB[t_] → rbB0 Exp[-ⅈ 2 k vB t], rfX[t_] -> rfX0 Exp[ⅈ 2 k vx t],
   rfY[t_] -> rfY0 Exp[ⅈ 2 k vY t],  rfS[t_] -> rfS0 Exp[ⅈ 2 k vS t], tbP[t_] → tbP0 ,
   tbB[t_] → tbB0, tfB[t_] → tfB0, tbX[t_] → tbX0, tbY[t_] → tbY0, tbS[t_] → tbS0};
LinearField := {EinP[t_] → EinP0 Exp[ ⅈ βP t], EinX[t_] -> EinX0 Exp[ ⅈ βX t],
   EinY[t_] -> EinY0 Exp[ ⅈ βY t], EinS[t_] -> EinS0 Exp[ ⅈ βS t]};
```

```
Expand[E1Internal[t] /. LinearMotion]
```

$$e^{2 i k t v_P + 2 i k (t - \tau P - \tau X) v_X}\, \mathrm{rfP0}\, \mathrm{rfX0}\, \mathrm{tbB0}\, \mathrm{tfB0}\, \mathrm{E1}[t - 2\,\tau P - 2\,\tau X]\, \Pi_{B\leftarrow P}\, \Pi_{B\leftarrow X}\, \Pi_{P\leftarrow B}\, \Pi_{X\leftarrow B} +$$
$$e^{-2 i k (t - \tau P - 2 \tau X) v_B + 2 i k t v_P + 2 i k (t - \tau P - \tau X) v_X}\, \mathrm{rbB0}\, \mathrm{rfP0}\, \mathrm{rfX0}\, \mathrm{tbB0}\, \mathrm{E13}[t - \tau P - \tau S - 2\,\tau X]\, \Pi_{B\leftarrow S}\, \Pi_{B\leftarrow X}\, \Pi_{P\leftarrow B}\, \Pi_{X\leftarrow B} +$$
$$e^{2 i k (t - \tau P) v_B + 2 i k (t - \tau P - 2 \tau Y) v_B + 2 i k t v_P + 2 i k (t - \tau P - \tau Y) v_Y}\, \mathrm{rfB0}^2\, \mathrm{rfP0}\, \mathrm{rfY0}\, \mathrm{E1}[t - 2\,\tau P - 2\,\tau Y]\, \Pi_{B\leftarrow P}\, \Pi_{B\leftarrow Y}\, \Pi_{P\leftarrow B}\, \Pi_{Y\leftarrow B} +$$
$$e^{2 i k (t - \tau P) v_B + 2 i k t v_P + 2 i k (t - \tau P - \tau Y) v_Y}\, \mathrm{rfB0}\, \mathrm{rfP0}\, \mathrm{rfY0}\, \mathrm{tbB0}\, \mathrm{E13}[t - \tau P - \tau S - 2\,\tau Y]\, \Pi_{B\leftarrow S}\, \Pi_{B\leftarrow Y}\, \Pi_{P\leftarrow B}\, \Pi_{Y\leftarrow B}$$

```
Expand[E13Internal[t] /. LinearMotion]
```

$$e^{-2 i k (t - \tau S) v_B + 2 i k t v_S + 2 i k (t - \tau S - \tau X) v_X}\, \mathrm{rbB0}\, \mathrm{rfS0}\, \mathrm{rfX0}\, \mathrm{tfB0}\, \mathrm{E1}[t - \tau P - \tau S - 2\,\tau X]\, \Pi_{B\leftarrow P}\, \Pi_{B\leftarrow X}\, \Pi_{S\leftarrow B}\, \Pi_{X\leftarrow B} +$$
$$e^{-2 i k (t - \tau S) v_B - 2 i k (t - \tau S - 2 \tau X) v_B + 2 i k t v_S + 2 i k (t - \tau S - \tau X) v_X}$$
$$\mathrm{rbB0}^2\, \mathrm{rfS0}\, \mathrm{rfX0}\, \mathrm{E13}[t - 2\,\tau S - 2\,\tau X]\, \Pi_{B\leftarrow S}\, \Pi_{B\leftarrow X}\, \Pi_{S\leftarrow B}\, \Pi_{X\leftarrow B} +$$
$$e^{2 i k (t - \tau S - 2 \tau Y) v_B + 2 i k t v_S + 2 i k (t - \tau S - \tau Y) v_Y}\, \mathrm{rfB0}\, \mathrm{rfS0}\, \mathrm{rfY0}\, \mathrm{tfB0}\, \mathrm{E1}[t - \tau P - \tau S - 2\,\tau Y]\, \Pi_{B\leftarrow P}\, \Pi_{B\leftarrow Y}\, \Pi_{S\leftarrow B}\, \Pi_{Y\leftarrow B} +$$
$$e^{2 i k t v_S + 2 i k (t - \tau S - \tau Y) v_Y}\, \mathrm{rfS0}\, \mathrm{rfY0}\, \mathrm{tbB0}\, \mathrm{tfB0}\, \mathrm{E13}[t - 2\,\tau S - 2\,\tau Y]\, \Pi_{B\leftarrow S}\, \Pi_{B\leftarrow Y}\, \Pi_{S\leftarrow B}\, \Pi_{Y\leftarrow B}$$

```
Expand[E1External[t] /. LinearMotion /. LinearField]
```

$$e^{i t \beta_P}\, \mathrm{EinP0}\, \mathrm{tbP0} + e^{2 i k t v_P + i (t - \tau P - \tau X) \beta_X}\, \mathrm{EinX0}\, \mathrm{rfP0}\, \mathrm{tbB0}\, \mathrm{tbX0}\, \Pi_{B\leftarrow X}\, \Pi_{P\leftarrow B} +$$
$$e^{2 i k (t - \tau P) v_B + 2 i k t v_P + i (t - \tau P - \tau Y) \beta_Y}\, \mathrm{EinY0}\, \mathrm{rfB0}\, \mathrm{rfP0}\, \mathrm{tbY0}\, \Pi_{B\leftarrow Y}\, \Pi_{P\leftarrow B}$$

```
Expand[E13External[t] /. LinearMotion /. LinearField]
```

$$e^{i t \beta_S}\, \mathrm{EinS0}\, \mathrm{tbS0} + e^{-2 i k (t - \tau S) v_B + 2 i k t v_S + i (t - \tau S - \tau X) \beta_X}\, \mathrm{EinX0}\, \mathrm{rbB0}\, \mathrm{rfS0}\, \mathrm{tbX0}\, \Pi_{B\leftarrow X}\, \Pi_{S\leftarrow B} +$$
$$e^{2 i k t v_S + i (t - \tau S - \tau Y) \beta_Y}\, \mathrm{EinY0}\, \mathrm{rfS0}\, \mathrm{tbY0}\, \mathrm{tfB0}\, \Pi_{B\leftarrow Y}\, \Pi_{S\leftarrow B}$$

■ Propagation times, τ's

■ $\tau_N \equiv \frac{L(N \leftrightarrow B)}{c}$ **: distance between mirror N and the beam splitter**
$\tau = 2\times(\max[\tau_P, \tau_S] + \max[\tau_X, \tau_Y])$
$\Delta_{XY} = \tau_X - \tau_Y$
$\Delta_{PS} = \tau_P - \tau_S$

```
τP + 2 τX + τS  /. {τP → τ - τX - τY - τS} /. {τY → τX - ΔXY}
τP + 2 τY + τS /. {τP → τ - τX - τY - τS} /. {τX → ΔXY + τY}
2 τP + 2 τX  /.  {2 τP + 2 τX → τ + τP + τX - τY - τS} /. {τP → ΔPS + τS,  τX → ΔXY + τY}
2 τP + 2 τY  /.  {2 τP + 2 τY → τ + τP - τX + τY - τS} /. {τP → ΔPS + τS,  τX → ΔXY + τY}
2 τX + 2 τS  /.  {2 τX + 2 τS → τ - τP + τX - τY + τS} /. {τP → ΔPS + τS,  τX → ΔXY + τY}
2 τY + 2 τS  /.  {2 τY + 2 τS → τ - τP - τX + τY + τS} /. {τP → ΔPS + τS,  τX → ΔXY + τY}
```

$\tau + \Delta_{XY}$

$\tau - \Delta_{XY}$

$\tau + \Delta_{PS} + \Delta_{XY}$

$\tau + \Delta_{PS} - \Delta_{XY}$

$\tau - \Delta_{PS} + \Delta_{XY}$

$\tau - \Delta_{PS} - \Delta_{XY}$

■ Choice of τ

The choice of τ is arbitrary. For the case of FP cavity, one round trip time was a convenient choice and it worked. For the case of DRM, there are several cavity lengths, and the choice is not unique. Most natural choice is the average trip time,

$\tau_P + \tau_S + \tau_X + \tau_Y$

Unfortunately, it turns out that one needs to do "extrapolation" (explained below) is this value is used, and "extrapolation" makes simulation unstable. In order to avoid that,

$\tau = 2\times(\max[\tau_P, \tau_S] + \max[\tau_X, \tau_Y])$

is used in the following calculation, with which all calculations can be done by "interpolation".

Linearlination of internal fields

The formula expressing E_P (=E1) and E_S (=E13) at time t by using past internal fields and external input fields

$$\begin{aligned}\text{Ep}[t] =\ & \text{Mpxp}[t]\,\text{Ep}[t-\tau-\delta\text{pxp}] + \text{Mpyp}[t]\,\text{Ep}[t-\tau-\delta\text{pyp}] \\ & + \text{Mpxs}[t]\,\text{Ep}[t-\tau-\delta\text{pxs}] + \text{Mpys}[t]\,\text{Ep}[t-\tau-\delta\text{pys}] \\ & + \text{EpExt}[t]\end{aligned}$$

$$\begin{aligned}\text{Es}[t] =\ & \text{Msxp}[t]\,\text{Ep}[t-\tau-\delta\text{sxp}] + \text{Msyp}[t]\,\text{Ep}[t-\tau-\delta\text{syp}] \\ & + \text{Msxs}[t]\,\text{Es}[t-\tau-\delta\text{sxs}] + \text{Msys}[t]\,\text{Ep}[t-\tau-\delta\text{sys}] \\ & + \text{EsExt}[t]\end{aligned}$$

Fields at time t-τ-δ can be expressed by fields at time t and t-τ as follows by linear interpolation.

$$\text{Eps}[t-\tau-\delta] = \text{Eps}[t-\tau] - \Delta\text{Eps}\,\delta = \text{Eps}[t-\tau] - (\text{Eps}[t] - \text{Eps}[t-\tau])\,\frac{\delta}{\tau}$$

The choice of τ discussed above is that all δ's become negative, i.e., evaluation can be done by "interpolating" fields between time t and t-τ.

$$\begin{aligned}\text{Ep}[t] =\ & \text{Mpxp}[t]\,(\text{Ep}[t-\tau] - \Delta\text{Ep}\,\delta\text{pxp}) + \text{Mpyp}[t]\,(\text{Ep}[t-\tau] - \Delta\text{Ep}\,\delta\text{pyp}) + \text{Mpxs}[t]\,(\text{Es}[t-\tau]-\Delta\text{Es}\,\delta\text{pxs}) + \text{Mpys}[t]\,(\text{Es}[t-\tau]-\Delta\text{Es}\,\delta\text{pys}) + \text{EpExt}[t] \\ =\ & (\text{Mpxp}[t] + \text{Mpyp}[t])\,\text{Ep}[t-\tau] + (\text{Mpxs}[t] + \text{Mpys}[t])\,\text{Es}[t-\tau] \\ & - \text{Mpxp}\,\frac{\delta\text{pxp}}{\tau}\,(\text{Ep}[t]-\text{Ep}[t-\tau]) - \text{Mpyp}\,\frac{\delta\text{pyp}}{\tau}\,(\text{Ep}[t]-\text{Ep}[t-\tau]) - \text{Mpxs}\,\frac{\delta\text{pxs}}{\tau}\,(\text{Es}[t]-\text{Es}[t-\tau]) - \text{Mpys}\,\frac{\delta\text{pys}}{\tau}\,(\text{Es}[t]-\text{Es}[t-\tau]) + \text{EpExt}[t] \\ =\ & -\left(\text{Mpxp}\,\frac{\delta\text{pxp}}{\tau} + \text{Mpyp}\,\frac{\delta\text{pyp}}{\tau}\right)\text{Ep}[t] - \left(\text{Mpxs}\,\frac{\delta\text{pxs}}{\tau} + \text{Mpys}\,\frac{\delta\text{pys}}{\tau}\right)\text{Es}[t] \\ & + \left(\text{Mpxp} + \text{Mpyp} + \text{Mpxp}\,\frac{\delta\text{pxp}}{\tau} + \text{Mpyp}\,\frac{\delta\text{pyp}}{\tau}\right)\text{Ep}[t-\tau] + \left(\text{Mpxs} + \text{Mpys} + \text{Mpxs}\,\frac{\delta\text{pxs}}{\tau} + \text{Mpxs}\,\frac{\delta\text{pxs}}{\tau}\right)\text{Es}[t-\tau] + \text{EpExt}[t] \\ =\ & -\text{mpp}\,\text{Ep}[t] - \text{mps}\,\text{Es}[t] + \text{Mpp}\,\text{Ep}[t-\tau] + \text{Mps}\,\text{Es}[t-\tau] + \text{EpExt}[t]\end{aligned}$$

$$\begin{aligned}\text{mpp} &= \text{Mpxp}\,\frac{\delta\text{pxp}}{\tau} + \text{Mpyp}\,\frac{\delta\text{pyp}}{\tau} \\ \text{mps} &= \text{Mpxs}\,\frac{\delta\text{pxs}}{\tau} + \text{Mpys}\,\frac{\delta\text{pys}}{\tau} \\ \text{Mpp} &= \text{Mpxp} + \text{Mpyp} + \text{mpp} \\ \text{Mps} &= \text{Mpxs} + \text{Mpys} + \text{mps}\end{aligned}$$

$$\begin{pmatrix} 1+\text{mpp} & \text{mps} \\ \text{msp} & 1+\text{mss} \end{pmatrix}[t] \begin{pmatrix} \text{Ep} \\ \text{Es} \end{pmatrix}[t] = \begin{pmatrix} \text{Mpp} & \text{Mps} \\ \text{Msp} & \text{Mss} \end{pmatrix}[t] \begin{pmatrix} \text{Ep} \\ \text{Es} \end{pmatrix}[t-\tau] + \begin{pmatrix} \text{EpExt} \\ \text{EsExt} \end{pmatrix}[t]$$

M_{PXP} is the coupling of the previous $E_P(t-\tau)$ to the current $E_P(t)$, through the reflection by mirror X.
mpp also couples the past and present E_P, but is directly caused by the difference of the length between $L_{X\leftrightarrow B}$ and $L_{Y\leftrightarrow B}$. msp is an important coupling to observe. If a field is resonant in the cavity (all propagators are 1) and if two ITMs have same refractive indexes ($\text{rf}_X = \text{rf}_Y$), then msp is proportional to $\tau_X - \tau_Y$. Due to this coupling, the field E_P can leak into the signal recycling cavity even when the dark port condition is fully satisfied.

$$\begin{aligned}E_P\,\text{Internal} =\ & A_{\text{PXP}}\,\text{Exp}[i\,\omega_{\text{PXP}}\,t]\left(E_P[t-\tau] - (E_P[t] - E_P[t-\tau])\,\frac{\delta_{\text{XPX}}}{\tau}\right) \\ & + A_{\text{PYP}}\,\text{Exp}[i\,\omega_{\text{PYP}}\,t])\,E_P[t-\tau] + (A_{\text{PXS}}\,\text{Exp}[i\,\omega_{\text{PXS}}\,t] + A_{\text{PYS}}\,\text{Exp}[i\,\omega_{\text{PYS}}\,t])\,E_S[t-\tau]\end{aligned}$$

$$\text{Mpp} = A_{\text{PXP}}\,\text{Exp}[i\,\omega_{\text{PXP}}\,t] + A_{\text{PYP}}\,\text{Exp}[i\,\omega_{\text{PYP}}\,t] + \text{mpp} = \text{AM}_{\text{pp}} + \text{mpp}$$

$$\text{mpp} = A_{\text{PXP}}\,\text{Exp}[i\,\omega_{\text{PXP}}\,t]\,\frac{\delta_{\text{PXP}}}{\tau} + A_{\text{PYP}}\,\text{Exp}[i\,\omega_{\text{PYP}}\,t]\,\frac{\delta_{\text{PYP}}}{\tau}$$
$$\simeq A_{\text{PXP}}\,\frac{\delta_{\text{PXP}}}{\tau} + A_{\text{PYP}}\,\frac{\delta_{\text{PYP}}}{\tau} + i\,t\left(A_{\text{PXP}}\,\omega_{\text{PXP}}\,\frac{\delta_{\text{PXP}}}{\tau} + A_{\text{PYP}}\,\omega_{\text{PYP}}\,\frac{\delta_{\text{PYP}}}{\tau}\right)$$

■ Moving the left hand side matrix to right

$$\begin{pmatrix}\text{Ep}\\ \text{Es}\end{pmatrix}[t] = \begin{pmatrix}1+\text{mpp} & \text{mps}\\ \text{msp} & 1+\text{mss}\end{pmatrix}^{-1}[t]\begin{pmatrix}\text{Mpp} & \text{Mps}\\ \text{Msp} & \text{Mss}\end{pmatrix}[t]\begin{pmatrix}\text{Ep}\\ \text{Es}\end{pmatrix}[t-\tau] + \begin{pmatrix}1+\text{mpp} & \text{mps}\\ \text{msp} & 1+\text{mss}\end{pmatrix}^{-1}[t]\begin{pmatrix}\text{EpExt}\\ \text{EsExt}\end{pmatrix}[t]$$

```
InvS = Series[Inverse[( ONEmpp0 + ⅈ t mpp1    mps0 + ⅈ t mps1
                        msp0 + ⅈ t msp1     ONEmss0 + ⅈ t mss1 )], {t, 0, 1}];
```

ONEmpp0 = 1 + mpp0, ONEmss0 = 1 + mss0

```
InvS /. {t → 0}
```

$$\left\{\left\{\frac{\text{ONEmss0}}{-\text{mps0 msp0} + \text{ONEmpp0 ONEmss0}}, -\frac{\text{mps0}}{-\text{mps0 msp0} + \text{ONEmpp0 ONEmss0}}\right\}, \left\{-\frac{\text{msp0}}{-\text{mps0 msp0} + \text{ONEmpp0 ONEmss0}}, \frac{\text{ONEmpp0}}{-\text{mps0 msp0} + \text{ONEmpp0 ONEmss0}}\right\}\right\}$$

