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Refer to:	LIGO-T030275-00
Date:	December 2, 2003

Subject: **Angular Stability in a Triangular Fabry-Perot Cavity**

Summary:

This work is an extension of the angular instabilities in Fabry-Perot cavities described in T030120 to include the LIGO mode cleaner. Basically, the same conclusion is reached: the mode cleaner is intrinsically unstable at high powers.

Geometry of a Triangular Cavity:

Following notes from Dave Ottoway we use the ray matrix formalism (see for example Siegman, Chap. 15, "ABCD matrices") to calculate the position of the optical axis as function of the mirror angles. In the ray matrix formalism the coordinates of a beam are described by its position, x, and its slope, x', ie.,

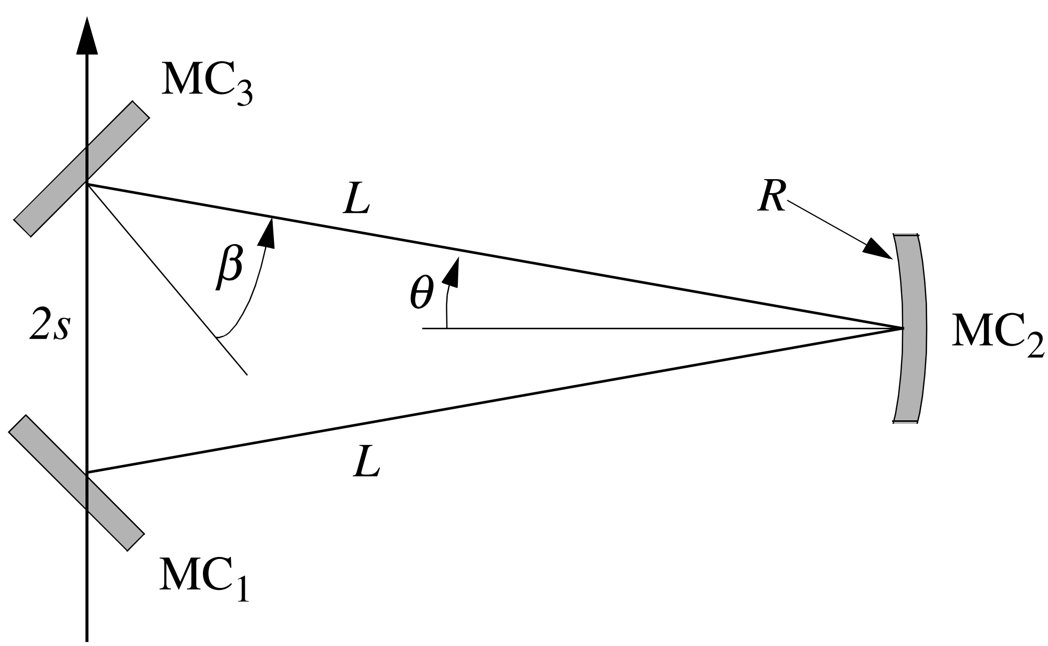

Figure 1: Triangular Cavity.

$$\boldsymbol{x}(z) = \begin{pmatrix} x(z) \\ n(z)\frac{d}{dz}x(z) \end{pmatrix} \tag{2}$$

where z is the coordinate along the beam axis and $n(z)$ denotes the refractive index. Since all our beams propagate in vacuum we set $n(z) = 1$ from now on. The free space propagation along L and $2s$ is then described by

$$D_L = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}, \quad D_s = \begin{pmatrix} 1 & 2s \\ 0 & 1 \end{pmatrix}. \tag{3}$$

The reflection from the curved mirror is described by

$$M_2 = \begin{pmatrix} 1 & 0 \\ -2/R_e & 1 \end{pmatrix} \tag{4}$$

with $R_e = R\cos\theta$ in the horizontal plane and $R_e = R/\cos\theta$ in the vertical plane. The reflection from the flat mirrors MC_1 and MC_3 is described by the identity matrix, ie., $M_1 = M_3 = \mathbf{1}$. Since the coordinates flip left-to-right in the horizontal plane after each refleciton from a mirror, we also need to multiply each mirror operator by $M_{\text{flip}}(h) = -\mathbf{1}$ when calculating misalignments in the horizontal plane and by $M_{\text{flip}}(v) = \mathbf{1}$ when calculating misalignments in the vertical plane.