```
FullSimplify[SeriesCoefficient[InvS[[1, 1]], 1]]
FullSimplify[SeriesCoefficient[InvS[[1, 2]], 1]]
FullSimplify[SeriesCoefficient[InvS[[2, 1]], 1]]
FullSimplify[SeriesCoefficient[InvS[[2, 2]], 1]]
```

$$-\frac{i\,(\text{mps0 msp0 mss1} - (\text{mps1 msp0} + \text{mps0 msp1})\,\text{ONEmss0} + \text{mpp1 ONEmss0}^2)}{(\text{mps0 msp0} - \text{ONEmpp0 ONEmss0})^2}$$

$$\frac{i\,(-\text{mps0}^2\,\text{msp1} + \text{mps0 mss1 ONEmpp0} + \text{mpp1 mps0 ONEmss0} - \text{mps1 ONEmpp0 ONEmss0})}{(\text{mps0 msp0} - \text{ONEmpp0 ONEmss0})^2}$$

$$\frac{i\,(-\text{mps1 msp0}^2 + \text{msp0 mss1 ONEmpp0} + \text{mpp1 msp0 ONEmss0} - \text{msp1 ONEmpp0 ONEmss0})}{(\text{mps0 msp0} - \text{ONEmpp0 ONEmss0})^2}$$

$$-\frac{i\,(\text{mpp1 mps0 msp0} + \text{ONEmpp0}\,(-\text{mps1 msp0} - \text{mps0 msp1} + \text{mss1 ONEmpp0}))}{(\text{mps0 msp0} - \text{ONEmpp0 ONEmss0})^2}$$

```
Expand[(A0 + ⅈ t A1) * (B0 + ⅈ t B1)]
```

$\text{A0 B0} + i\,\text{A1 B0}\,t + i\,\text{A0 B1}\,t - \text{A1 B1}\,t^2$

```
Mpp := Apxp Exp[ ⅈ ωpxp t ] + Apyp Exp[ⅈ ωpyp t ] + mpp;
Mps := Apxs Exp[ ⅈ ωpxs t ] + Apys Exp[ⅈ ωpys t ] + mps;
Msp := Asxp Exp[ ⅈ ωsxp t ] + Asyp Exp[ⅈ ωsyp t ] + msp;
Mss := Asxs Exp[ ⅈ ωsxs t ] + Asys Exp[ⅈ ωsys t ] + mss;
mpp := Apxp Exp[ ⅈ ωpxp t ] δpxp + Apyp Exp[ⅈ ωpyp t ] δpyp;
mps := Apxs Exp[ ⅈ ωpxs t ] δpxs + Apys Exp[ⅈ ωpys t ] δpys;
msp := Asxp Exp[ ⅈ ωsxp t ] δsxp + Asyp Exp[ⅈ ωsyp t ] δsyp;
mss := Asxs Exp[ ⅈ ωsxs t ] δsxs + Asys Exp[ⅈ ωsys t ] δsys;
MEp := App Exp[ⅈ ωpp t] + Apx Exp[ⅈ ωpx t] + Apy Exp[ⅈ ωpy t ];
MEs := Ass Exp[ⅈ ωss t] + Asx Exp[ⅈ ωsx t] + Asy Exp[ⅈ ωsy t ];
```

■ **valication**

```
Apxp := r_P r_X t_B^2 π_{P←B} π_{B←X} π_{X←B} π_{B←P};
Apyp := r_P r_Y r_B^2 π_{P←B} π_{B←Y} π_{Y←B} π_{B←P};
Apxs := r_B r_P r_X t_B π_{P←B} π_{B←X} π_{X←B} π_{B←S};
Apys := -r_B r_P r_Y t_B π_{P←B} π_{B←Y} π_{Y←B} π_{B←S};
Asxp := r_B r_S r_X t_B π_{S←B} π_{B←X} π_{X←B} π_{B←P};
Asyp := -r_B r_S r_Y t_B π_{S←B} π_{B←Y} π_{Y←B} π_{B←P};
Asxs := r_B^2 r_S r_X π_{S←B} π_{B←X} π_{X←B} π_{B←S};
Asys := r_S r_Y t_B^2 π_{S←B} π_{B←Y} π_{Y←B} π_{B←S};
π_{P←B} := Exp[-ⅈ k Lp ];
π_{B←P} := Exp[-ⅈ k Lp ];
π_{B←X} := Exp[-ⅈ k Lx ];
π_{X←B} := Exp[-ⅈ k Lx ];
π_{B←Y} := Exp[-ⅈ k Ly ];
π_{Y←B} := Exp[-ⅈ k Ly ];
π_{S←B} := Exp[-ⅈ k Ls ];
π_{B←S} := Exp[-ⅈ k Ls ];
ttaauu = 2 (Ls + Lx );
Δps := (Lp - Ls) / ttaauu;
Δxy := (Lx - Ly) / ttaauu;
δcorr := Δxy - Δps;
δpxp := Δps + Δxy - δcorr;
δpyp := Δps - Δxy - δcorr;
δpxs := Δxy - δcorr;
δpys := -Δxy - δcorr;
δsxp := Δxy - δcorr;
δsyp := -Δxy - δcorr;
δsxs := Δxy - Δps - δcorr;
δsys := -Δxy - Δps - δcorr;
```

```
LPRM = 37.5 10^-6;
TPRM = 0.0086;

LSRM = 37.5 10^-6;
TSRM = 0.005;

RBS = 0.49985;
TBS = 0.49985;

LITMX = 37.5 10^-6;
TITMX = 0.005;

LITMY = 37.5 10^-6;
TITMY = 0.005;

(* cavity quantities *)
LengPRM = 0.2;
LengITMX = 2;
LengITMY = 1.7;
LengSRM = 0.4058;

SBFreq = 69.625263 10^6;
Cspeed = 2.99792458 10^8;

delZ = 6.22125057 10^-12;
```

```
numeric := {
  rB -> Sqrt[ RBS ],
  tB -> Sqrt[ TBS ],
  rX -> Sqrt[ 1 - LITMX - TITMX ],
  rY -> Sqrt[ 1 - LITMY - TITMY ],
  rS -> Sqrt[ 1 - LSRM - TSRM ] Exp[ i 2 π (2 delZ)/(1.064 10^-6) ],
  rP -> Sqrt[ 1 - LPRM - TPRM ],
  tP → Sqrt[ TPRM ],
  k -> 2 Pi SBFreq / Cspeed 0, (* for now CR *)
  cc → Cspeed,
  Lp → LengPRM, Lx → LengITMX, Ly → LengITMY, Ls → LengSRM}
```

```
MM00 = ( Apxp + Apyp   Apxs + Apys
         Asxp + Asyp   Asxs + Asys ) /. numeric
```

{{0.992863, 0.}, {5.55112 × 10^-17 + 0. i, 0.994664 + 0.000073084 i}}

```
mm11 = ( Apxp δpxp + Apyp δpyp   Apxs δpxs + Apys δpys
         Asxp δsxp + Asyp δsyp   Asxs δsxs + Asys δsys ) /. numeric
```

{{-0.146837, 0.0619043}, {0.0620166 + 4.55674 × 10^-6 i, -0.0620166 - 4.55674 × 10^-6 i}}

```
(Apxp Apyp Apxs Apys Asxp Asyp Asxs Asys) /. numeric
```

```
{{0.496431, 0.496431, 0.496431, -0.496431, 0.497332 + 0.000036542 i,
  -0.497332 - 0.000036542 i, {{0.497332 + 0.000036542 i, 0.497332 + 0.000036542 i}}}}
```

```
(δpxp δpyp δpxs δpys δsxp δsyp δsxs δsys) /. numeric
```

```
{{-0.0855433, -0.210242, -0.0427716, -0.16747, -0.0427716, -0.16747, 0, -0.124699}}
```

$$\text{MM11Amp} = \text{Inverse}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \text{mm11}\right].(\text{MM00} + \text{mm11})$$

$$\{\{0.991594 - 3.17408\times 10^{-9}\, i,\ 0.000414761 - 5.67857\times 10^{-6}\, i\},\ \{0.000555764 + 4.37451\times 10^{-8}\, i,\ 0.994284 + 0.0000782618\, i\}\}$$

```
{EigVal0, EigVec0} = Eigensystem[MM00]
{EigVal1, EigVec1} = Eigensystem[MM11]
```

```
{{0.860184 - 0.498603 i, 0.859352 + 0.49812 i},
 {{0.482234 - 0.000753907 i, 0.876042 + 0. i}, {0.875672 + 0. i, -0.482905 + 0.000754955 i}}}

{{0.887288 - 0.470133 i, 0.825569 + 0.52815 i},
 {{0.47953 + 0.0000812605 i, 0.877525 + 0. i}, {0.874212 + 0. i, -0.485544 - 0.000370173 i}}}
```

```
Abs[EigVal0[[1]]]
Abs[EigVal0[[2]]]
Abs[EigVal1[[1]]]
Abs[EigVal1[[2]]]
```

```
0.994244

0.993282

1.00414

0.980054
```

E1internal

$$e^{2 i k t v_P + 2 i k (t-\tau P-\tau X) v_X}\, \text{rfP0 rfX0 tbB0 tfB0 E1}[t - 2\,\tau P - 2\,\tau X]\, \Pi_{B\leftarrow P}\, \Pi_{B\leftarrow X}\, \Pi_{P\leftarrow B}\, \Pi_{X\leftarrow B} + e^{-2 i k (t-\tau P-2\tau X) v_B + 2 i k t v_P + 2 i k (t-\tau P-\tau X) v_X}\, \text{rbB0 rfP0 rfX0 tbB0 E13}[t - \tau P - \tau S - 2\,\tau X]\, \Pi_{B\leftarrow S}\, \Pi_{B\leftarrow X}\, \Pi_{P\leftarrow B}\, \Pi_{X\leftarrow B} + e^{2 i k (t-\tau P) v_B + 2 i k (t-\tau P-2\tau Y) v_B + 2 i k t v_P + 2 i k (t-\tau P-\tau Y) v_Y}\, \text{rfB0}^2\, \text{rfP0 rfY0 E1}[t - 2\,\tau P - 2\,\tau Y]\, \Pi_{B\leftarrow P}\, \Pi_{B\leftarrow Y}\, \Pi_{P\leftarrow B}\, \Pi_{Y\leftarrow B} + e^{2 i k (t-\tau P) v_B + 2 i k t v_P + 2 i k (t-\tau P-\tau Y) v_Y}\, \text{rfB0 rfP0 rfY0 tbB0 E13}[t - \tau P - \tau S - 2\,\tau Y]\, \Pi_{B\leftarrow S}\, \Pi_{B\leftarrow Y}\, \Pi_{P\leftarrow B}\, \Pi_{Y\leftarrow B}$$

■ phase

```
Simplify[Series[Factor[2 ⅈ k t vP + 2 ⅈ k (t - τP - τX) vX] / (2 ⅈ k), {t, 0, 1}]]
Simplify[Series[
  Factor[2 ⅈ k (t - τP) vB + 2 ⅈ k (t - τP - 2 τY) vB + 2 ⅈ k t vP + 2 ⅈ k (t - τP - τY) vY] / (2 ⅈ k),
  {t, 0, 1}]]
Simplify[Series[Factor[-2 ⅈ k (t - τP - 2 τX) vB + 2 ⅈ k t vP + 2 ⅈ k (t - τP - τX) vX] / (2 ⅈ k),
  {t, 0, 1}]]
Simplify[Series[Factor[2 ⅈ k (t - τP) vB + 2 ⅈ k t vP + 2 ⅈ k (t - τP - τY) vY] / (2 ⅈ k),
  {t, 0, 1}]]
```

$-(\tau P + \tau X)\, v_X + (v_P + v_X)\, t + O[t]^2$

$-(\tau P + \tau Y)\,(2\, v_B + v_Y) + (2\, v_B + v_P + v_Y)\, t + O[t]^2$

$((\tau P + 2\,\tau X)\, v_B - (\tau P + \tau X)\, v_X) + (-v_B + v_P + v_X)\, t + O[t]^2$

$(-\tau P\, v_B - (\tau P + \tau Y)\, v_Y) + (v_B + v_P + v_Y)\, t + O[t]^2$

E1[t-2 τP-2 τX] = E1[t-$\tau - \triangle_{PS} - \triangle_{XY}$] : δpxp = Δps + Δxy
E1[t-2 τP-2 τY] = E1[t-τ - $\Delta_{PS} + \Delta_{XY}$] : δpyp = Δps - Δxy
E13[t-τP-τS-2 τX]= E13[$t - \tau - \Delta_{XY}$] : δpxs = Δxy
E13[t-τP-τS-2 τY] = E13[$t - \tau + \Delta_{XY}$] : δpys = - Δxy

■ summary

Ep = E1, Es = E13

E_P Internal = (A_{PXP} Exp[$i\, \omega_{PXP}$ t] + A_{PYP} Exp[$i\, \omega_{PYP}$ t]) $E_P[t-\tau]$ + (A_{PXS} Exp[$i\, \omega_{PXS}$ t] + A_{PYS} Exp[$i\, \omega_{PYS}$ t]) $E_S[t-\tau]$
$A_{PX\,P}$ = rfP0 rfX0 tB0^2 ΦBP ΦBX ΦPB ΦXB Exp[-i 2 k $(\tau P + \tau X)\, v_X$]
ω_{PXP} = 2 k ($v_P + v_X$)
δpxp = Δps + Δxy
A_{PYP} = rfB0^2 rfP0 rfY0 ΦBP ΦBY ΦPB ΦYB Exp[-i 2 k $(\tau P + \tau Y)\,(2\, v_B + v_Y)$]
ω_{PYP} = 2 k $(2\, v_B + v_P + v_Y)$
δpyp = Δps - Δxy
A_{PXS} = rbB0 rfP0 rfX0 tB0 ΦBP ΦBX ΦSB ΦXB E13 Exp[-i 2 k $(-(\tau P + 2\,\tau X)\, v_B + (\tau P + \tau X)\, v_X)$]
ω_{PXS} = 2 k $(-v_B + v_P + v_X)$
δpxs = Δxy
A_{PYS} = rfB0 rfP0 rfY0 tB0 ΦBP ΦBY ΦSB ΦYB Exp[-i 2 k $(\tau P\, v_B + (\tau P + \tau Y)\, v_Y)$]
ω_{PYS} = 2 k $(v_B + v_P + v_y)$
δpys = - Δxy

```
Apxp := rP0 rX0 tB0^2 propP^2 propX^2;
Apyp := rP0 rY0 rB0^2 propP^2 propY^2;
Apxs := -rB0 rP0 rX0 tB0 propP propX^2 propS;
Apys := rB0 rP0 rY0 tB0 propP propY^2 propS;
```

E13internal

$e^{-2 i k (t-\tau S) v_B + 2 i k t v_S + 2 i k (t-\tau S-\tau X) v_x}$ rbB0 rfS0 rfX0 tfB0 E1[t − τP − τS − 2 τX] $\Pi_{B\leftarrow P}\Pi_{B\leftarrow X}\Pi_{S\leftarrow B}\Pi_{X\leftarrow B}$ + $e^{-2 i k (t-\tau S) v_B - 2 i k (t-\tau S-2\tau X) v_B + 2 i k t v_S + 2 i k (t-\tau S-\tau X) v_x}$ rbB0² rfS0 rfX0 E13[t − 2 τS − 2 τX] $\Pi_{B\leftarrow S}\Pi_{B\leftarrow X}\Pi_{S\leftarrow B}\Pi_{X\leftarrow B}$ + $e^{2 i k (t-\tau S-2\tau Y) v_B + 2 i k t v_S + 2 i k (t-\tau S-\tau Y) v_Y}$ rfB0 rfS0 rfY0 tfB0 E1[t − τP − τS − 2 τY] $\Pi_{B\leftarrow P}\Pi_{B\leftarrow Y}\Pi_{S\leftarrow B}\Pi_{Y\leftarrow B}$ + $e^{2 i k t v_S + 2 i k (t-\tau S-\tau Y) v_Y}$ rfS0 rfY0 tbB0 tfB0 E13[t − 2 τS − 2 τY] $\Pi_{B\leftarrow S}\Pi_{B\leftarrow Y}\Pi_{S\leftarrow B}\Pi_{Y\leftarrow B}$

phase

```
Simplify[
 Series[Factor[-2 i k (t - τS) vB + 2 i k t vS + 2 i k (t - τS - τX) vx] / (2 i k), {t, 0, 1}]]
Simplify[Series[Factor[2 i k (t - τS - 2 τY) vB + 2 i k t vS + 2 i k (t - τS - τY) vY] / (2 i k),
  {t, 0, 1}]]
Simplify[Series[Factor[-2 i k (t - τS) vB - 2 i k (t - τS - 2 τX) vB +
     2 i k t vS + 2 i k (t - τS - τX) vx] / (2 i k), {t, 0, 1}]]
Simplify[Series[Factor[2 i k t vS + 2 i k (t - τS - τY) vY] / (2 i k), {t, 0, 1}]]
```

$(\tau S\, v_B - (\tau S + \tau X)\, v_x) + (-v_B + v_S + v_x)\, t + O[t]^2$

$(-(\tau S + 2\,\tau Y)\, v_B - (\tau S + \tau Y)\, v_Y) + (v_B + v_S + v_Y)\, t + O[t]^2$

$(\tau S + \tau X)\,(2\, v_B - v_x) + (-2\, v_B + v_S + v_x)\, t + O[t]^2$

$-(\tau S + \tau Y)\, v_Y + (v_S + v_Y)\, t + O[t]^2$

E1[t-τP-τS-2 τX] = E1[t-τ − Δ_{XY}] : δsxp = Δxy
E1[t-τP-τS-2 τY]= E1[t-τ + Δ_{XY}] : δsyp = - Δxy
E13[t-2 τS-2 τX]= E13[$t-\tau+\Delta_{PS}-\Delta_{XY}$] : δsxs = Δxy - Δ_{PS}
E13[t-2 τS-2 τY] = E13[$t-\tau+\Delta_{PS}+\Delta_{XY}$] : δsys = - Δxy- Δ_{PS}

summary

Ep = E1, Es = E13

E_S Internal = (A_{SXP} Exp[$i\,\omega_{SXP}$ t] + A_{SYP} Exp[$i\,\omega_{SYP}$ t]) $E_P[t-\tau]$ + (A_{SXS} Exp[$i\,\omega_{SXS}$ t] + A_{SYS} Exp[$i\,\omega_{SYS}$ t]) $E_S[t-\tau]$
$A_{SX\,P}$ = rbB0 rfS0 rfX0 tB0 ΦBS ΦBX ΦPB ΦXB Exp[-i 2 k $(-\tau S\, v_B + (\tau S + \tau X)\, v_x)$]
ω_{SXP} = 2 k $(-v_B + v_S + v_x)$
δsxp = Δxy
A_{SYP} = rfB0 rfS0 rfY0 tB0 ΦBS ΦBY ΦPB ΦYB Exp[-i 2 k $((\tau S + 2\,\tau Y)\, v_B + (\tau S + \tau Y)\, v_Y)$]
ω_{SYP} = 2 k $(v_B + v_S + v_Y)$
δsyp = - Δxy
A_{SXS} = rbB0² rfS0 rfX0 ΦBS ΦBX ΦSB ΦXB Exp[-i 2 k $(\tau S + \tau X)\,(-2\, v_B + v_x)$]
ω_{SXS} = 2 k $(-2\, v_B + v_S + v_x)$
δsxs = Δxy - Δ_{PS}
A_{SYS} = rfS0 rfY0 tB0² ΦBS ΦBY ΦSB ΦYB Exp[-i 2 k $(\tau S + \tau Y)\, v_Y$]
ω_{SYS} = 2 k $(v_S + v_Y)$
δsys = - Δxy- Δ_{PS}

E1external

$$e^{i\,t\,\beta_P}\,\text{EinP0 tP0} - e^{2\,i\,k\,t\,v_P + i\,(t-\tau P-\tau X)\,\beta_X}\,\text{EinX0 rP0 tB0 tX0 }\Phi\text{BP }\Phi\text{XB} + e^{2\,i\,k\,(t-\tau P)\,v_B + 2\,i\,k\,t\,v_P + i\,(t-\tau P-\tau Y)\,\beta_Y}\,\text{EinY0 rB0 rP0 tY0 }\Phi\text{BP }\Phi\text{YB}$$

phase

```
Simplify[Series[Factor[2 ⅈ k (t - τP) vB + 2 ⅈ k t vP] / (2 ⅈ k), {t, 0, 1}]]
```

$$-\tau P\, v_B + (v_B + v_P)\, t + O[t]^2$$

summary

Ep = E1, Es = E13

E_P External = A_{PP} Exp[$i\,\omega_{PP}$ t] EinP0 + A_{PX} Exp[$i\,\omega_{PX}$ t] EinX0 + A_{PY} Exp[$i\,\omega_{PY}$ t] EinY0
$A_{P\,P}$ = tP0
$\omega_{PP} = \beta_P$
$A_{P\,X}$ = -rP0 tB0 tX0 ΦBP ΦXB Exp[-i (τP+τX) β_X]
$\omega_{PX} = \beta_X + 2\,k\,v_P$
$A_{P\,Y}$ = rB0 rP0 tY0 ΦBP ΦYB Exp[-i ((τP + τY) β_Y + 2 k τP v_B)]
$\omega_{PY} = \beta_Y + 2\,k\,(\,v_B + v_P\,)$

E13external

$$e^{i\,t\,\beta_S}\,\text{EinS0 tS0} - e^{-2\,i\,k\,(t-\tau S)\,v_B + 2\,i\,k\,t\,v_S + i\,(t-\tau S-\tau X)\,\beta_X}\,\text{EinX0 rBConj0 rS0 tX0 }\Phi\text{BS }\Phi\text{XB} - e^{2\,i\,k\,t\,v_S + i\,(t-\tau S-\tau Y)\,\beta_Y}\,\text{EinY0 rS0 tB0 tY0 }\Phi\text{BS }\Phi\text{YB}$$

phase

```
Simplify[Series[Factor[-2 ⅈ k (t - τS) vB + 2 ⅈ k t vS] / (2 ⅈ k), {t, 0, 1}]]
```

$$\tau S\, v_B + (-v_B + v_S)\, t + O[t]^2$$

summary

Ep = E1, Es = E13

E_S External = A_{SS} Exp[i ω_{SS} t] EinS0 + A_{SX} Exp[i ω_{SX} t] EinX0 + A_{SY} Exp[i ω_{SY} t] EinY0
A_{SS} = tS0
$\omega_{SS} = \beta_S$
A_{SX} = -rBConj0 rS0 tX0 ΦBS ΦXB Exp[-i ((τS+τX) β_X - 2 k τS v_B)]
$\omega_{SX} = \beta_X + 2$ k ($v_S - v_B$)
A_{SY} = -rS0 tB0 tY0 ΦBS ΦYB Exp[-i (τS + τY) β_Y]
$\omega_{SY} = \beta_Y + 2$ k v_S

■ Summart of fields

E_P Internal = (A_{PXP} Exp[i ω_{PXP} t] + A_{PYP} Exp[i ω_{PYP} t]) $E_P[t-\tau]$ + (A_{PXS} Exp[i ω_{PXS} t] + A_{PYS} Exp[i ω_{PYS} t]) $E_S[t-\tau]$
E_S Internal = (A_{PXP} Exp[i ω_{PXP} t] + A_{PYP} Exp[i ω_{PYP} t]) $E_P[t-\tau]$ + (A_{PXS} Exp[i ω_{PXS} t] + A_{PYS} Exp[i ω_{PYS} t]) $E_S[t-\tau]$
E_P External = A_{PP} Exp[i ω_{PP} t] EinP0 + A_{PX} Exp[i ω_{PX} t] EinX0 + A_{PY} Exp[i ω_{PY} t] EinY0
E_S External = A_{SS} Exp[i ω_{SS} t] EinS0 + A_{SX} Exp[i ω_{SX} t] EinX0 + A_{SY} Exp[i ω_{SY} t] EinY0

■ Details list

"t" in the exponent = t - t(now).

```
EPint[t_] := (CA_PXP Exp[ⅈ Cω_PXP t] + CA_PYP Exp[ⅈ Cω_PYP t]) Ep[t0 - τ] +
   (CA_PXS Exp[ⅈ Cω_PXS t] + CA_PYS Exp[ⅈ Cω_PYS t]) Es[t0 - τ];
A_PXP := r0_P r0_X t0_B^2 Π_P←B Π_B←X Π_X←B Π_B←P Exp[-ⅈ 2 k (τ_P + τ_X) v_X] Exp[ -ⅈ V_EP (Δ_PS + Δ_XY)];
ω_PXP := 2 k (v_P + v_X);
A_PYP := r0_P r0_Y r0_B^2 Π_P←B Π_B←Y Π_Y←B Π_B←P Exp[-ⅈ 2 k (τ_P + τ_Y) (2 v_B + v_Y)] Exp[ -ⅈ V_EP (Δ_PS - Δ_XY)];
ω_PYP := 2 k (2 v_B + v_P + v_Y);
A_PXS :=
  rb0_B r0_P r0_X t0_B Π_P←B Π_B←X Π_X←B Π_B←S Exp[-ⅈ 2 k ((τ_P + τ_X) v_X - (τ_P + 2 τ_X) v_B)] Exp[ -ⅈ V_ES Δ_XY];
ω_PXS := 2 k (v_P + v_X - v_B);
A_PYS := r0_B r0_P r0_Y t0_B Π_P←B Π_B←Y Π_Y←B Π_B←S Exp[-ⅈ 2 k (τ_P v_B + (τ_P + τ_Y) v_Y)] Exp[ ⅈ V_ES Δ_XY];
ω_PYS := 2 k (v_P + v_Y + v_B);
```

```
ESint[t_] := (CA_SXP Exp[ⅈ Cω_SXP t] + CA_SYP Exp[ⅈ Cω_SYP t]) Ep[t0 - τ] +
   (CA_SXS Exp[ⅈ Cω_SXS t] + CA_SYS Exp[ⅈ Cω_SYS t]) Es[t0 - τ];
A_SXP := rb0_B r0_S r0_X t0_B Π_S←B Π_B←X Π_X←B Π_B←P Exp[-ⅈ 2 k ((τ_S + τ_X) v_X - τ_S v_B)] Exp[ -ⅈ V_EP Δ_XY];
ω_SXP := 2 k (v_S + v_X - v_B);
A_SYP :=
  r0_B r0_S r0_Y t0_B Π_S←B Π_B←Y Π_Y←B Π_B←P Exp[-ⅈ 2 k ((τ_S + 2 τ_Y) v_B + (τ_S + τ_Y) v_Y)] Exp[ ⅈ V_EP Δ_XY];
ω_SYP := 2 k (v_B + v_S + v_Y);
A_SXS := rb0_B^2 r0_S r0_X Π_S←B Π_B←X Π_X←B Π_B←S Exp[-ⅈ 2 k (τ_S + τ_X) (v_X - 2 v_B)] Exp[ ⅈ V_ES (Δ_PS - Δ_XY)];
ω_SXS := 2 k (v_S + v_X - 2 v_B);
A_SYS := r0_S r0_Y t0_B^2 Π_S←B Π_B←Y Π_Y←B Π_B←S Exp[-ⅈ 2 k (τ_S + τ_Y) v_Y] Exp[ ⅈ V_ES (Δ_PS + Δ_XY)];
ω_SYS := 2 k (v_S + v_Y);
```

```
EPext[t_] := CA_PP Exp[ⅈ Cω_PP t] Ein0_P + CA_PX Exp[ⅈ Cω_PX t] Ein0_X + CA_PY Exp[ⅈ Cω_PY t] Ein0_Y;
A_PP := t0_P;
ω_PP := β_P;
A_PX := -r0_P t0_B t0_X Π_P←B Π_B←X Exp[-ⅈ (τ_P + τ_X) β_X];
ω_PX := β_X + 2 k v_P;
A_PY := r0_B r0_P t0_Y Π_P←B Π_B←Y Exp[-ⅈ ((τ_P + τ_Y) β_Y + 2 k τ_P v_B)];
ω_PY := β_Y + 2 k (v_B + v_P);
```

```
ESext[t_] := CA_SS Exp[ⅈ Cω_SS t] Ein0_P + CA_SX Exp[ⅈ Cω_SX t] Ein0_X + CA_PY Exp[ⅈ Cω_SY t] Ein0_Y;
A_SS := t0_S;
ω_PP := β_S;
A_SX := -rb0_B r0_S t0_X Π_S←B Π_B←X Exp[-ⅈ ((τ_S + τ_X) β_X - 2 k τ_S v_B)];
ω_SX := β_X + 2 k (v_S - v_B);
A_SY := -t0_B r0_S t0_Y Π_S←B Π_B←Y Exp[-ⅈ (τ_S + τ_Y) β_Y];
ω_SY := β_Y + 2 k v_S;
```

Some calculation

```
vMTo0 := {v_P → 0, v_S → 0, v_X → 0, v_Y → 0, v_B → 0}
```

```
VETo0 := {V_ES → 0, V_EP → 0}
```

```
FullSimplify[(A_PXP + A_PYP) - (A_SXS + A_SYS) /. vMTo0 /. VETo0]
```

$$r0_P\, \Pi_{B\leftarrow P}\, \Pi_{P\leftarrow B}\, (r0_X\, t0_B^2\, \Pi_{B\leftarrow X}\, \Pi_{X\leftarrow B} + r0_B^2\, r0_Y\, \Pi_{B\leftarrow Y}\, \Pi_{Y\leftarrow B}) - r0_S\, \Pi_{B\leftarrow S}\, \Pi_{S\leftarrow B}\, (r0_X\, rb0_B^2\, \Pi_{B\leftarrow X}\, \Pi_{X\leftarrow B} + r0_Y\, t0_B^2\, \Pi_{B\leftarrow Y}\, \Pi_{Y\leftarrow B})$$

```
FullSimplify[((A_PXP + A_PYP) - (A_SXS + A_SYS))^2 + 4 (A_PXS + A_PYS) (A_SXP + A_SYP) /. vMTo0 /. VETo0]
```

$$4\, r0_P\, r0_S\, t0_B^2\, \Pi_{B\leftarrow P}\, \Pi_{B\leftarrow S}\, \Pi_{P\leftarrow B}\, \Pi_{S\leftarrow B}\, (r0_X\, rb0_B\, \Pi_{B\leftarrow X}\, \Pi_{X\leftarrow B} - r0_B\, r0_Y\, \Pi_{B\leftarrow Y}\, \Pi_{Y\leftarrow B})^2 + (r0_P\, \Pi_{B\leftarrow P}\, \Pi_{P\leftarrow B}\, (r0_X\, t0_B^2\, \Pi_{B\leftarrow X}\, \Pi_{X\leftarrow B} + r0_B^2\, r0_Y\, \Pi_{B\leftarrow Y}\, \Pi_{Y\leftarrow B}) - r0_S\, \Pi_{B\leftarrow S}\, \Pi_{S\leftarrow B}\, (r0_X\, rb0_B^2\, \Pi_{B\leftarrow X}\, \Pi_{X\leftarrow B} + r0_Y\, t0_B^2\, \Pi_{B\leftarrow Y}\, \Pi_{Y\leftarrow B}))^2$$

Expression for iteration

- $\vec{E}(t) = \overleftrightarrow{\text{Min}}(t) \cdot \vec{E}(t-\tau) + \overleftrightarrow{\text{Mext}}(t) \cdot \vec{E}_{\text{ext}}$

  $\vec{E} = \begin{pmatrix} E_P \\ E_S \end{pmatrix}, \quad \vec{E}_{\text{ext}} = \begin{pmatrix} \text{EinP0} \\ \text{EinS0} \\ \text{EinX0} \\ \text{EinY0} \end{pmatrix}$

  $E_{\text{ext}}(t) = \text{Eext0} \times \text{Exp}[\, i\, \beta\, t\,]$
  r(t) = r(0) Exp[i 2 k v t]

v_B is $\frac{1}{\sqrt{2}}$ of the beam splitter mirror velocity due to the mirror tilt.
t is measured "now" as the origine.

Recursive formulation and summation

Recursive formula

$$\vec{E}(t) = \prod_{k=0}^{N-1} \overleftrightarrow{\text{Min}}(t-k\,\tau) \cdot \vec{E}(t-N\,\tau) + \sum_{j=1}^{N-1} \prod_{k=0}^{j-1} \overleftrightarrow{\text{Min}}(t-k\,\tau) \cdot \overleftrightarrow{\text{Mext}}(t-\text{j}\tau) \cdot \vec{E}_{\text{ext}}(t-\text{j}\tau) + \overleftrightarrow{\text{Mext}}(t) \cdot \vec{E}_{\text{ext}}(\text{t})$$

$\vec{E}(t) = \overleftrightarrow{\text{Min}}(t) \cdot \vec{E}(t-\tau) + \overleftrightarrow{\text{Mext}}(t) \cdot \vec{E}_{\text{ext}}(t)$
$= \overleftrightarrow{\text{Min}}(t) \cdot \left(\overleftrightarrow{\text{Min}}(t-\tau) \cdot \vec{E}(t-2\,\tau) + \overleftrightarrow{\text{Mext}}(t-\tau) \cdot \vec{E}_{\text{ext}}(t-\tau)\right) + \overleftrightarrow{\text{Mext}}(t) \cdot \vec{E}_{\text{ext}}(t)$
$= \overleftrightarrow{\text{Min}}(t) \cdot \overleftrightarrow{\text{Min}}(t-\tau) \cdot \vec{E}(t-2\,\tau) + \overleftrightarrow{\text{Min}}(t) \cdot \overleftrightarrow{\text{Mext}}(t-\tau) \cdot \vec{E}_{\text{ext}}(t-\tau) + \overleftrightarrow{\text{Mext}}(t) \cdot \vec{E}_{\text{ext}}(\text{t})$
$= \overleftrightarrow{\text{Min}}(t) \cdot \overleftrightarrow{\text{Min}}(t-\tau) \cdot \left(\overleftrightarrow{\text{Min}}(t-2\,\tau) \cdot \vec{E}(t-3\,\tau) + \overleftrightarrow{\text{Mext}}(t-2\,\tau) \cdot \vec{E}_{\text{ext}}(t-2\,\tau)\right) + \overleftrightarrow{\text{Min}}(t) \cdot \overleftrightarrow{\text{Mext}}(t-\tau) \cdot \vec{E}_{\text{ext}}(t-\tau) + \overleftrightarrow{\text{Mext}}(t) \cdot \vec{E}_{\text{ext}}(t)$
$= \overleftrightarrow{\text{Min}}(t) \cdot \overleftrightarrow{\text{Min}}(t-\tau) \cdot \overleftrightarrow{\text{Min}}(t-2\,\tau) \cdot \vec{E}(t-3\,\tau) + \overleftrightarrow{\text{Min}}(t) \cdot \overleftrightarrow{\text{Min}}(t-\tau) \cdot \overleftrightarrow{\text{Mext}}(t-2\,\tau) \cdot \vec{E}_{\text{ext}}(t-2\,\tau) + \overleftrightarrow{\text{Min}}(t) \cdot \overleftrightarrow{\text{Mext}}(t-\tau) \cdot \vec{E}_{\text{ext}}(t-\tau) + \overleftrightarrow{\text{Mext}}(t) \cdot \vec{E}_{\text{ext}}(\text{t})$
$= \prod_{i=0}^{N-1} \overleftrightarrow{\text{Min}}(t-i\,\tau) \cdot \vec{E}(t-N\,\tau) + \sum_{j=1}^{N-1} \prod_{i=0}^{j-1} \overleftrightarrow{\text{Min}}(t-i\,\tau) \cdot \overleftrightarrow{\text{Mext}}(t-\text{j}\tau) \cdot \vec{E}_{\text{ext}}(t-j\,\tau) + \overleftrightarrow{\text{Mext}}(t) \cdot \vec{E}_{\text{ext}}(\text{t})$

$$\vec{E}(t) = \prod_{k=0}^{N-1} \overleftrightarrow{\text{Min}}(t-k\,\tau) \cdot \vec{E}(t-N\,\tau) + \sum_{j=1}^{N-1} \prod_{k=0}^{j-1} \overleftrightarrow{\text{Min}}(t-k\,\tau) \cdot \overleftrightarrow{\text{Mext}}(t-\text{j}\tau) \cdot \vec{E}_{\text{ext}}(t-\text{j}\tau) + \overleftrightarrow{\text{Mext}}(t) \cdot \vec{E}_{\text{ext}}(\text{t})$$

```
Min[t_] := ( A_PXYP Exp[ⅈ ω_PXYP t]   A_PXYS Exp[ⅈ ω_PXYS t] )
           ( A_SXYP Exp[ⅈ ω_SXYP t]   A_SXYS Exp[ⅈ ω_SXYS t] );
Min0 := ( A_PXYP  A_PXYS )
        ( A_SXYP  A_SXYS );
Min1 := ⅈ ( A_PXYP ω_PXYP   A_PXYS ω_PXYS )
          ( A_SXYP ω_SXYP   A_SXYS ω_SXYS ) t;
A_PXYP := A_PXP + A_PYP;
ω_PXYP := (A_PXP ω_PXP + A_PYP ω_PYP) / (A_PXP + A_PYP);
A_PXYS := A_PXS + A_PYS;
ω_PXYS := (A_PXS ω_PXS + A_PYS ω_PYS) / (A_PXS + A_PYS);
A_SXYP := A_SXP + A_SYP;
ω_SXYP := (A_SXP ω_SXP + A_SYP ω_SYP) / (A_SXP + A_SYP);
A_SXYS := A_SXS + A_SYS;
ω_SXYS := (A_SXS ω_SXS + A_SYS ω_SYS) / (A_SXS + A_SYS);