Misalignments are simply introduced by adding vectors of the form

$$\boldsymbol{\alpha}_i(h) = \begin{pmatrix} 0 \\ 2\alpha_i \end{pmatrix}, \tag{5}$$

$$a_1(v) = \begin{pmatrix} 0 \\ 2\cos\beta\ \alpha_1 \end{pmatrix}, \quad a_2(v) = \begin{pmatrix} 0 \\ 2\alpha_1 \end{pmatrix} \quad \text{and} \quad a_3(v) = \begin{pmatrix} 0 \\ 2\cos\beta\ \alpha_3 \end{pmatrix} \tag{6}$$

where α_i is the misalignment of the *i*-th mirror, respectively. Since the incident angle is non-normal a vertical tilt of the MC_1 and MC_3 mirrors by α_1 and α_3, respectively, will give a beam deflection of only $2\cos\beta\ \alpha_i$ with $\beta = (\pi/2 - \theta)/2$. Since we assume $\theta \ll 1$, the factor $\cos\theta$ in $a_2(v)$ has been neglected.

We now define the vectors, $\boldsymbol{v}_i$, that describe the beam after each mirror:

$$\begin{aligned} \boldsymbol{v}_1 &= M_{\text{flip}}D_L(M_{\text{flip}}M_2D_L(M_{\text{flip}}D_s\,\boldsymbol{v}_1 + \boldsymbol{\alpha}_3) + \boldsymbol{\alpha}_2) + \boldsymbol{\alpha}_1 \\ &= (\mathbf{1} - M_{\text{flip}}MD_LM_{\text{flip}}M_2D_LM_{\text{flip}}D_s)^{-1}(M_{\text{flip}}D_L(M_{\text{flip}}M_2D_L\boldsymbol{\alpha}_3 + \boldsymbol{\alpha}_2) + \boldsymbol{\alpha}_1) \\ \boldsymbol{v}_3 &= M_{\text{flip}}D_s\boldsymbol{v}_1 + \boldsymbol{\alpha}_3 \\ \boldsymbol{v}_2 &= M_{\text{flip}}M_2D_L\boldsymbol{v}_3 + \boldsymbol{\alpha}_2 \end{aligned} \tag{7}$$

Taking only the position coordinates of the vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ we can form a new vector, $\Delta\boldsymbol{x}$, which describes the beam position on each of the mirorrs. For MC_1 and MC_3 the first components of the vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_3$ describe the displacement perpendicular to the beam. Hence, the true displacement on these mirrors is enhanced by a factor of $1/\cos\beta$. Finally, we rotate into a basis of common and differential motion for MC_1 and MC_3, and get:

$$\Delta\boldsymbol{x}(h) = \begin{pmatrix} (x_1+x_3)/(\sqrt{2}\cos\beta) \\ (x_1-x_3)/(\sqrt{2}\cos\beta) \\ x_2 \end{pmatrix}, \quad \Delta\boldsymbol{x}(v) = \begin{pmatrix} (x_1+x_3)/\sqrt{2} \\ (x_1-x_3)/\sqrt{2} \\ x_2 \end{pmatrix} \quad \text{and} \tag{8}$$

$$\Delta\alpha = \begin{pmatrix} (\alpha_1+\alpha_3)/\sqrt{2} \\ (\alpha_1-\alpha_3)/\sqrt{2} \\ \alpha_2 \end{pmatrix}. \tag{9}$$

We now rewrite equation (7) into

$$\Delta\boldsymbol{x}/L = K\,\Delta\alpha \tag{10}$$

$$\text{with} \quad K(h) = \frac{1}{L(s+L-R_e)}\begin{pmatrix} -2s(L-R_e)\sec\beta & 0 & -sR_e\sec\beta \\ 0 & -2L\sec\beta(s+L-R_e) & 0 \\ -sR_e & 0 & (s+L)R_e \end{pmatrix} \tag{11}$$

$$\text{and} \quad K(v) = \frac{1}{L}\begin{pmatrix} -2(L-R_e)\cos\beta & 0 & \sqrt{2}R_e \\ 0 & -2sL\cos\beta/(s+L) & 0 \\ 2R_e\cos\beta & 0 & R_e \end{pmatrix}. \tag{12}$$

In the limit $s \to 0$, $\theta \to 0$, $\beta \to \pi/4$ and $R_e \to L/(1-g)$ with $g = 1 - L/R$ we obtain

$$K(h) \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2\sqrt{2} & 0 \\ 0 & 0 & -1/g \end{pmatrix} \quad \text{and} \tag{13}$$

$$K(v) \to \frac{1}{1-g}\begin{pmatrix} \sqrt{2}g & 0 & \sqrt{2} \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}. \tag{14}$$

We see that common angular misalignment of MC_1 and MC_3 in the horizontal plane produces no displacement of the beam in $K(h)$. This can be easily understood by recognizing that two mirrors fixed at right angle act like a corner cube. Similarly, a differential misalignment of MC_1 and MC_3 in the vertical plane will have no effect on the beam positions, since the beam path between the two mirrors is of negligible length and the two misalignments compensate each other. We can therefore neglect these degrees-of-freedom from our further analysis.