Eext[t_] := ( AE_P Exp[ⅈ ωE_P t] )
            ( AE_S Exp[ⅈ ωE_S t] );
AE_P := Ein_P 0 A_PP + Ein_X 0 A_PX + Ein_Y 0 A_PY;
ωE_p := (Ein_P 0 A_PP ω_PP + Ein_X 0 A_PX ω_PX + Ein_Y 0 A_PY ω_PY) / CAE_P;  (* /. CAE_P→AE_P *)
AE_S := Ein_S 0 A_SS + Ein_X 0 A_SX + Ein_Y 0 A_SY;
ωE_S := (Ein_S 0 A_SS ω_SS + Ein_X 0 A_SX ω_SX + Ein_Y 0 A_SY ω_SY) / CAE_S;
```

■ Summation

$$\vec{E}(t) = \begin{pmatrix} E_P(t) \\ E_S(t) \end{pmatrix}$$

$$\vec{E}(t) = \prod_{k=0}^{N-1} \overleftrightarrow{\mathrm{Min}}(t - k\,\tau) \cdot \vec{E}(t - N\,\tau) + \sum_{j=1}^{N-1} \prod_{k=0}^{j-1} \overleftrightarrow{\mathrm{Min}}(t - k\,\tau) \cdot \overleftrightarrow{\mathrm{Mext}}(t - \mathrm{j}\tau) \cdot \vec{E}_{\mathrm{ext}}(t - \mathrm{j}\tau) + \overleftrightarrow{\mathrm{Mext}}(t) \cdot \vec{E}_{\mathrm{ext}}(\mathrm{t})$$

■ Bacic strategy

$\mathrm{Min}(\mathrm{t} - \mathrm{n}\tau) \simeq \mathrm{Min0} - \mathrm{n}\ \tau\ \mathrm{Min1}$

$$\mathrm{Min0} \sim \begin{pmatrix} P \leftarrow P & P \leftarrow S \\ S \leftarrow P & S \leftarrow S \end{pmatrix}$$

$$\mathrm{Min0} = \mathrm{U\ M0}\ U^{-1} = \mathrm{U} \begin{pmatrix} \mathrm{G1} & 0 \\ 0 & \mathrm{G2} \end{pmatrix} U^{-1}$$

Internal

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = i \begin{pmatrix} A_{PXYP}\ \omega_{PXYP} & A_{PXYS}\ \omega_{PXYS} \\ A_{SXYP}\ \omega_{SXYP} & A_{SXYS}\ \omega_{SXYS} \end{pmatrix} \tau$$

$$\prod_{k=0}^{N-1} \text{Min}(t - k\,\tau) = \text{UmatIn}.(\text{M0}^N - \text{M0M1N}[N-1]).\text{InvUMatIn}$$

External

$$\sum_{j=1}^{N-1} \prod_{k=0}^{j-1} \overleftrightarrow{\text{Min}}(t - k\,\tau) \cdot \overleftrightarrow{\text{Mext}}(t - j\tau) = \sum_{j=1}^{N-1} \text{UmatIn}.(\text{M0}^j - \text{M0M1N}[j-1]).\text{InvUmatIn}.\overleftrightarrow{\text{Mext}}(t - j\tau)$$

$$\prod_{k=0}^{j-1} \text{Min}(t - k\,\tau) = \text{UmatIn}.(\text{M0}^j - \text{M0M1N}[j-1]).\text{InvUMatIn}$$

$$\text{Mexp}(t - j\,\tau) = \begin{pmatrix} AE_P\ \text{Exp}[-i\,\omega E_P\,\tau\,j] \\ AE_S\ \text{Exp}[-i\,\omega E_S\,\tau\,j] \end{pmatrix} = \begin{pmatrix} AE_P\ \text{EPT}^j \\ AE_S\ \text{EST}^j \end{pmatrix},$$

$$\text{EPT} = \text{Exp}[-i\,\omega E_P\,\tau]$$
$$\text{EST} = \text{Exp}[-i\,\omega E_S\,\tau]$$

Mathematics

Diagonalization

$$\text{Mx}[n\_] := \begin{pmatrix} A & B \\ C & D \end{pmatrix} + n \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

```
{EigVal, EigVec} = Eigensystem[Mx[0]];
DiagonalMatrix[EigVal] // MatrixForm
Umat = Transpose[EigVec] // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2}\left(A + D - \sqrt{A^2 + 4\,B\,C - 2\,A\,D + D^2}\right) & 0 \\ 0 & \frac{1}{2}\left(A + D + \sqrt{A^2 + 4\,B\,C - 2\,A\,D + D^2}\right) \end{pmatrix}$$

$$\begin{pmatrix} -\frac{-A+D+\sqrt{A^2+4\,B\,C-2\,A\,D+D^2}}{2\,C} & -\frac{-A+D-\sqrt{A^2+4\,B\,C-2\,A\,D+D^2}}{2\,C} \\ 1 & 1 \end{pmatrix}$$

Change eigenvalues so that (1,1) ((2,2)) component becomes A (D) when $\sqrt{(A-D)^2 + 4\,BC} \simeq (A-D)$

$$Z_{ABCD} := (A - D)^2 + 4\,B\,C$$

```
Umat = {{1, (A-D-√CZ_ABCD)/(2 C)}, {-(A-D-√CZ_ABCD)/(2 B), 1}};

FullSimplify[Inverse[Umat] /. CZ_ABCD -> Z_ABCD] // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2-\frac{2(A-D)}{A+\sqrt{4BC+(A-D)^2}-D}} & \frac{B}{\sqrt{4BC+(A-D)^2}} \\ -\frac{C}{\sqrt{4BC+(A-D)^2}} & \frac{1}{2-\frac{2(A-D)}{A+\sqrt{4BC+(A-D)^2}-D}} \end{pmatrix}$$

$$\frac{1}{2-\frac{2(A-D)}{A+\sqrt{4BC+(A-D)^2}-D}} = \frac{A-D+\sqrt{(A-D)^2+4BC}}{2\sqrt{(A-D)^2+4BC}} = \frac{1}{2} + \frac{A-D}{2\sqrt{Z_{ABCD}}}$$

■ **Inv Umat**

```
InvUmat = {{1/2 (1 + (A-D)/√CZ_ABCD), B/√CZ_ABCD}, {-C/√CZ_ABCD, 1/2 (1 + (A-D)/√CZ_ABCD)}};

FullSimplify[ Umat.InvUmat /. CZ_ABCD -> Z_ABCD ]
FullSimplify[ InvUmat.Umat /. CZ_ABCD -> Z_ABCD ]
```

{{1, 0}, {0, 1}}

{{1, 0}, {0, 1}}

■ **Make sure :** $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \text{Umat}.\begin{pmatrix} G1 & 0 \\ 0 & G2 \end{pmatrix}.\text{InvUmat}$

```
FullSimplify[InvUmat.{{A, B}, {C, D}}.Umat /. CZ_ABCD → Z_ABCD]
```

$$\left\{\left\{\frac{1}{2}\left(A+\sqrt{4BC+(A-D)^2}+D\right), 0\right\}, \left\{0, \frac{1}{2}\left(A-\sqrt{4BC+(A-D)^2}+D\right)\right\}\right\}$$

■ **Rotations**

```
M0 = FullSimplify[InvUmat.{{A, B}, {C, D}}.Umat /. CZ_ABCD -> Z_ABCD]
```

$$\left\{\left\{\frac{1}{2}\left(A+\sqrt{4BC+(A-D)^2}+D\right), 0\right\}, \left\{0, \frac{1}{2}\left(A-\sqrt{4BC+(A-D)^2}+D\right)\right\}\right\}$$

```
G1 = M0[[1, 1]]
G2 = M0[[2, 2]]
```

$$\frac{1}{2}\left(A+\sqrt{4\,B\,C+(A-D)^2}+D\right)$$

$$\frac{1}{2}\left(A-\sqrt{4\,B\,C+(A-D)^2}+D\right)$$

$$\mathbf{G1}=\frac{1}{2}\left(A+D+\sqrt{4\,B\,C+(A-D)^2}\right);$$

$$\mathbf{G2}=\frac{1}{2}\left(A+D-\sqrt{4\,B\,C+(A-D)^2}\right);$$

$$\mathbf{M1 = FullSimplify}\left[\mathbf{InvUmat}.\begin{pmatrix} a & b \\ c & d \end{pmatrix}.\mathbf{Umat}\ /.\ CZ_{ABCD} \to Z_{ABCD}\right]$$

$$\left\{\left\{\frac{2\,B\,c+2\,b\,C+a\left(A+\sqrt{4\,B\,C+(A-D)^2}-D\right)+d\left(-A+\sqrt{4\,B\,C+(A-D)^2}+D\right)}{2\sqrt{4\,B\,C+(A-D)^2}},\right.\right.$$
$$\left.\frac{A\,B\,c+A\,b\,C-2\,a\,B\,C+2\,B\,C\,d-B\,c\sqrt{4\,B\,C+(A-D)^2}+b\,C\sqrt{4\,B\,C+(A-D)^2}-B\,c\,D-b\,C\,D}{2\,C\sqrt{4\,B\,C+(A-D)^2}}\right\},$$
$$\left\{\frac{A\,B\,c+A\,b\,C-2\,a\,B\,C+2\,B\,C\,d+B\,c\sqrt{4\,B\,C+(A-D)^2}-b\,C\sqrt{4\,B\,C+(A-D)^2}-B\,c\,D-b\,C\,D}{2\,B\sqrt{4\,B\,C+(A-D)^2}},\right.$$
$$\left.\left.\frac{-2\,B\,c-2\,b\,C+d\left(A+\sqrt{4\,B\,C+(A-D)^2}-D\right)+a\left(-A+\sqrt{4\,B\,C+(A-D)^2}+D\right)}{2\sqrt{4\,B\,C+(A-D)^2}}\right\}\right\}$$

$$\mathbf{M1}=\left\{\left\{\frac{2\,B\,c+2\,b\,C+a\left(A+\sqrt{4\,B\,C+(A-D)^2}-D\right)+d\left(-A+\sqrt{4\,B\,C+(A-D)^2}+D\right)}{2\sqrt{4\,B\,C+(A-D)^2}},\right.\right.$$
$$\left.\frac{A\,B\,c+A\,b\,C-2\,a\,B\,C+2\,B\,C\,d-B\,c\sqrt{4\,B\,C+(A-D)^2}+b\,C\sqrt{4\,B\,C+(A-D)^2}-B\,c\,D-b\,C\,D}{2\,C\sqrt{4\,B\,C+(A-D)^2}}\right\},$$
$$\left\{\frac{A\,B\,c+A\,b\,C-2\,a\,B\,C+2\,B\,C\,d+B\,c\sqrt{4\,B\,C+(A-D)^2}-b\,C\sqrt{4\,B\,C+(A-D)^2}-B\,c\,D-b\,C\,D}{2\,B\sqrt{4\,B\,C+(A-D)^2}},\right.$$
$$\left.\left.\frac{-2\,B\,c-2\,b\,C+d\left(A+\sqrt{4\,B\,C+(A-D)^2}-D\right)+a\left(-A+\sqrt{4\,B\,C+(A-D)^2}+D\right)}{2\sqrt{4\,B\,C+(A-D)^2}}\right\}\right\};$$

■ Products

■ calculation

Uinv Mx[n] U = M0 - n M1, M0 = $\begin{pmatrix} G1 & 0 \\ 0 & G2 \end{pmatrix}$, M1 = $\begin{pmatrix} m11 & m12 \\ m21 & m22 \end{pmatrix}$

Mx[0] Mx[1] Mz[2] ... Mx[N] = U Uinv Mx[0] U Uinv Mx[1] U ... Uinv Mx[N] U Uinv
= U (M0) (M0 -M1) (M0 -2M1) ... (M0-N M1) Uinv
= U ($M0^{N+1}$ - M0 M1 $M0^{N-1}$ -2 $M0^2$ M1$M0^{N-2}$ - ... -N $M0^N$ M1) Uinv

$$\text{Sum}\left[k \begin{pmatrix} CG1^k & 0 \\ 0 & CG2^k \end{pmatrix} . \begin{pmatrix} m11 & m12 \\ m21 & m22 \end{pmatrix} . \begin{pmatrix} CG1^{N-k} & 0 \\ 0 & CG2^{N-k} \end{pmatrix}, \{k, 1, N\}\right]$$

$$\sum_{k=1}^{N} \{\{CG1^N\ k\ m11,\ CG1^k\ CG2^{-k+N}\ k\ m12\},\ \{CG1^{-k+N}\ CG2^k\ k\ m21,\ CG2^N\ k\ m22\}\}$$

```
FullSimplify[ Sum[CG1^N k m11, {k, 1, N}] ]
FullSimplify[ Sum[CG1^k CG2^(-k+N) k m12, {k, 1, N}] ]
FullSimplify[ Sum[CG1^(-k+N) CG2^k k m21, {k, 1, N}] ]
FullSimplify[ Sum[CG2^N k m22, {k, 1, N}] ]
```

$$\frac{1}{2} CG1^N\ m11\ N\ (1 + N)$$

$$\frac{CG1\ CG2^N\ m12\ \left(CG2 - \left(\frac{CG1}{CG2}\right)^N (CG2 - CG1\ N + CG2\ N)\right)}{(CG1 - CG2)^2}$$

$$\frac{CG1^N\ CG2\ m21\ \left(CG1 - \left(\frac{CG2}{CG1}\right)^N (CG1 + CG1\ N - CG2\ N)\right)}{(CG1 - CG2)^2}$$

$$\frac{1}{2} CG2^N\ m22\ N\ (1 + N)$$

Summary

Uinv Mx[n] U = M0 - n M1, M0 = $\begin{pmatrix} G1 & 0 \\ 0 & G2 \end{pmatrix}$, M1 = $\begin{pmatrix} m11 & m12 \\ m21 & m22 \end{pmatrix}$

Mx[0] Mx[1] Mz[2] ... Mx[N] = U Uinv Mx[0] U Uinv Mx[1] U ... Uinv Mx[N] U Uinv
= U (M0) (M0 -M1) (M0 -2M1) ... (M0-N M1) Uinv
= U ($M0^{N+1}$ - M0 M1 $M0^{N-1}$ -2 $M0^2$ M1$M0^{N-2}$ - ... -N $M0^N$ M1) Uinv
= U [$M0^{N+1}$ - M0M1N[N] + ...] Uinv

$$\text{M0M1N[n\_]} := \begin{pmatrix} \frac{1}{2} CG1^n\ n\ (n+1)\ m11 & \frac{CG1\ (CG2\ (CG2^n - CG1^n) - CG1^n\ (CG2 - CG1)\ n)}{(CG1-CG2)^2}\ m12 \\ \frac{CG2\ (CG1\ (CG1^n - CG2^n) - CG2^n\ (CG1 - CG2)\ n)}{(CG1-CG2)^2}\ m21 & \frac{1}{2} CG2^n\ n\ (n+1)\ m22 \end{pmatrix};$$

Numerical verification

$$\text{TM0} := \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix};$$

$$\text{TM2} := \begin{pmatrix} 3 & 7 \\ 2 & 9 \end{pmatrix};$$

```
TMN[n_] := TM0 - n ϵ TM2;
```

$$\text{Prod[n\_]} := \left\{ EE = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \text{Do[EE = TMN[k].EE, \{k, n, 0, -1\}]; EE} \right\}$$

```
Prod[1]
```

$$\{\{\{1 - 3\epsilon,\ -7\epsilon\},\ \{-4\epsilon,\ 2\ (2 - 9\epsilon)\}\}\}$$

```
Series[ Prod[10], {ϵ, 0, 1}]
Series[ Prod[5], {ϵ, 0, 1}]
```

$\{\{\{1-165\,\epsilon+O[\epsilon]^2, -14252\,\epsilon+O[\epsilon]^2\}, \{-36868\,\epsilon+O[\epsilon]^2, 2048-506880\,\epsilon+O[\epsilon]^2\}\}\}$

$\{\{\{1-45\,\epsilon+O[\epsilon]^2, -399\,\epsilon+O[\epsilon]^2\}, \{-516\,\epsilon+O[\epsilon]^2, 64-4320\,\epsilon+O[\epsilon]^2\}\}\}$

```
M0M1N[10] /. {CG1 → 1, CG2 → 2, m11 → 3, m12 → 7, m21 → 2, m22 → 9}
M0M1N[5] /. {CG1 → 1, CG2 → 2, m11 → 3, m12 → 7, m21 → 2, m22 → 9}
```

{{165, 14252}, {36868, 506880}}

{{45, 399}, {516, 4320}}

■ Misc calculations

$$\frac{A-D-\sqrt{Z_{ABCD}}}{2C} = \frac{A-D-(A-D)\sqrt{1+\frac{4BC}{(A-D)^2}}}{2C} =$$

$$\frac{A-D}{2C}\left(1-\sqrt{1+\frac{4BC}{(A-D)^2}}\right) = \frac{A-D}{2C}\,\frac{-\frac{4BC}{(A-D)^2}}{1+\sqrt{1+\frac{4BC}{(A-D)^2}}} = -\frac{2B}{A-D}\,\frac{1}{1+\sqrt{1+\frac{4BC}{(A-D)^2}}}$$

$$2(Bc+bC)+(a-d)(A-D)+(a+d)(A-D)\zeta_{ABCD} = 2(Bc+bC)+(A-D)(a-d+(a+d)\zeta_{ABCD})$$

$$(A-D)(Bc+bC)-2BC(a-d)-(Bc-bC)(A-D)\zeta_{ABCD} = (A-D)(Bc+bC-(Bc-bC)\zeta_{ABCD})-2BC(a-d)$$

$$(A-D)(Bc+bC)-2BC(a-d)+(Bc-bC)(A-D)\zeta_{ABCD} = (A-D)(Bc+bC+(Bc-bC)\zeta_{ABCD})-2BC(a-d)$$

$$-2(Bc+bC)-(a-d)(A-D)+(a+d)(A-D)\zeta_{ABCD} = -2(Bc+bC)-(A-D)((a-d)-(a+d)\zeta_{ABCD})$$

$$\frac{Bc(1-\zeta_{ABCD})}{2C\zeta_{ABCD}}+\frac{b(1+\zeta_{ABCD})}{2\zeta_{ABCD}}-\frac{B(a-d)}{(A-D)\zeta_{ABCD}}$$
$$= \frac{-Bc\frac{2B}{(A-D)^2}}{\zeta_{ABCD}(1+\zeta_{ABCD})}+\frac{b(1+\zeta_{ABCD})}{2\zeta_{ABCD}}-\frac{B(a-d)}{(A-D)\zeta_{ABCD}}$$
$$= \frac{1}{\zeta_{ABCD}}\left(\frac{1+\zeta_{ABCD}}{2}b-\frac{2B^2c}{(A-D)^2(1+\zeta_{ABCD})}-\frac{B(a-d)}{(A-D)}\right)$$

$$\frac{1}{\zeta_{ABCD}}\left(\frac{(A-D)(Bc(1+\zeta_{ABCD})+bC(1-\zeta_{ABCD}))}{2B(A-D)}-\frac{C(a-d)}{(A-D)}\right)$$
$$= \frac{1}{\zeta_{ABCD}}\left(\frac{(1+\zeta_{ABCD})}{2}c+\frac{bC\frac{(-2C)}{(A-D)^2}}{(1+\zeta_{ABCD})}-\frac{C(a-d)}{(A-D)}\right)$$
$$= \frac{1}{\zeta_{ABCD}}\left(\frac{(1+\zeta_{ABCD})}{2}c-\frac{2bC^2}{(A-D)^2(1+\zeta_{ABCD})}-\frac{C(a-d)}{(A-D)}\right)$$

■ Mathematics Summary

■ Summary formula

(* generic matrix with small perturbation *)

$$\text{Mx}[\text{n\_}] := \begin{pmatrix} A & B \\ C & D \end{pmatrix} - n \begin{pmatrix} a & b \\ c & d \end{pmatrix};$$

(* definition *)

$$\text{CZ}_{\text{ABCD}} := (A - D)^2 + 4\,B\,C;$$

$$\text{CSZ}_{\text{ABCD}} := \sqrt{1 + \frac{4\,B\,C}{(A - D)^2}};$$

(* Diagonalizing the main matrix, InvUMat * $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ * Umat = $\begin{pmatrix} G1 & 0 \\ 0 & G2 \end{pmatrix}$

$\sqrt{Z_{\text{ABCD}}}$ = (A-D) $\sqrt{1+\frac{4\,B\,C}{(A-D)^2}}$ = (A-D) SZ_{ABCD} when A≠D. This makes G1->A,

G2->D when Umat-> $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

*)

$$\text{Umat} = \begin{pmatrix} 1 & \frac{-2\,B}{(A-D)\,(1+SZ_{\text{ABCD}})} \\ \frac{2\,C}{(A-D)\,(1+SZ_{\text{ABCD}})} & 1 \end{pmatrix};$$

$$\text{InvUmat} = \begin{pmatrix} \frac{1}{2}\left(1 + \frac{1}{SZ_{\text{ABCD}}}\right) & \frac{B}{(A-D)\,SZ_{\text{ABCD}}} \\ -\frac{C}{(A-D)\,SZ_{\text{ABCD}}} & \frac{1}{2}\left(1 + \frac{1}{SZ_{\text{ABCD}}}\right) \end{pmatrix};$$

$$G1 = \frac{1}{2}\,(A + D + (A - D)\,SZ_{\text{ABCD}});$$

$$G2 = \frac{1}{2}\,(A + D - (A - D)\,SZ_{\text{ABCD}});$$

(* small matrix after rotation, M1 = InvUMat * $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ * UMat *)

$$M1 = \Bigg\{\Bigg\{\frac{2\,(B\,c + b\,C) + (A - D)\,(a - d + (a + d)\,SZ_{\text{ABCD}})}{2\,(A - D)\,SZ_{\text{ABCD}}},$$

$$\frac{1}{SZ_{\text{ABCD}}}\left(\frac{1 + SZ_{\text{ABCD}}}{2}\,b - \frac{2\,B^2\,c}{(A - D)^2\,(1 + SZ_{\text{ABCD}})} - \frac{B\,(a - d)}{(A - D)}\right)\Bigg\},$$

$$\Bigg\{\frac{1}{SZ_{\text{ABCD}}}\left(\frac{1 + SZ_{\text{ABCD}}}{2}\,c - \frac{2\,b\,C^2}{(A - D)^2\,(1 + SZ_{\text{ABCD}})} - \frac{C\,(a - d)}{(A - D)}\right),$$

$$\frac{-2\,(B\,c + b\,C) - (A - D)\,((a - d) - (a + d)\,SZ_{\text{ABCD}})}{2\,(A - D)\,SZ_{\text{ABCD}}}\Bigg\}\Bigg\};$$

(* Mx[0] Mx[1] ... Mx[N]

= Umat (M0) (M0-M1) ... (M0-N M1) InvUmat

= Umat ($M0^{N+1}$ - M0M1N[N] + ...) InvUmat

$$M0 = \begin{pmatrix} CG1 & 0 \\ 0 & CG2 \end{pmatrix},\quad M1 = \begin{pmatrix} m11 & m12 \\ m21 & m22 \end{pmatrix}$$

*)

$$\text{M0M1N}[\text{n\_}] := \begin{pmatrix} \frac{1}{2}\,CG1^n\,n\,(n+1)\,m11 & \frac{CG1\,(CG2\,(CG2^n - CG1^n) - CG1^n\,(CG2 - CG1)\,n)}{(CG1-CG2)^2}\,m12 \\ \frac{CG2\,(CG1\,(CG1^n - CG2^n) - CG2^n\,(CG1 - CG2)\,n)}{(CG1-CG2)^2}\,m21 & \frac{1}{2}\,CG2^n\,n\,(n+1)\,m22 \end{pmatrix};$$

■ Explicit Validation

$$\text{FullSimplify}\left[\text{InvUmat}.\begin{pmatrix} A & B \\ C & D \end{pmatrix}.\text{Umat} - \begin{pmatrix} G1 & 0 \\ 0 & G2 \end{pmatrix} \text{/. } SZ_{ABCD} \to CSZ_{ABCD}\right]$$

```
{{0, 0}, {0, 0}}
```

$$\text{FullSimplify}\left[\text{Umat.M1.InvUmat} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{/. } SZ_{ABCD} \to CSZ_{ABCD}\right]$$

```
{{0, 0}, {0, 0}}
```

■ M0M1N as a polynomial

$$\text{M0N1N1N}[n\_] := CG1^n \begin{pmatrix} \frac{1}{2}\, n\, (n+1)\, m11 & -\frac{CG1\,(CG2-(CG1-CG2)\, n)}{(CG1-CG2)^2}\, m12 \\ \frac{CG1\, CG2}{(CG1-CG2)^2}\, m21 & 0 \end{pmatrix};$$

$$\text{M0N1N1N0}[n\_] := CG1^n \begin{pmatrix} 0 & -\frac{CG1\, CG2}{(CG1-CG2)^2}\, m12 \\ \frac{CG1\, CG2}{(CG1-CG2)^2}\, m21 & 0 \end{pmatrix};$$

$$\text{M0N1N1N1}[n\_] := n\, CG1^n \begin{pmatrix} \frac{1}{2}\, m11 & \frac{CG1}{CG1-CG2}\, m12 \\ 0 & 0 \end{pmatrix};$$

$$\text{M0N1N1N2}[n\_] := n^2\, CG1^n \begin{pmatrix} \frac{1}{2}\, m11 & 0 \\ 0 & 0 \end{pmatrix};$$

$$\text{M0N1N2N}[n\_] := CG2^n \begin{pmatrix} 0 & \frac{CG1\, CG2}{(CG1-CG2)^2}\, m12 \\ -\frac{CG2\,(CG1+(CG1-CG2)\, n)}{(CG1-CG2)^2}\, m21 & \frac{1}{2}\, n\, (n+1)\, m22 \end{pmatrix};$$

$$\text{M0N1N2N0}[n\_] := CG2^n \begin{pmatrix} 0 & \frac{CG1\, CG2}{(CG1-CG2)^2}\, m12 \\ -\frac{CG2\, CG1}{(CG1-CG2)^2}\, m21 & 0 \end{pmatrix};$$

$$\text{M0N1N2N1}[n\_] := n\, CG2^n \begin{pmatrix} 0 & 0 \\ -\frac{CG2}{CG1-CG2}\, m21 & \frac{1}{2}\, m22 \end{pmatrix};$$

$$\text{M0N1N2N2}[n\_] := n^2\, CG2^n \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}\, m22 \end{pmatrix};$$

```
M0N1NX[n_] := CG1^n (CA10 + n CA11 + n^2 CA12) + CG2^n (CA20 + n CA21 + n^2 CA22);
```

$$CA10 := \begin{pmatrix} 0 & -\frac{CG1\ CG2}{(CG1-CG2)^2}\ m12 \\ \frac{CG1\ CG2}{(CG1-CG2)^2}\ m21 & 0 \end{pmatrix};$$

$$CA11 := \begin{pmatrix} \frac{1}{2}\ m11 & \frac{CG1}{CG1-CG2}\ m12 \\ 0 & 0 \end{pmatrix};$$

$$CA12 := \begin{pmatrix} \frac{1}{2}\ m11 & 0 \\ 0 & 0 \end{pmatrix};$$

$$CA20 := \begin{pmatrix} 0 & \frac{CG1\ CG2}{(CG1-CG2)^2}\ m12 \\ -\frac{CG2\ CG1}{(CG1-CG2)^2}\ m21 & 0 \end{pmatrix};$$

$$CA21 := \begin{pmatrix} 0 & 0 \\ -\frac{CG2}{CG1-CG2}\ m21 & \frac{1}{2}\ m22 \end{pmatrix};$$

$$CA22 := \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}\ m22 \end{pmatrix};$$

■ Misc rotation by Umat and InvUmat

Umat * M0^N * InvUmat

$$\begin{pmatrix} U11 & U12 \\ U21 & U22 \end{pmatrix} . \begin{pmatrix} CG1^N & 0 \\ 0 & CG2^N \end{pmatrix} . \begin{pmatrix} IU11 & IU12 \\ IU21 & IU22 \end{pmatrix}$$

$\{\{CG1^N\ IU11\ U11 + CG2^N\ IU21\ U12,\ CG1^N\ IU12\ U11 + CG2^N\ IU22\ U12\},$
$\{CG1^N\ IU11\ U21 + CG2^N\ IU21\ U22,\ CG1^N\ IU12\ U21 + CG2^N\ IU22\ U22\}\}$

Umat * CAxxx * InvUmat

$$\begin{pmatrix} U11 & U12 \\ U21 & U22 \end{pmatrix} . \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} . \begin{pmatrix} IU11 & IU12 \\ IU21 & IU22 \end{pmatrix}$$

{{IU11 U11, IU12 U11}, {IU11 U21, IU12 U21}}

$$\begin{pmatrix} U11 & U12 \\ U21 & U22 \end{pmatrix} . \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} . \begin{pmatrix} IU11 & IU12 \\ IU21 & IU22 \end{pmatrix}$$

{{IU21 U12, IU22 U12}, {IU21 U22, IU22 U22}}

$$\begin{pmatrix} U11 & U12 \\ U21 & U22 \end{pmatrix} . \begin{pmatrix} 0 & X12 \\ X21 & 0 \end{pmatrix} . \begin{pmatrix} IU11 & IU12 \\ IU21 & IU22 \end{pmatrix}$$

{{IU21 U11 X12 + IU11 U12 X21, IU22 U11 X12 + IU12 U12 X21},
{IU21 U21 X12 + IU11 U22 X21, IU22 U21 X12 + IU12 U22 X21}}

$$\begin{pmatrix} U11 & U12 \\ U21 & U22 \end{pmatrix} . \begin{pmatrix} X11 & X12 \\ 0 & 0 \end{pmatrix} . \begin{pmatrix} IU11 & IU12 \\ IU21 & IU22 \end{pmatrix}$$

```
{{IU11 U11 X11 + IU21 U11 X12, IU12 U11 X11 + IU22 U11 X12},
 {IU11 U21 X11 + IU21 U21 X12, IU12 U21 X11 + IU22 U21 X12}}
```

$$\begin{pmatrix} U11 & U12 \\ U21 & U22 \end{pmatrix} . \begin{pmatrix} 0 & 0 \\ X21 & X22 \end{pmatrix} . \begin{pmatrix} IU11 & IU12 \\ IU21 & IU22 \end{pmatrix}$$

```
{{IU11 U12 X21 + IU21 U12 X22, IU12 U12 X21 + IU22 U12 X22},
 {IU11 U22 X21 + IU21 U22 X22, IU12 U22 X21 + IU22 U22 X22}}
```

External sources

$$\text{M0N1NX}[n] = \text{CG1}^n\,(\text{CA10} + n\,\text{CA11} + n^2\,\text{CA12}) + \text{CG2}^n\,(\text{CA20} + n\,\text{CA21} + n^2\,\text{CA22})$$

$$\sum_{j=1}^{N-1} \prod_{k=0}^{j-1} \overleftrightarrow{\text{Min}}(t - k\,\tau) \cdot \overleftrightarrow{\text{Mext}}(t - j\tau) = \sum_{j=1}^{N-1} \text{UmatIn}.(\text{M0}^j - \text{M0M1N}[j-1]).\text{InvUmatIn}.\overleftrightarrow{\text{Mext}}(t - j\tau)$$

$$\prod_{k=0}^{j-1} \text{Min}(t - k\,\tau) = \text{UmatIn}.(\text{M0}^j - \text{M0M1N}[j-1]).\text{InvUMatIn}$$

$$\text{Mexp}\,(t - j\,\tau) = \begin{pmatrix} \text{AE}_P\,\text{Exp}[-i\,\omega\text{E}_P\,\tau\,j] \\ \text{AE}_S\,\text{Exp}[-i\,\omega\text{E}_S\,\tau\,j] \end{pmatrix} = \begin{pmatrix} \text{AE}_P\,\text{Exp}[-i\,\omega\text{E}_P\,\tau\,j] \\ \text{AE}_S\,\text{Exp}[-i\,\omega\text{E}_S\,\tau\,j] \end{pmatrix} = \begin{pmatrix} \text{AE}_P\,\text{EPT}^j \\ \text{AE}_S\,\text{EST}^j \end{pmatrix},$$

$$\text{EPT} = \text{Exp}[-i\,\omega\text{E}_P\,\tau]$$
$$\text{EST} = \text{Exp}[-i\,\omega\text{E}_S\,\tau]$$

$$\sum_{j=1}^{N-1} \text{UmatIn}\,(\text{M0}^j - \text{M0M1N}[j-1])\,\text{InvUmatIn}.\text{Mext}(t - j\tau)$$

$$= \sum_{j=0}^{N-2} \text{UmatIn}\left(\begin{pmatrix} \text{CG1}^{j+1} & 0 \\ 0 & \text{GC2}^{j+1} \end{pmatrix} - \text{M0M1N}[j]\right) \text{InvUmatIn}.\begin{pmatrix} \text{AE}_P\,\text{EPT}^{j+1} \\ \text{AE}_S\,\text{EST}^{j+1} \end{pmatrix}$$

$$= \sum_{j=0}^{N-2} \text{UmatIn}\left(\text{CG1}^j\left(\left(\begin{pmatrix} \text{CG1} & 0 \\ 0 & 0 \end{pmatrix} - \text{CA10}\right) - j\,\text{CA11} - j^2\,\text{CA12}\right) + \text{CG2}^j\left(\left(\begin{pmatrix} 0 & 0 \\ 0 & \text{CG2} \end{pmatrix} - \text{CA20}\right) - j\,\text{CA21} - j^2\,\text{CA22}\right)\right)$$
$$\text{InvUmatIn}.\begin{pmatrix} \text{AE}_P\,\text{EPT}^{j+1} \\ \text{AE}_S\,\text{EST}^{j+1} \end{pmatrix}$$

$$= \sum_{j=0}^{N-2} (\text{CG1}^j(\text{DA10} - j\,\text{DA11} - j^2\,\text{DA12}) + \text{CG2}^j(\text{DA20} - j\,\text{DA21} - j^2\,\text{DA22})).\begin{pmatrix} \text{AE}_P\,\text{EPT}^{j+1} \\ \text{AE}_S\,\text{EST}^{j+1} \end{pmatrix}$$

$$DA10 = UmatIn.\left(CG1\begin{pmatrix}1&0\\0&0\end{pmatrix} - CA10\right) InvUmatIn,$$

$$DA11 = UmatIn.CA11.InvUmatIn,\ \ DA12 = UmatIn.CA12.InvUmatIn$$

$$DA20 = UmatIn.\left(CG2\begin{pmatrix}0&0\\0&1\end{pmatrix} - CA20\right) InvUmatIn,$$

$$DA21 = UmatIn.CA21.InvUmatIn,\ \ DA22 = UmatIn.CA22.InvUmatIn$$

$$\begin{pmatrix}AE_P\ EPT^{j+1}\\AE_S\ EST^{j+1}\end{pmatrix} = \begin{pmatrix}EPT^{j+1}&0\\0&EST^{j+1}\end{pmatrix}\begin{pmatrix}AE_P\\AE_S\end{pmatrix}$$

$$\sum_{j=0}^{N-2}(CG1^j(DA10 - j\,DA11 - j^2\,DA12) + CG2^j(DA20 - j\,DA21 - j^2\,DA22)).\begin{pmatrix}AE_P\ EPT^{j+1}\\AE_S\ EST^{j+1}\end{pmatrix} =$$

$$\sum_{j=0}^{N-2}(CG1^j(DA10 - j\,DA11 - j^2\,DA12) + CG2^j(DA20 - j\,DA21 - j^2\,DA22)).\begin{pmatrix}EPT^j&0\\0&EST^j\end{pmatrix}\begin{pmatrix}AE_P\ EPT\\AE_S\ EST\end{pmatrix}$$

$$= \sum_{j=0}^{N-2}\left(DA10\begin{pmatrix}(CG1*EPT)^j&0\\0&(CG1*EST)^j\end{pmatrix} - DA11\begin{pmatrix}j(CG1*EPT)^j&0\\0&j(CG1*EST)^j\end{pmatrix} - \right.$$

$$DA12\begin{pmatrix}j^2(CG1*EPT)^j&0\\0&j^2(CG1*EST)^j\end{pmatrix} + DA20\begin{pmatrix}(CG2*EPT)^j&0\\0&(CG2*EST)^j\end{pmatrix} -$$

$$\left. DA11\begin{pmatrix}j(CG2*EPT)^j&0\\0&j(CG2*EST)^j\end{pmatrix} - DA12\begin{pmatrix}j^2(CG2*EPT)^j&0\\0&j^2(CG2*EST)^j\end{pmatrix}\right)\begin{pmatrix}AE_P\ EPT\\AE_S\ EST\end{pmatrix}$$

- **Summation for double summation**

$$\sum_{j=0}^{N} a^j$$

$$\frac{-1 + a^{1+N}}{-1 + a}$$

$$\text{Simplify}\left[\text{Series}\left[\frac{-1 + a^{1+N}}{-1 + a} \text{ /. } \{a \to 1 - \epsilon\}, \{\epsilon, 0, 3\}\right] \Big/ (1 + N)\right]$$

$$1 - \frac{N\,\epsilon}{2} + \frac{1}{6}(-1 + N)\,N\,\epsilon^2 - \frac{1}{24}((-2 + N)(-1 + N)\,N)\,\epsilon^3 + O[\epsilon]^4$$

$$\text{Simplify}\left[\text{Series}\left[\left(\frac{-1 + a^{1+N}}{-1 + a} \text{ /. } \{a \to 1 - \epsilon\}\right) \Big/ (1 + N) - \text{Exp}\left[-\frac{\epsilon\,N}{2}\right], \{\epsilon, 0, 3\}\right]\right]$$

$$\frac{1}{24}(-4 + N)\,N\,\epsilon^2 - \frac{1}{48}(N(4 - 6N + N^2))\,\epsilon^3 + O[\epsilon]^4$$

$$\sum_{j=0}^{N} j\, a^j$$

$$\frac{a\,(1 - a^N - a^N N + a^{1+N} N)}{(-1 + a)^2}$$

```
Simplify[Series[
  a (1 - a^N - a^N N + a^(1+N) N)/(-1 + a)^2 / (1/2 N (N + 1)) - Exp[-(1 + 2 N) / 3 ϵ] /. {a → 1 - ϵ}, {ϵ, 0, 3}]]
```

$$\frac{1}{36}\,(-8 - 11\,N + N^2)\,\epsilon^2 + \frac{(-152 + 87\,N + 363\,N^2 - 28\,N^3)\,\epsilon^3}{1620} + O[\epsilon]^4$$

```
Simplify[Series[
  a (1 - a^N (1 + N - a N))/(1 - a)^2 / (1/2 N (N + 1)) - Exp[-(1 + 2 N) / 3 ϵ] /. {a → 1 - ϵ}, {ϵ, 0, 3}]]
```

$$\frac{1}{36}\,(-8 - 11\,N + N^2)\,\epsilon^2 + \frac{(-152 + 87\,N + 363\,N^2 - 28\,N^3)\,\epsilon^3}{1620} + O[\epsilon]^4$$

```
-28 / 1620
```

$$-\frac{7}{405}$$

$$\sum_{j=0}^{N} j^2\, a^j$$

$$\frac{a\,(-1 - a + a^N + a^{1+N} + 2\,a^N N - 2\,a^{1+N} N + a^N N^2 - 2\,a^{1+N} N^2 + a^{2+N} N^2)}{(-1 + a)^3}$$