Radiation Pressure and Torsion Pendulum:

We follow the analysis outlined in T030120. In the first section we derived the K-matrices which describe the geometry of a triangular cavity the same way as the K-matrix in the analysis of T030210. There is one slight difference, however, the torque from radiation pressure depends on the incident angle by

$$T_{1,2} = \frac{2P}{c}\cos\beta\,\Delta x_{1,2} \quad \text{and} \quad T_3 = \frac{2P}{c}\cos\theta\,\Delta x_3. \tag{15}$$

We can absorbe this difference into the K-matrix by forming

$$\bar{K} = T_\Delta K \quad \text{with } T_\Delta = \begin{pmatrix} \cos\beta & 0 & 0 \\ 0 & \cos\beta & 0 \\ 0 & 0 & \cos\theta \end{pmatrix}. \tag{16}$$

Thus, we can study the stability of a triangular cavity by computing the eigenvalues of the modified $\bar{K}$-matrices. Neglecting the common motion of MC_1 and MC_3 and using the same approximations as above, $\theta \to 0$ and $\beta \to \pi/4$, we get in the horizontal plane:

$$k_1(h) = -2 \quad \text{and} \quad k_2(h) = -\frac{1}{g}. \tag{17}$$

We see that both eigenvalues are negative and, therefore, a triangular cavity is intrinsically stable in the horizontal plane. The eigenvectors are then simply the differential motion of MC_1 and MC_3, and the motion of MC_2 alone

$$\kappa_1(h) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \kappa_2(h) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{18}$$

Neglecting the differential motion of MC_1 and MC_3 the eigenvalues and eigenvectors in the vertical plane become

$$\begin{array}{ll} k_1(v) = \dfrac{1+g-\sqrt{5-2g+g^2}}{2(1-g)} & \kappa_1(v) = \dfrac{1}{2}\begin{pmatrix} -1+g-\sqrt{5-2g+g^2} \\ 2 \end{pmatrix} \\ k_2(v) = \dfrac{1+g+\sqrt{5-2g+g^2}}{2(1-g)} & \kappa_2(v) = \dfrac{1}{2}\begin{pmatrix} -1+g+\sqrt{5-2g+g^2} \\ 2 \end{pmatrix} \end{array}. \tag{19}$$

The determinants are

$$\det K(h) = \frac{2}{g} \quad \text{and} \quad \det K(v) = -\frac{1}{1-g}. \tag{20}$$

Looking at equation (10) from T030120 we see that the vertical case is identical to the simple Fabry-Perot, if we make the substitutions $g_1 \to 1$ and $g_2 \to g$. We also see that for a stable triangular cavity there is always one intrinsically stable and one intrinsically unstable eigenmode in the vertical plane. On the other hand the two eignemodes in the horizontal plane are always intrinsically stable.

LIGO mode cleaners:

For LIGO we are using the following small optics suspension parameters:

Table 1: LIGO small optics suspension parameters

Parameter	Description	LIGO / Adv. LIGO	Unit
M	mirror mass (SOS)	0.243 / 3.04	kg
R	mirror radius (SOS)	0.0375 / 0.075	m
h	mirror thickness (SOS)	0.025 / 0.08	m
ω	pitch angular frequency (SOS)	$2\pi\,0.80$ / $2\pi\,1.08$	rad/s
ω	yaw angular frequency (SOS)	$2\pi\,0.87$ / $2\pi\,1.08$	rad/s
Θ	angular moment (SOS)	9.81×10^{-5} / 5.8×10^{-3}	kg m^2

The mode cleaners have the following parameters (Adv. LIGO cavity parameters were chosen to be the same as the 2K):

Table 2: LIGO mode cleaner parameter

Parameter	4K MC		2K MC		Adv. LIGO	
	Value	Unit	Value	Unit	Value	Unit
L	12.24	m	15.251	m	15.25	m
MC2 curvature	17.25	m	21.50	m	21.5	m
g	0.290		0.291		0.291	
P_{crit}	12.6	kW	10.1	kW	1.09	MW
Parameter	**horizontal**		**vertical**			
k_1	–2.00		–0.586			
k_2	–3.44		2.41			

where P_{crit} is calculated from Eqn. (9) in T030120 using the largest positive k_i. Since the mode cleaner parameters are almost identical we just give the eigenvalues for horizontal and vertical of the LIGO 4K. With a power build-up factor of about 500 the critical power at the input to the mode cleaner is 25W and 20W for 4K and 2K, respectively.

References:

[1] John A. Sidles and Daniel Sigg, *“Optical Torques in Suspended Fabry-Perot Interferometers,”* LIGO-P030055-B.

[2] Daniel Sigg, *“Angular Instabilities in High Power Fabry-Perot Cavities,”* LIGO-T030120-00.