```
Simplify[
 Series[a (-1 - a + a^N + a^(1+N) + 2 a^N N - 2 a^(1+N) N + a^N N^2 - 2 a^(1+N) N^2 + a^(2+N) N^2)/(-1 + a)^3 / (1/6 N (1 + 3 N + 2 N^2)) -
    Exp[-(3 (N (1 + N)) ϵ)/(2 + 4 N)] /. {a → 1 - ϵ}, {ϵ, 0, 3}]]
```

$$-\frac{(4 + 34\,N + 91\,N^2 + 54\,N^3 - 3\,N^4)\,\epsilon^2}{40\,(1 + 2\,N)^2} + \frac{(-24 - 172\,N - 324\,N^2 + 199\,N^3 + 925\,N^4 + 501\,N^5 - 25\,N^6)\,\epsilon^3}{240\,(1 + 2\,N)^3} + O[\epsilon]^4$$

```
Simplify[
 Series[a (1 + a - a^N (a + (1 + N (1 - a))^2)) / (1 - a)^3 / ((1/6) N (1 + 3 N + 2 N^2)) - Exp[-(3 (N (1 + N)) ϵ)/(2 + 4 N)] /.
   {a → 1 - ϵ}, {ϵ, 0, 3}]]
```

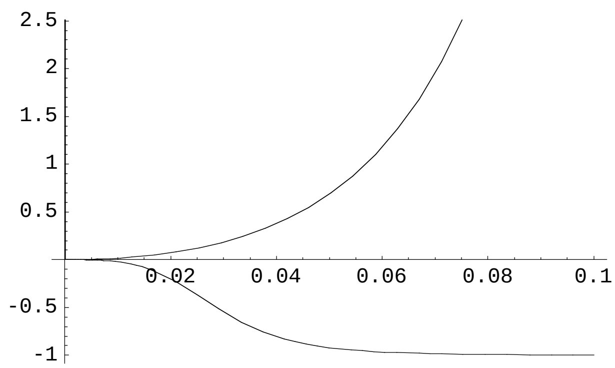
$$-\frac{(4+34N+91N^2+54N^3-3N^4)\,\epsilon^2}{40\,(1+2N)^2}+\frac{(-24-172N-324N^2+199N^3+925N^4+501N^5-25N^6)\,\epsilon^3}{240\,(1+2N)^3}+O[\epsilon]^4$$

```
25/(240 8)
```

$$\frac{5}{384}$$

```
dAJ[N_, ϵ_] := (1 - (1 - ϵ)^(1+N))/ϵ / ((1 + N) Exp[-(ϵ N)/2]) - 1;
dJAJ[N_, ϵ_] := ((1 - ϵ) (1 - (1 - ϵ)^N (1 + ϵ N)))/ϵ^2 / ((1/2) N (N + 1) Exp[-((1 + 2 N) ϵ)/3]) - 1;
dJJAJ[N_, ϵ_] :=
  ((1 - ϵ) (2 - ϵ - (1 - ϵ)^N (1 - ϵ + (1 + N ϵ)^2)))/ϵ^3 / ((1/6) N (1 + 3 N + 2 N^2) Exp[-(3 (N (1 + N)) ϵ)/(2 (1 + 2 N))]) - 1;
dJJAJ2[N_, ϵ_] := ((1 - ϵ) (2 - ϵ - (1 - ϵ)^N (1 - ϵ + (1 + N ϵ)^2)))/ϵ^3 /
    ((1/6) N (1 + 3 N + 2 N^2) (Exp[-(3 (N (1 + N)) ϵ)/(2 (1 + 2 N))] + 3/160 (N ϵ)^2)) - 1;
```

```
Plot[{dJJAJ[100, ϵ], dJJAJ2[100, ϵ]}, {ϵ, 0, 0.1}]
```



- Graphics -

```
dAJ[10, 0.1]
dAJ[100, 0.01]
dAJ[1000, 0.001]
dJAJ[10, 0.1]
dJAJ[100, 0.01]
dJAJ[1000, 0.001]
dJJAJ[10, 0.1]
dJJAJ[100, 0.01]
dJJAJ[1000, 0.001]
```

```
0.0284864

0.0408622

0.0420582

-0.0027208

0.0264814

0.02906

-0.0179656

0.01654

0.0196469
```

■ Summary

$$a = 1 - \epsilon$$

$$\sum_{j=0}^{N} a^j = \frac{1-a^{1+N}}{1-a} \simeq (1+N)\left(\mathrm{Exp}\left[-\frac{\epsilon N}{2}\right] + \frac{1}{24}(N\epsilon)^2 - \frac{1}{48}(N\epsilon)^3\right)$$

$$\sum_{j=0}^{N} j\, a^j = \frac{a\,(1-a^N\,(1+N-a\,N))}{(1-a)^2} \simeq \frac{1}{2} N\,(N+1)\left(\mathrm{Exp}\left[-\frac{(1+2N)\,\epsilon}{3}\right] + \frac{1}{36}(N\epsilon)^2 - \frac{7}{405}(N\epsilon)^3\right)$$

$$\sum_{j=0}^{N} j^2 a^j = \frac{a\,(1+a-a^N\,(a+(1+N\,(1-a))^2))}{(1-a)^3} \simeq \frac{1}{6} N\,(1+N)\,(1+2N)\left(\mathrm{Exp}\left[-\frac{3N\,(1+N)\,\epsilon}{2\,(1+2N)}\right] + \frac{3}{160}(N\epsilon)^2 - \frac{5}{384}(N\epsilon)^3\right)$$

■ Verification

```
FullSimplify[ M1 /. {A → 1, B → 2, C → 3, D → 4, a → -10, b → 21, c → -40, d → 50}]
FullSimplify[
 InvUmat.(A B
          C D).Umat /. {A → 1, B → 2, C → 3, D → 4, a → -10, b → 21, c → -40, d → 50}]
FullSimplify[InvUmat.(a b
                      c d).Umat /.
  {A → 1, B → 2, C → 3, D → 4, a → -10, b → 21, c → -40, d → 50}]
G1 /. {A → 1, B → 2, C → 3, D → 4, a → -10, b → 21, c → -40, d → 50}
G2 /. {A → 1, B → 2, C → 3, D → 4, a → -10, b → 21, c → -40, d → 50}
```

$$\left\{\left\{20-\frac{73}{\sqrt{33}},\ \frac{143}{6}-\frac{257}{2\sqrt{33}}\right\},\ \left\{\frac{1}{44}\left(-1573-257\sqrt{33}\right),\ 20+\frac{73}{\sqrt{33}}\right\}\right\}$$

$$\left\{\left\{\frac{1}{2}\left(5-\sqrt{33}\right),\ 0\right\},\ \left\{0,\ \frac{1}{2}\left(5+\sqrt{33}\right)\right\}\right\}$$

$$\left\{\left\{20-\frac{73}{\sqrt{33}},\ \frac{143}{6}-\frac{257}{2\sqrt{33}}\right\},\ \left\{\frac{1}{44}\left(-1573-257\sqrt{33}\right),\ 20+\frac{73}{\sqrt{33}}\right\}\right\}$$

$$\frac{1}{2}\left(5-\sqrt{33}\right)$$

$$\frac{1}{2}\left(5+\sqrt{33}\right)$$

```
M1 /. {A → 1, B → 0, C → 0, D → 4, a → -10, b → 21, c → -40, d → 50}
InvUmat.(A B
         C D).Umat /. {A → 1, B → 0, C → 0, D → 4, a → -10, b → 21, c → -40, d → 50}
InvUmat.(a b
         c d).Umat /. {A → 1, B → 0, C → 0, D → 4, a → -10, b → 21, c → -40, d → 50}
G1 /. {A → 1, B → 0, C → 0, D → 4, a → -10, b → 21, c → -40, d → 50}
G2 /. {A → 1, B → 0, C → 0, D → 4, a → -10, b → 21, c → -40, d → 50}
```

{{-10, 21}, {-40, 50}}

{{1, 0}, {0, 4}}

{{-10, 21}, {-40, 50}}

1

4

```
TM0 := (1 0
        0 2);
TM2 := (3 7
        2 9);
TMN[n_] := TM0 + n ∊ TM2;
Prod[n_] := {EE = (1 0
                   0 1); Do[ EE = TMN[k].EE, {k, n, 0, -1}]; EE}
```

```
Series[ Prod[10], {ϵ, 0, 1}]
Series[ Prod[5], {ϵ, 0, 1}]
```

$\{\{\{1+165\,\epsilon+O[\epsilon]^2,\ 14252\,\epsilon+O[\epsilon]^2\},\ \{36868\,\epsilon+O[\epsilon]^2,\ 2048+506880\,\epsilon+O[\epsilon]^2\}\}\}$

$\{\{\{1+45\,\epsilon+O[\epsilon]^2,\ 399\,\epsilon+O[\epsilon]^2\},\ \{516\,\epsilon+O[\epsilon]^2,\ 64+4320\,\epsilon+O[\epsilon]^2\}\}\}$

```
M0M1N[10] /. {CG1 → 1, CG2 → 2, m11 → 3, m12 → 7, m21 → 2, m22 → 9}
M0M1N[5] /. {CG1 → 1, CG2 → 2, m11 → 3, m12 → 7, m21 → 2, m22 → 9}
```

{{165, 14252}, {36868, 506880}}

{{45, 399}, {516, 4320}}

```
(M0N1N1N[10] + M0N1N2N[10]) /. {CG1 → 1, CG2 → 2, m11 → 3, m12 → 7, m21 → 2, m22 → 9}
(M0N1N1N[5] + M0N1N2N[5]) /. {CG1 → 1, CG2 → 2, m11 → 3, m12 → 7, m21 → 2, m22 → 9}
```

{{165, 14252}, {36868, 506880}}

{{45, 399}, {516, 4320}}

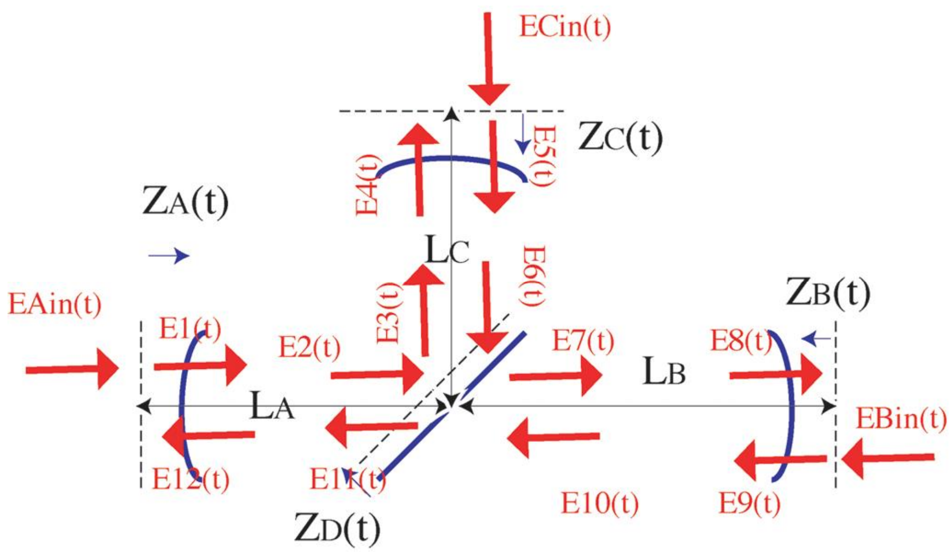
```
(M0N1N1N0[10] + M0N1N1N1[10] + M0N1N1N2[10] + M0N1N2N0[10] + M0N1N2N1[10] +
   M0N1N2N2[10]) /. {CG1 → 1, CG2 → 2, m11 → 3, m12 → 7, m21 → 2, m22 → 9}
```

{{165, 14252}, {36868, 506880}}

Fields in Power recycled Michelson cavity

I append this old note for completeness. Consistency is not guaranteed.

■ Field definition



■ Formulation - basic relationship

■ Basic formulation

■ A input calculations

```
E1lhs[t_] := tA[t] EAin[t] + rA[t] E12[t];
E2[t_] := Φ[A → D] E1[t - τA];
E3[t_] := rD[t] E2[t];
E4[t_] := Φ[D -> C] E3[t - τC];
E5[t_] := rC[t] E4[t] + tC[t] ECin[t];
E6[t_] := Φ[C -> D] E5[t - τC];
E7[t_] := tD[t] E2[t];
E8[t_] := Φ[D -> B] E7[t - τB];
E9[t_] := rB[t] E8[t] + tB[t] EBin[t];
E10[t_] := Φ[B -> D] E9[t - τB];
E11[t_] := rD[t] E6[t] + tD[t] E10[t];
E12[t_] := Φ[D -> A] E11[t - τA];
```

```
E1lhs[t]
```

```
EAin[t] tA[t] + rA[t] Φ[D → A]
  (tD[t - τA] Φ[B → D] (EBin[t - τA - τB] tB[t - τA - τB] + E1[t - 2 τA - 2 τB] rB[t - τA - τB]
        tD[t - τA - 2 τB] Φ[A → D] Φ[D → B]) + rD[t - τA] Φ[C → D] (ECin[t - τA - τC] tC[t - τA - τC] +
       E1[t - 2 τA - 2 τC] rC[t - τA - τC] rD[t - τA - 2 τC] Φ[A → D] Φ[D → C]))
```

```
E1Internal[t_] := E1lhs[t] /. {EAin[_] → 0, ECin[_] → 0, EBin[_] → 0}
```

```
E1External[t_] := Simplify[E1lhs[t] - E1Internal[t]]
```

```
E1Internal[t]
E1External[t]
```

```
rA[t] Φ[D → A]
 (E1[t - 2 τA - 2 τB] rB[t - τA - τB] tD[t - τA] tD[t - τA - 2 τB] Φ[A → D] Φ[B → D] Φ[D → B] +
   E1[t - 2 τA - 2 τC] rC[t - τA - τC] rD[t - τA] rD[t - τA - 2 τC] Φ[A → D] Φ[C → D] Φ[D → C])

EAin[t] tA[t] + rA[t] (EBin[t - τA - τB] tB[t - τA - τB] tD[t - τA] Φ[B → D] +
     ECin[t - τA - τC] rD[t - τA] tC[t - τA - τC] Φ[C → D]) Φ[D → A]
```

```
LinearMotion :=
  {rA[t_] → rA0 Exp[ⅈ 2 k vA t], rB[t_] -> rB0 Exp[ⅈ 2 k vB t], rC[t_] -> rC0 Exp[ⅈ 2 k vC t],
   rD[t_] -> rD0 Exp[ⅈ 2 k vD t], tA[t_] → tA0 , tB[t_] → tB0, tC[t_] → tC0, tD[t_] → tD0};
LinearField := {EAin[t_] → EAin0 Exp[ βA t],
   EBin[t_] -> EBin0 Exp[ βB t], ECin[t_] -> ECin0 Exp[ βC t]};
```

```
E1Internal[t] /. LinearMotion
```

$e^{2 i k t v_A}$ rA0 Φ[D → A] ($e^{2 i k v_B (t-\tau_A-\tau_B)}$ rB0 tD0^2 E1[t − 2 τ_A − 2 τ_B] Φ[A → D] Φ[B → D] Φ[D → B] +
$e^{2 i k v_D (t-\tau_A)+2 i k v_D (t-\tau_A-2\tau_C)+2 i k v_C (t-\tau_A-\tau_C)}$ rC0 rD0^2 E1[t − 2 τ_A − 2 τ_C] Φ[A → D] Φ[C → D] Φ[D → C])

```
Simplify[ⅇ^(2 ⅈ t k vA) ⅇ^(2 ⅈ k vB (t-τA-τB))]
```

$e^{2 i k (t v_A+v_B (t-\tau_A-\tau_B))}$

```
Simplify[ⅇ^(2 ⅈ t k vA) ⅇ^(2 ⅈ k vD (t-τA)+2 ⅈ k vD (t-τA-2 τC)+2 ⅈ k vC (t-τA-τC))]
```

$e^{2 i k (t v_A+(v_C+2 v_D) (t-\tau_A-\tau_C))}$

```
2 τA + 2 τB /. {τA → (τ - τB - τC) / 2} /. {τB → τC + 2 Δ}
```

$2 \Delta + \tau$

```
2 τA + 2 τC /. {τA -> (τ - τB - τC) / 2} /. {τB -> τC + 2 Δ}
```

$-2 \Delta + \tau$

E1Int = ACAamp · Exp[$i\, \phi_{ACA}$] + ABAamp · Exp[$i\, \phi_{ABA}$]

ABAamp = rA0 rB0 tD0^2 E1[t - 2 τ_A - 2 τ_B] Φ[A -> D] Φ[B -> D] Φ[D -> B] Φ[D -> A]
$\phi_{ABA} = 2\,k t\,(v_A + v_B) - 2\,k\,v_B(\tau_A + \tau_B)$

ACAamp = rA0 rC0 rD0^2 E1[t - 2 τ_A - 2 τ_C] Φ[A -> D] Φ[C -> D] Φ[D -> C] Φ[D -> A]
$\phi_{ACA} = 2\,t\,k(v_A + v_C + 2\,v_D) - 2\,k(\tau_A + \tau_C)\,(v_C + 2\,v_D)$

$\tau = 2\,\tau_A + \tau_B + \tau_C$
$2\Delta = \tau_B - \tau_C$

E1[t - 2 τ_A - 2 τ_B] = E1[t - τ - 2 Δ]
E1[t - 2 τ_A - 2 τ_C] = E1[t - τ + 2 Δ]

```
E1External[t] /. LinearMotion /. LinearField
```

$e^{t\,\beta_A}$ EAin0 tA0 + $e^{2\,i\,k\,t\,v_A}$ rA0
($e^{\beta_B\,(t-\tau_A-\tau_B)}$ EBin0 tB0 tD0 Φ[B → D] + $e^{2\,i\,k\,v_D\,(t-\tau_A)+\beta_C\,(t-\tau_A-\tau_C)}$ ECin0 rD0 tC0 Φ[C → D]) Φ[D → A]

$$2\,i\,k\,t\,v_A + \beta_B\,(t - \tau_A - \tau_B)$$

$$2\,i\,k\,t\,v_A + \beta_B\,(t - \tau_A - \tau_B)$$

$$2\,i\,k\,t\,v_A + 2\,i\,k\,v_D\,(t - \tau_A) + \beta_C\,(t - \tau_A - \tau_C)$$

$$2\,i\,k\,t\,v_A + 2\,i\,k\,v_D\,(t - \tau_A) + \beta_C\,(t - \tau_A - \tau_C)$$

E1Ext = AAamp · Exp[ϕ_{AA}] + ACamp · Exp[ϕ_{AC}] + ABamp · Exp[ϕ_{AB}]
AAamp = EAin0 · tA0
$\phi_{AA} = \beta_A\, t$
ACamp = ECin0 rA0 rD0 tC0 Φ[C->D] Φ[D->A]
ϕ_{AC} = t (i 2 k ($v_A + v_D$)+ β_C) - (i 2k v_D τ_A + β_C ($\tau_A + \tau_C$))
ABamp = EBin0 rA0 tB0 tD0 Φ[B->D] Φ[D->A]
ϕ_{AB} = t (i 2k v_A + β_B) - β_B ($\tau_A + \tau_B$)

■ Summary of recursion relation

$\tau = 2\,\tau_A + \tau_B\; + \tau_C$
$2\Delta = \tau_B - \tau_C$
$V_{\mathrm{E1}} = \mathrm{dE}_1 / \mathrm{dt}\; / \mathrm{E1}$

E1[t] = E1Int[t] + E1Ext[t]

E1Int = (C_{ACA} · Exp[$i\,\xi_{\mathrm{ACA}}\,t$] + C_{ABA} · Exp[$i\,\xi_{\mathrm{ABA}}$ t]) E1[t-τ]

C_{ACA} = Φ[A -> D] Φ[C -> D] Φ[D -> C] Φ[D -> A] × rA0 rC0 rD0^2 Exp[-i 2k ($v_C + 2\,v_D$) ($\tau_A + \tau_C$)] (1 + 2Δ V_{E1})
$\xi_{\mathrm{ACA}} = 2\,k(v_A + v_C + 2\,v_D)$

C_{ABA} = Φ[A -> D] Φ[B -> D] Φ[D -> B] Φ[D -> A] × rA0 rB0 tD0^2 Exp[-i 2k v_B ($\tau_A + \tau_B$)] (1 - 2Δ V_{E1})
$\xi_{\mathrm{ABA}} = 2\,k\;(v_A + v_B)$

E1Ext = C_A · Exp[ζ_A t] + C_C · Exp[ζ_C t] + C_B · Exp[$\zeta_B\,t$]

C_A = EAin0 · tA0
$\zeta_A = \beta_A$
C_C = Φ[C->D] Φ[D->A] × ECin0 rA0 rD0 tC0 Exp[-i 2k $v_D\;\tau_A - \beta_C\,(\tau_A + \tau_C)$]
$\zeta_C = i\;2\;\mathrm{k}\;(v_A + v_D) + \beta_C$
C_B = Φ[B->D] Φ[D->A] × EBin0 rA0 tB0 tD0 Exp[- $\beta_B\,(\tau_A + \tau_B)$]
$\zeta_B = i\;2\mathrm{k}\;v_A\; + \beta_B$

Manual calculation

A inputs

E1(t) = $t_A(t) \cdot \text{EAin}(t) + r_A(t) \cdot \text{E12}(t)$
E2(t) = Φ[A->D] E1$(t - \tau_A)$
E3(t) = $r_D(t) \cdot \text{E2}(t)$
E4(t) = Φ[D->C] E3$(t - \tau_C)$
E5(t) = $r_C(t) \cdot \text{E4}(t)$
E6(t) = Φ[C->D] · E5$(t - \tau_C)$
E7(t) = $t_D(t) \cdot \text{E2}(t)$
E8(t) = Φ[D->B] · E7$(t - \tau_B)$
E9(t) = $r_B(t) \cdot \text{E8}(t)$
E10(t) = Φ[B->D] · E9$(t - \tau_B)$
E11(t) = $r_D(t) \cdot \text{E6}(t) + t_D(t) \cdot \text{E10}(t)$
E12(t) = Φ[D->A] · E11$(t - \tau_A)$

E1(t)
$= t_A(t) \cdot \text{EAin}(t) + r_A(t) \cdot \text{E12}(t)$
$= t_A(t) \cdot \text{EAin}(t) + r_A(t) \cdot \Phi[D \to A] \cdot \text{E11}(t - \tau_A)$
$= t_A(t) \cdot \text{EAin}(t) + r_A(t) \cdot \Phi[D \to A] \cdot (r_D(t - \tau_A) \cdot \text{E6}(t - \tau_A) + t_D(t - \tau_A) \cdot \text{E10}(t - \tau_A))$
$= t_A(t) \cdot \text{EAin}(t) + r_A(t) \cdot \Phi[D \to A] \cdot (r_D(t - \tau_A) \cdot \Phi[C \to D] \cdot \text{E5}(t - \tau_A - \tau_C) + t_D(t - \tau_A) \cdot \Phi[B \to D] \cdot \text{E9}(t - \tau_A - \tau_B))$
$= t_A(t) \cdot \text{EAin}(t) + r_A(t) \cdot \Phi[D \to A] \cdot$
$(r_D(t - \tau_A) \cdot \Phi[C \to D] \cdot r_C(t - \tau_A - \tau_C) \cdot \text{E4}(t - \tau_A - \tau_C) + t_D(t - \tau_A) \cdot \Phi[B \to D] \cdot r_B(t - \tau_A - \tau_B) \cdot \text{E8}(t - \tau_A - \tau_B))$
$= t_A(t) \cdot \text{EAin}(t) + r_A(t) \cdot \Phi[D \to A] \cdot (r_D(t - \tau_A) \cdot \Phi[C \to D] \cdot r_C(t - \tau_A - \tau_C) \cdot \Phi[D \to C]\, \text{E3}(t - \tau_A - \tau_C - \tau_C) +$
$t_D(t - \tau_A) \cdot \Phi[B \to D] \cdot r_B(t - \tau_A - \tau_B) \cdot \Phi[D \to B] \cdot \text{E7}(t - \tau_A - \tau_B - \tau_B))$
$= t_A(t) \cdot \text{EAin}(t) +$
$r_A(t) \cdot \Phi[D \to A] \cdot (r_D(t - \tau_A) \cdot \Phi[C \to D] \cdot r_C(t - \tau_A - \tau_C) \cdot \Phi[D \to C]\, r_D(t - \tau_A - \tau_C - \tau_C) \cdot \text{E2}(t - \tau_A - \tau_C - \tau_C) +$
$t_D(t - \tau_A) \cdot \Phi[B \to D] \cdot r_B(t - \tau_A - \tau_B) \cdot \Phi[D \to B] \cdot t_D(t - \tau_A - \tau_B - \tau_B) \cdot \text{E2}(t - \tau_A - \tau_B - \tau_B))$
$= t_A(t) \cdot \text{EAin}(t) + r_A(t) \cdot \Phi[D \to A] \cdot$
$(r_D(t - \tau_A) \cdot \Phi[C \to D] \cdot r_C(t - \tau_A - \tau_C) \cdot \Phi[D \to C]\, r_D(t - \tau_A - \tau_C - \tau_C) \cdot \Phi[A \to D]\, \text{E1}(t - \tau_A - \tau_C - \tau_C - \tau_A) +$
$t_D(t - \tau_A) \cdot \Phi[B \to D] \cdot r_B(t - \tau_A - \tau_B) \cdot \Phi[D \to B] \cdot t_D(t - \tau_A - \tau_B - \tau_B) \cdot \Phi[A \to D]\, \text{E1}(t - \tau_A - \tau_B - \tau_B - \tau_A))$

E5(t) = $t_C(t) \cdot \text{ECin}(t) + r_C(t) \cdot \text{E4}(t)$
E6(t) = Φ[C->D] · E5$(t - \tau_C)$
E1(t) = $r_A(t) \cdot \text{E12}(t)$
E2(t) = Φ[A->D] E1$(t - \tau_A)$
E3(t) = $r_D(t) \cdot \text{E2}(t)$
E4(t) = Φ[D->C] E3$(t - \tau_C)$
E7(t) = $t_D(t) \cdot \text{E2}(t)$
E8(t) = Φ[D->B] · E7$(t - \tau_B)$
E9(t) = $r_B(t) \cdot \text{E8}(t)$
E10(t) = Φ[B->D] · E9$(t - \tau_B)$
E11(t) = $r_D(t) \cdot \text{E6}(t) + t_D(t) \cdot \text{E10}(t)$
E12(t) = Φ[D->A] · E11$(t - \tau_A)$

EA1 summary

E1(t)
$= t_A(t) \cdot \text{EAin}(t) + r_A(t) \cdot \Phi[D \to A] \cdot$
$(r_D(t-\tau_A) \cdot \Phi[C \to D] \cdot r_C(t-\tau_A-\tau_C) \cdot \Phi[D \to C]\, r_D(t-\tau_A-\tau_C-\tau_C) \cdot \Phi[A \to D]\, \text{E1}(t-\tau_A-\tau_C-\tau_C-\tau_A) +$
$t_D(t-\tau_A) \cdot \Phi[B \to D] \cdot r_B(t-\tau_A-\tau_B) \cdot \Phi[D \to B] \cdot t_D(t-\tau_A-\tau_B-\tau_B) \cdot \Phi[A \to D]\, \text{E1}(t-\tau_A-\tau_B-\tau_B-\tau_A))$

Linear calculation

$R_X[t] = R_X[0] \cdot \exp[i\, 2\, k\, X[t]] = R_X[0] \cdot \exp[\text{ i 2k } (X_0 + v_X \text{ t})]$

Extrnal input field
$\text{Fin[t]} = \text{Fin[0]} \cdot \exp[\beta_F\, t]$

E1(t)
$= t_A(t) \cdot \text{EAin}(t) + r_A(t) \cdot \Phi[D \to A] \cdot$
$(r_D(t-\tau_A) \cdot \Phi[C \to D] \cdot r_C(t-\tau_A-\tau_C) \cdot \Phi[D \to C]\, r_D(t-\tau_A-\tau_C-\tau_C) \cdot \Phi[A \to D]\, \text{E1}(t-\tau_A-\tau_C-\tau_C-\tau_A) +$
$t_D(t-\tau_A) \cdot \Phi[B \to D] \cdot r_B(t-\tau_A-\tau_B) \cdot \Phi[D \to B] \cdot t_D(t-\tau_A-\tau_B-\tau_B) \cdot \Phi[A \to D]\, \text{E1}(t-\tau_A-\tau_B-\tau_B-\tau_A))$
$= t_A(t) \cdot \text{EAin}(t) + R(t) \cdot \text{E1}(t-\tau)$

$\tau_J = L_J / c$
$\tau_{\text{JK}} = \tau_J + \tau_K$
$\tau = 2\, \tau_A + \tau_B + \tau_C$
$\Delta = (L_B - L_C)/2$
$\tau_\Delta = \Delta / c = (\tau_{\text{ABBA}} - \tau)/2 = (\tau - \tau_{\text{ACCA}})/2$

$\text{E1}(t-\tau_A-\tau_C-\tau_C-\tau_A) = \text{E1}(t-\tau_{\text{ACCA}}) = \text{E1}(t-\tau+2\,\tau_\Delta) = \text{E1}(t-\tau)(1 + V_{\text{E1}}(t-\tau)\, 2\, \tau_\Delta) \simeq$
$\text{E1}(t-\tau)(1 + V_{\text{E1}}(t)\, 2\, \tau_\Delta)$
$\text{E1}(t-\tau_A-\tau_B-\tau_B-\tau_A) = \text{E1}(t-\tau_{\text{ABBA}}) = \text{E1}(t-\tau-2\,\tau_\Delta) = \text{E1}(t-\tau)(1 - V_{\text{E1}}(t-\tau)\, 2\, \tau_\Delta) \simeq$
$\text{E1}(t-\tau)(1 - V_{\text{E1}}(t)\, 2\, \tau_\Delta)$

R(t)
$= r_A(t) \cdot \Phi[D \to A] \cdot$
$(r_D(t-\tau_A) \cdot \Phi[C \to D] \cdot r_C(t-\tau_A-\tau_C) \cdot \Phi[D \to C]\, r_D(t-\tau_A-\tau_C-\tau_C) \cdot \Phi[A \to D] \cdot (1 + V_{\text{E1}}(t)\, 2\, \tau_\Delta) +$
$t_D(t-\tau_A) \cdot \Phi[B \to D] \cdot r_B(t-\tau_A-\tau_B) \cdot \Phi[D \to B] \cdot t_D(t-\tau_A-\tau_B-\tau_B) \cdot \Phi[A \to D] \cdot (1 - V_{\text{E1}}(t)\, 2\, \tau_\Delta))$

AC(t)
$= r_A(t) \cdot \Phi[D \to A] \cdot r_D(t-\tau_A) \cdot \Phi[C \to D] \cdot r_C(t-\tau_A-\tau_C) \cdot \Phi[D \to C]\, r_D(t-\tau_A-\tau_C-\tau_C) \cdot \Phi[A \to D] \cdot (1 + V_{\text{E1}}(t)\, 2\, \tau_\Delta)$
$= \text{AC0} \cdot \exp(i\, \phi_{\text{AC}})$
$\text{AC0} = r_A(t) \cdot \Phi[D \to A] \cdot r_D(t) \cdot \Phi[C \to D] \cdot r_C(t) \cdot \Phi[D \to C]\, r_D(t) \cdot \Phi[A \to D] \cdot (1 + V_{\text{E1}}(t)\, 2\, \tau_\Delta)$
$\phi_{\text{AC}} / 2\text{k} = v_D \cdot (t-\tau_A) + v_C \cdot (t-\tau_A-\tau_C) + v_D \cdot (t-\tau_A-2\,\tau_C) = t \cdot (2\, v_D + v_C) - (2\, v_D + v_C) \cdot (\tau_A + \tau_C)$

AB(t)
$= r_A(t) \cdot \Phi[D \to A] \cdot t_D(t-\tau_A) \cdot \Phi[B \to D] \cdot r_B(t-\tau_A-\tau_B) \cdot \Phi[D \to B] \cdot t_D(t-\tau_A-\tau_B-\tau_B) \cdot \Phi[A \to D] \cdot (1 - V_{\text{E1}}(t)\, 2\, \tau_\Delta)$
$= \text{AB0} \cdot \exp(i\, \phi_{\text{AB}}(t))$
$\text{AB0} = r_A(t) \cdot \Phi[D \to A] \cdot t_D(t) \cdot \Phi[B \to D] \cdot r_B(t) \cdot \Phi[D \to B] \cdot t_D(t) \cdot \Phi[A \to D] \cdot (1 - V_{\text{E1}}(t)\, 2\, \tau_\Delta)$
$\phi_{\text{AB}} / 2\, k = v_B \cdot (t-\tau_A-\tau_B)$

C inputs

E1(t) = $r_A(t) \cdot \text{E12}(t)$
E2(t) = Φ[A->D] E1$(t-\tau_A)$
E3(t) = $r_D(t) \cdot \text{E2}(t)$
E4(t) = Φ[D->C] E3$(t-\tau_C)$
E5(t) = $t_C(t) \cdot \text{ECin}(t) + r_C(t) \cdot \text{E4}(t)$
E6(t) = Φ[C->D] · E5$(t-\tau_C)$
E7(t) = $t_D(t) \cdot \text{E2}(t)$
E8(t) = Φ[D->B] · E7$(t-\tau_B)$
E9(t) = $r_B(t) \cdot \text{E8}(t)$
E10(t) = Φ[B->D] · E9$(t-\tau_B)$
E11(t) = $r_D(t) \cdot \text{E6}(t) + t_D(t) \cdot \text{E10}(t)$
E12(t) = Φ[D->A] · E11$(t-\tau_A)$

E1(t)
$= r_A(t) \cdot \text{E12}(t)$
$= r_A(t) \cdot \Phi[D \to A] \cdot \text{E11}(t-\tau_A)$
$= r_A(t) \cdot \Phi[D \to A] \cdot (r_D(t-\tau_A) \cdot \text{E6}(t-\tau_A) + t_D(t-\tau_A) \cdot \text{E10}(t-\tau_A))$
$= r_A(t) \cdot \Phi[D \to A] \cdot (r_D(t-\tau_A) \cdot \Phi[C \to D] \cdot \text{E5}(t-\tau_A-\tau_C) + t_D(t-\tau_A) \cdot \Phi[B \to D] \cdot \text{E9}(t-\tau_A-\tau_B))$
$= r_A(t) \cdot \Phi[D \to A] \cdot (r_D(t-\tau_A) \cdot \Phi[C \to D] \cdot (t_C(t-\tau_A-\tau_C) \cdot \text{ECin}(t-\tau_A-\tau_C) + r_C(t-\tau_A-\tau_C) \cdot \text{E4}(t-\tau_A-\tau_C)) +$
$t_D(t-\tau_A) \cdot \Phi[B \to D] \cdot \text{E9}(t-\tau_A-\tau_B))$
$= r_A(t) \cdot \Phi[D \to A] \cdot r_D(t-\tau_A) \cdot \Phi[C \to D] \cdot t_C(t-\tau_A-\tau_C) \cdot \text{ECin}(t-\tau_A-\tau_C) +$
$r_A(t) \cdot \Phi[D \to A] \cdot (r_D(t-\tau_A) \cdot \Phi[C \to D] \cdot (r_C(t-\tau_A-\tau_C) \cdot \Phi[D \to C]\, \text{E3}(t-\tau_A-\tau_C-\tau_C)) +$
$t_D(t-\tau_A) \cdot \Phi[B \to D] \cdot r_B(t-\tau_A-\tau_B) \cdot \text{E8}(t-\tau_A-\tau_B))$
$= r_A(t) \cdot \Phi[D \to A] \cdot r_D(t-\tau_A) \cdot \Phi[C \to D] \cdot t_C(t-\tau_A-\tau_C) \cdot \text{ECin}(t-\tau_A-\tau_C) +$
$r_A(t) \cdot \Phi[D \to A] \cdot (r_D(t-\tau_A) \cdot \Phi[C \to D] \cdot (r_C(t-\tau_A-\tau_C) \cdot \Phi[D \to C]\, r_D(t-\tau_A-\tau_C-\tau_C) \cdot \text{E2}(t-\tau_A-\tau_C-\tau_C)) +$
$t_D(t-\tau_A) \cdot \Phi[B \to D] \cdot r_B(t-\tau_A-\tau_B) \cdot \Phi[D \to B] \cdot \text{E7}(t-\tau_A-\tau_B-\tau_B))$
$= r_A(t) \cdot \Phi[D \to A] \cdot r_D(t-\tau_A) \cdot \Phi[C \to D] \cdot t_C(t-\tau_A-\tau_C) \cdot \text{ECin}(t-\tau_A-\tau_C) +$
$r_A(t) \cdot \Phi[D \to A] \cdot$
$(r_D(t-\tau_A) \cdot \Phi[C \to D] \cdot (r_C(t-\tau_A-\tau_C) \cdot \Phi[D \to C]\, r_D(t-\tau_A-\tau_C-\tau_C) \cdot \Phi[A \to D]\, \text{E1}(t-\tau_A-\tau_C-\tau_C-\tau_A)) +$
$t_D(t-\tau_A) \cdot \Phi[B \to D] \cdot r_B(t-\tau_A-\tau_B) \cdot \Phi[D \to B] \cdot t_D(t-\tau_A-\tau_B-\tau_B) \cdot \text{E2}(t-\tau_A-\tau_B-\tau_B))$
$= r_A(t) \cdot \Phi[D \to A] \cdot r_D(t-\tau_A) \cdot \Phi[C \to D] \cdot t_C(t-\tau_A-\tau_C) \cdot \text{ECin}(t-\tau_A-\tau_C) +$
$r_A(t) \cdot \Phi[D \to A] \cdot$
$(r_D(t-\tau_A) \cdot \Phi[C \to D] \cdot (r_C(t-\tau_A-\tau_C) \cdot \Phi[D \to C]\, r_D(t-\tau_A-\tau_C-\tau_C) \cdot \Phi[A \to D]\, \text{E1}(t-\tau_A-\tau_C-\tau_C-\tau_A)) +$
$t_D(t-\tau_A) \cdot \Phi[B \to D] \cdot r_B(t-\tau_A-\tau_B) \cdot \Phi[D \to B] \cdot t_D(t-\tau_A-\tau_B-\tau_B) \cdot \Phi[A \to D]\, \text{E1}(t-\tau_A-\tau_B-\tau_B-\tau_A))$

EC1 summary

E1(t) $= r_A(t) \cdot \Phi[D \to A] \cdot r_D(t-\tau_A) \cdot \Phi[C \to D] \cdot t_C(t-\tau_A-\tau_C) \cdot \text{ECin}(t-\tau_A-\tau_C) +$
$r_A(t) \cdot \Phi[D \to A] \cdot$
$(r_D(t-\tau_A) \cdot \Phi[C \to D] \cdot (r_C(t-\tau_A-\tau_C) \cdot \Phi[D \to C]\, r_D(t-\tau_A-\tau_C-\tau_C) \cdot \Phi[A \to D]\, \text{E1}(t-\tau_A-\tau_C-\tau_C-\tau_A)) +$
$t_D(t-\tau_A) \cdot \Phi[B \to D] \cdot r_B(t-\tau_A-\tau_B) \cdot \Phi[D \to B] \cdot t_D(t-\tau_A-\tau_B-\tau_B) \cdot \Phi[A \to D]\, \text{E1}(t-\tau_A-\tau_B-\tau_B-\tau_A))$

■ EC1 Linear calculation

$R_X[\,t\,] = R_X[0] \cdot \exp[\,i\,2\,k\,X[t]\,] = R_X[0] \cdot \exp[\,\text{i}\,2\text{k}\,(\,X_0 + v_X\,\text{t}\,)\,]$

Extrnal input field
Fin[t] = Fin[0] $\cdot \exp[\,\beta_F\,t\,]$

$$\text{E1(t)} = r_A(t) \cdot \Phi[D \to A] \cdot r_D(t-\tau_A) \cdot \Phi[C \to D] \cdot t_C(t-\tau_A-\tau_C) \cdot \text{ECin}(t-\tau_A-\tau_C) + r_A(t) \cdot \Phi[D \to A] \cdot (\,r_D(t-\tau_A) \cdot \Phi[C \to D] \cdot (\,r_C(t-\tau_A-\tau_C) \cdot \Phi[D \to C]\,r_D(t-\tau_A-\tau_C-\tau_C) \cdot \Phi[A \to D]\,\text{E1}(t-\tau_A-\tau_C-\tau_C-\tau_A)) + t_D(t-\tau_A) \cdot \Phi[B \to D] \cdot r_B(t-\tau_A-\tau_B) \cdot \Phi[D \to B] \cdot t_D(t-\tau_A-\tau_B-\tau_B) \cdot \Phi[A \to D]\,\text{E1}(t-\tau_A-\tau_B-\tau_B-\tau_A))$$

$$\text{E1External(t)} = r_A(t) \cdot \Phi[D \to A] \cdot r_D(t-\tau_A) \cdot \Phi[C \to D] \cdot t_C(t-\tau_A-\tau_C) \cdot \text{ECin}(t-\tau_A-\tau_C)$$
$$= r_A(t) \cdot \Phi[D \to A] \cdot r_D(t) \cdot \exp(-i\,2\,k\,v_D\,\tau_A) \cdot \Phi[C \to D] \cdot t_C \cdot \text{ECin}(t) \cdot \exp(-\beta(\tau_A+\tau_C))$$
$$= r_A(t) \cdot \Phi[D \to A] \cdot r_D(t) \cdot \Phi[C \to D] \cdot t_C \cdot \exp(-\beta(\tau_A+\tau_C) - i\,2\,k\,v_D\,\tau_A) \cdot \text{ECin}(t)$$

E1Internal
$$= r_A(t) \cdot \Phi[D \to A] \cdot (\,r_D(t-\tau_A) \cdot \Phi[C \to D] \cdot (\,r_C(t-\tau_A-\tau_C) \cdot \Phi[D \to C]\,r_D(t-\tau_A-\tau_C-\tau_C) \cdot \Phi[A \to D]\,\text{E1}(t-\tau_A-\tau_C-\tau_C-\tau_A)) + t_D(t-\tau_A) \cdot \Phi[B \to D] \cdot r_B(t-\tau_A-\tau_B) \cdot \Phi[D \to B] \cdot t_D(t-\tau_A-\tau_B-\tau_B) \cdot \Phi[A \to D]\,\text{E1}(t-\tau_A-\tau_B-\tau_B-\tau_A))$$

■ Summart after linearlization

E1(t)
$$= \text{E1extern}(t) + r_A(t) \cdot \Phi[D \to A] \cdot (\,r_D(t-\tau_A) \cdot \Phi[C \to D] \cdot r_C(t-\tau_A-\tau_C) \cdot \Phi[D \to C]\,r_D(t-\tau_A-\tau_C-\tau_C) \cdot \Phi[A \to D]\,\text{E1}(t-\tau_A-\tau_C-\tau_C-\tau_A) + t_D(t-\tau_A) \cdot \Phi[B \to D] \cdot r_B(t-\tau_A-\tau_B) \cdot \Phi[D \to B] \cdot t_D(t-\tau_A-\tau_B-\tau_B) \cdot \Phi[A \to D]\,\text{E1}(t-\tau_A-\tau_B-\tau_B-\tau_A))$$
$$= \text{E1extern}(t) + (\,C_{\text{AC}} \cdot \exp(i\,2\text{k}\,v_{\text{AC}} \cdot t) + C_{\text{AB}} \cdot \exp(i\,2\text{k}\,v_{\text{AB}} \cdot \text{t})\,) \cdot \text{E1(t-}\tau)$$

E1extern(t) = E1Aextern(t) + E1Cextern(t) + E1Bextrern(t)

E1Aextern(t) = $t_A(t) \cdot \text{EAin}(t)$
E1Cextern(t) = $r_A(t) \cdot \Phi[D \to A] \cdot r_D(t) \cdot \Phi[C \to D] \cdot t_C \cdot \exp(-\beta_C(\tau_A+\tau_C) - i\,2\,k\,v_D\,\tau_A) \cdot \text{ECin}(t)$
E1Bextern(t) = $r_A(t) \cdot \Phi[D \to A] \cdot t_D(t) \cdot \Phi[B \to D] \cdot t_B \cdot \exp(-\beta_B(\tau_A+\tau_B)) \cdot \text{EBin}(t)$

$$C_{\text{AC}} = r_A(t) \cdot \Phi[D \to A] \cdot r_D(t) \cdot \Phi[C \to D] \cdot r_C(t) \cdot \Phi[D \to C]\,r_D(t) \cdot \Phi[A \to D] \cdot (1 + V_{\text{E1}}(t)\,2\,\tau_\Delta) \cdot \exp(-i\,2\,k\,v_{\text{AC}} \cdot (\tau_A+\tau_C))$$
$v_{\text{AC}} = 2\,v_D + v_C$

$$C_{\text{AB}} = r_A(t) \cdot \Phi[D \to A] \cdot t_D(t) \cdot \Phi[B \to D] \cdot r_B(t) \cdot \Phi[D \to B] \cdot t_D(t) \cdot \Phi[A \to D] \cdot (1 - V_{\text{E1}}(t)\,2\,\tau_\Delta) \cdot \exp(-i\,2\,k\,v_{\text{AB}} \cdot (\tau_A+\tau_B))$$
$v_{\text{AB}} = v_B$

■ Recursive formulation

■ Basic calculation

$E(t) = Eext(t) + R(t) \cdot E(t-\tau)$

$E(t) = Eext(t) + \sum_{n=1}^{N-1} Eext(t - n\tau) \prod_{m=0}^{n-1} R(t - m\tau) + \prod_{m=0}^{N-1} R(t - m\tau) \cdot E(t - N\tau)$

$E(t + N\tau) = Eext(t + N\tau) + \sum_{n=1}^{N-1} Eext(t + N\tau - n\tau) \prod_{m=0}^{n-1} R(t + N\tau - m\tau) + \prod_{m=0}^{N-1} R(t + N\tau - m\tau) \cdot E(t)$

$R(t) = C_{ACA} \cdot Exp[\, i\, \xi_{ACA}\, t] + C_{ABA} \cdot Exp[\, i\, \xi_{ABA}\, t]$

$PR(n) = \prod_{m=0}^{n-1} R(t + N\tau - m\tau) = \prod_{m=0}^{n-1} (C_{ACA} \cdot \exp(i\, \xi_{ACA} \cdot (t + N\tau - m\tau)) + C_{ABA} \cdot \exp(i \xi_{ABA} \cdot (t + N\tau - m\tau)))$
$= \prod_{m=0}^{n-1} (C_{ACA}(t + N\tau) \cdot \exp(-i\, \xi_{ACA}\, \tau m) + C_{ABA}(t + N\tau) \cdot \exp(-i\, \xi_{ABA}\, \tau m))$
$\simeq \prod_{m=0}^{n-1} C_{ACB}(t + N\tau) \cdot \exp(-i\, \xi_{ACB}(t + N\tau)\, m)$
$= C_{ACB}(t + N\tau)^n \exp(-i\, \xi_{ACB}(t + N\tau) \cdot n(n-1)/2)$

$C_{ACA}(t) = C_{ACA} \cdot \exp(i\, \xi_{ACA} \cdot t)$
$C_{ABA}(t) = C_{ABA} \cdot \exp(i\, \xi_{ABA} \cdot t)$

$C_{ABC}(t) = C_{ACA}(t) + C_{ABA}(t)$

$\xi_{ACB}(t) = \frac{\xi_{ACA}\, C_{ACA}(t) + \xi_{ABA}\, C_{ABA}(t)}{C_{ACA}(t) + C_{ABA}(t)}\, \tau$

$Eext(t) = E0 \exp(\beta\, t)$

$\sum_{n=1}^{N-1} Eext(t + N\tau - n\tau) \prod_{m=0}^{n-1} R(t + N\tau - m\tau) =$
$\sum_{n=1}^{N-1} E0 \cdot \exp(\zeta \cdot (t + N\tau)) \cdot \exp(-\zeta\, \tau\, n) \cdot C_{ACB}\, (t + N\tau)^n \exp(-i\, \xi_{ACB}(t + N\tau) \cdot n(n-1)/2)$
$= E0 \cdot \exp(\zeta \cdot (t + N\tau)) \cdot \sum_{n=1}^{N-1} \exp(-\zeta\, \tau\, n) \cdot C_{ACB}\, (t + N\tau)^n \exp(-i\, \xi_{ACB}(t + N\tau) \cdot n(n-1)/2)$
$= E0 \cdot \exp(\zeta \cdot (t + N\tau)) \cdot \sum_{n=1}^{N-1} \tilde{C}_{ACB}^{\,n} \exp(-i\, \xi_{ACB}(t + N\tau) \cdot n^2/2)$
$= E0 \cdot \exp(\zeta \cdot (t + N\tau)) \cdot \sum_{n=1}^{N-1} \tilde{C}_{ACB}^{\,n} (1 - i\, \xi_{ACB}(t + N\tau) \cdot n^2/2)$
$= E0 \cdot \exp(\zeta \cdot (t + N\tau)) \cdot \{ \frac{1 - \tilde{C}_{ACB}^{\,N}}{1 - \tilde{C}_{ACB}} - 1 - i\, \xi_{ACB}(t + N\tau)/2 \times SP2(\tilde{C}_{ACB}) \}$

$\tilde{C}_{ACB} = C_{ABC}(t + N\tau) \exp(-\zeta\, \tau + i\, \xi_{ABC}(t + N\tau)/2)$

$SP2(x) = \sum_{n=1}^{N-1} x^n \cdot n^2$

$$\sum_{n=1}^{N-1} x^n n^2$$

$$\frac{-x - x^2 + N^2 x^N + x^{1+N} + 2 N x^{1+N} - 2 N^2 x^{1+N} + x^{2+N} - 2 N x^{2+N} + N^2 x^{2+N}}{(-1 + x)^3}$$

```
FullSimplify[-x - x^2 + N^2 x^N + x^(1+N) + 2 N x^(1+N) - 2 N^2 x^(1+N) + x^(2+N) - 2 N x^(2+N) + N^2 x^(2+N)] / (-1 + x)^3
```

$$\frac{-x\,(1+x)+x^N\,(x+(N+x-N\,x)^2)}{(-1+x)^3}$$

$$SP2(x) = \frac{x\,(1+x)-x^N\,(x+(N+x-N\,x)^2)}{(1-x)^3}$$

■ Summary of recursive summation

$$E(t) = Eext(t) + R(t)\cdot E(t-\tau)$$

$$E(t+N\,\tau) = Eext(t+N\,\tau) + \sum_{n=1}^{N-1} Eext(t+N\,\tau-n\,\tau)\prod_{m=0}^{n-1} R(t+N\,\tau-m\tau) + \prod_{m=0}^{N-1} R(t+N\,\tau-m\tau)\cdot E(t)$$

$$R(t) = C_{ACA}\cdot Exp[\,i\,\xi_{ACA}\,t\,] + C_{ABA}\cdot Exp[\,i\,\xi_{ABA}\,t]$$

$$Eext(t) = E0\,exp(\,\zeta\,t\,)$$

$$E(t+N\tau) = \left\{\frac{1-\tilde{C}_{ACB}^{\,N}}{1-\tilde{C}_{ACB}} - i\,\xi_{ACB}(t+N\tau)/2\times SP2(\tilde{C}_{ACB})\right\} Eext(t+N\tau) + C_{ACB}(t+N\tau)^N\;exp(-i\,\xi_{ACB}(t+N\tau)\,N^2/2)\,E(t)$$

$$C_{ACA}(t) = C_{ACA}\cdot exp(i\,\xi_{ACA}\cdot t)$$
$$C_{ABA}(t) = C_{ABA}\cdot exp(i\,\xi_{ABA}\cdot t)$$

$$C_{ACB}(t) = C_{ACA}(t) + C_{ABA}(t)$$

$$\xi_{ACB}(t) = \frac{\xi_{ACA}\,C_{ACA}(t)+\xi_{ABA}\,C_{ABA}(t)}{C_{ACA}(t)+C_{ABA}(t)}\,\tau$$

$$\tilde{C}_{ACB} = C_{ACB}(t+N\,\tau)\,exp(\iota\,\xi_{ACB}(t+N\tau)\,/\,2)\cdot exp(-\zeta\,\tau)$$

$$SP2(x) = \frac{x\,(1+x)-x^N\,(x+(N+x-N\,x)^2)}{(1-x)^3}$$

■ Programming

■ Basics

■ Initialization before run

$\tau = 2\,\tau_A + \tau_B + \tau_C$
$\tau_{AC} = \tau_A + \tau_C$
$\tau_{AB} = \tau_A + \tau_B$
$2\Delta = \tau_B - \tau_C$

for each sideband
Φ[A->D->C->D->A] = Φ[A -> D] Φ[C -> D] Φ[D -> C] Φ[D -> A]
Φ[A->D->B->D->A] = Φ[A -> D] Φ[B -> D] Φ[D -> B] Φ[D -> A]
Φ[C->D->A] = Φ[C->D] Φ[D->A]
Φ[B->D->A] = Φ[B->D] Φ[D->A]

■ Initialization when mirror is moved

Using reference k
rA0, rB0, rC0, rD0, v_A, v_B, v_C, v_D

■ Calculation of fields

For each sideband

C_{ACA0} = Φ[A->D->C->D->A] × rA0 rC0 $\mathrm{rD0}^2$ Exp[$-i$ 2k $(\nu_C + 2\,\nu_D)\,\tau_{\mathrm{AC}}$]
C_{ABA0} = Φ[A->D->B->D->A] × rA0 rB0 $\mathrm{tD0}^2$ Exp[$-i$ 2k $\nu_B\,\tau_{\mathrm{AB}}$]

C_{C0} = Φ[C->D->A] × rA0 rD0 tC0 Exp[$-i$ 2k $\nu_D\ \tau_A$]
C_{B0} = Φ[B->D->A] × rA0 tB0 tD0

$\xi_{\mathrm{ACA}} = 2\,k(\nu_A + \nu_C + 2\,\nu_D)$
$\xi_{\mathrm{ABA}} = 2\,k\,(\nu_A + \nu_B)$

For each External source

$V_{\mathrm{E1}} = \mathrm{dE}_1 / \mathrm{dt} / \mathrm{E1}$
β for each

$C_{\mathrm{ACA}} = C_{\mathrm{ACA0}}\,(1 + 2\Delta\,V_{\mathrm{E1}})$
$C_{\mathrm{ABA}} = C_{\mathrm{ABA0}}\,(1 - 2\Delta\,V_{\mathrm{E1}})$

C_A = tA0 EAin0
$\zeta_A = \beta_A$
$C_C = C_{\mathrm{C0}}$ ECin0 Exp[$-\beta_C\,(\tau_A + \tau_C)$]
$\zeta_C = i\,2\,\mathrm{k}\,(\nu_A + \nu_D) + \beta_C$
$C_B = C_{\mathrm{B0}}$ EBin0 Exp[$-\beta_B\,(\tau_A + \tau_B)$]
$\zeta_B = i\,2\mathrm{k}\,\nu_A + \beta_B$

$$C_{\mathrm{ACA}}(\mathrm{N}\tau) = C_{\mathrm{ACA}} \cdot \exp(i\,\xi_{\mathrm{ACA}} \cdot \mathrm{N}\tau)$$
$$C_{\mathrm{ABA}}(\mathrm{N}\tau) = C_{\mathrm{ABA}} \cdot \exp(i\,\xi_{\mathrm{ABA}} \cdot \mathrm{N}\tau)$$

$$C_{\mathrm{ACB}}(\mathrm{N}\tau) = C_{\mathrm{ACA}}(\mathrm{N}\tau) + C_{\mathrm{ABA}}(\mathrm{N}\tau)$$

$$\xi_{\mathrm{ACB}}(\mathrm{N}\tau) = \frac{\xi_{\mathrm{ACA}}\,C_{\mathrm{ACA}}(\mathrm{N}\tau) + \xi_{\mathrm{ABA}}\,C_{\mathrm{ABA}}(\mathrm{N}\tau)}{C_{\mathrm{ACB}}(\mathrm{N}\tau)}\,\tau$$

$$\tilde{C}_{\mathrm{ACB}} = C_{\mathrm{ACB}}(N\,\tau)\exp(\iota\,\xi_{\mathrm{ACB}}(\mathrm{N}\tau)\,/\,2) \cdot \exp(-\zeta\,\tau)$$

$$\mathrm{E}(t+\mathrm{N}\tau) = \left\{ \frac{1-\tilde{C}_{\mathrm{ACB}}^{\,N}}{1-\tilde{C}_{\mathrm{ACB}}} - i\,\xi_{\mathrm{ACB}}(\mathrm{N}\tau)/\,2 \times \mathrm{SP2}(\tilde{C}_{\mathrm{ACB}}) \right\} \mathrm{Eext}(\mathrm{N}\tau) + C_{\mathrm{ACB}}(\mathrm{N}\tau)^N\ \exp(-i\,\xi_{\mathrm{ACB}}(\mathrm{N}\tau)\,N^2/2)\,\mathrm{E}(t)$$

$$\mathrm{SP2}(x) = \frac{x\,(1+x) - x^N\,(x + (N + x - N\,x)^2)}{(1-x)^3}$$