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Measuring The Spring Constant Of A Cantilever Leaf Spring

Giles Hammond

California Institute of Technology
LIGO Project – MS 51-33
Pasadena CA 91125
Phone (626) 395-2129
Fax (626) 304-9834
E-mail: info@ligo.caltech.edu

Massachusetts Institute of Technology
LIGO Project – MS 20B-145
Cambridge, MA 01239
Phone (617) 253-4824
Fax (617) 253-7014
E-mail: info@ligo.mit.edu

WWW: http://www.ligo.caltech.edu

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1. Introduction

This brief document outlines two possible methods for determining the spring constant of a cantilever leaf spring. The first method involves applying a variable static load to the spring and measuring the corresponding profile. By varying the load a best fit value for the deflection and stiffness can be established. The second method is a dynamic measurement that involves oscillating the spring under a given load. By measuring the frequency of oscillation an estimate for the stiffness can also be obtained.

A key to the terminology used in the document follows:

k=spring constant (N/m)
E=Young's Modulus (N/m^2)
L=spring length (m)
w=spring width (m)
t=spring thickness (m)
m=mass (kg)
P=load (N)
P_{flat}=flat load (N)
x=distance from clamped end (m)
I=second moment of area (m^4)
M=bending moment (Nm)
y=deflection
f=frequency
R^S=squared residuals (m^2)
σ=standard deviation (m)
δ=standard error

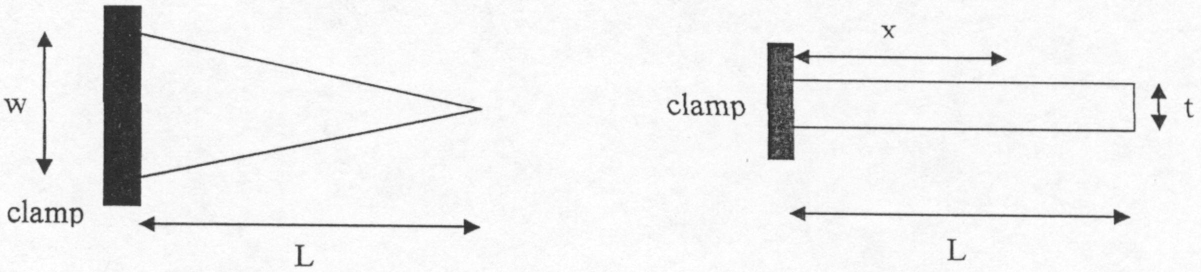

Figure 1. (a) plan view of the cantilever spring (b) side view of the cantilever spring

2. Deflection of the Leaf Spring

The deflection of the spring may be obtained by considering the bending moment acting at any point along the spring. Roark[1] gives an expression that relates the curvature of any point of the spring to the bending moment as

$$\frac{\partial^2 y}{\partial x^2} = \frac{M(x)}{EI(x)} \qquad (1)$$

where the bending moment M(x)=P(L-x) and the second moment of area varies along the length of the spring and is given by

$$I(x) = \frac{w(x)t^3}{12} = \frac{wt^3(L-x)}{12L} \qquad (2)$$

For example at x=0 the second moment of area is $wt^3/12$ while at x=L the second moment of area is zero (as the width of the spring tends to zero). Integrating equation (1), after substitution of (2), gives

$$\frac{\partial y}{\partial x} = \frac{12PLx}{Ewt^3} + A \qquad (3)$$

The constant of integration A=0 because the slope of the spring equals zero at x=0. Integrating equation (3) finally gives the deflection

$$y = \frac{6PLx^2}{Ewt^3} + B \qquad (4)$$

The integration constant B=0 also because the deflection equals zero at x=0. Thus the deflection of the spring at any point along its length is

$$y = \frac{6PLx^2}{Ewt^3} \qquad (5)$$

3. Method 1: Static Deflection

By applying a load to a point along the spring an estimate can be made of its spring constant. We can assume that for "small deflections" (see notes 3.1) the spring obeys Hooke's Law and thus P=ky. From equation (5) we find that the stiffness is given by

$$k = \frac{Ewt^3}{6L^3} \qquad (6)$$

where we have also assumed that x=L (i.e. the deflection is measured at the point at which the load is applied).

A measurement of the spring constant may be realised using a setup shown in figure 2. In this figure the clamped spring is loaded at its tip with a known load P. A coordinate measuring machine is used to measure the deflection as a function of the distance from the clamped end. The point at which the spring is flattest can be obtained by considering the residuals between the measured points $y^{meas}(x)$ (taken on the upper surface of the spring) and measurements of a known flat plate $y^{base}(x)$ (this could be the granite base plate of the coordinate measuring machine). Consider that N measurements of the spring deflection and base plate deflection are taken such that $y^{meas}(x)=y_1^{meas}(x)$, $y_2^{meas}(x),\ldots, y_N^{meas}(x)$ and $y^{base}(x)=y_1^{base}(x)$, $y_2^{base}(x),\ldots, y_N^{base}(x)$. The squared residuals of the measurement are defined as

$$R^s_{static} = \sum_{i=1}^{N} \left(y_i^{meas} - y_i^{base}\right)^2 \tag{7}$$

and the standard deviation of the measurement is given by

$$\sigma_{static} = \sqrt{\frac{R^S_{static}}{N-1}} \tag{8}$$

The calculation of the standard deviation can be conveniently achieved using EXCEL (see Appendix A and the attached file SPRING.XLS). The load can then be adjusted until the standard deviation of the spring is minimised (we can call this the "flat load"). Once this has been achieved the spring is at its flattest and measurements of the metrology of the experiment together with equation (6) can be used to estimate the spring stiffness. In order to estimate the errors affecting the measurement we can use an expression derived in Lyon[2]

$$\delta_k^2 = \sum_v \left(\frac{\partial k}{\partial v}\delta_v\right)^2 \tag{9}$$

where δ_k is the stiffness error, $\partial k/\partial v$ are derivatives over all variables in equation (6) and δ_v are the errors on each variable v. The error can be shown to be

$$\frac{\delta_k}{k} = \sqrt{\left(\frac{\delta_E}{E}\right)^2 + \left(\frac{\delta_w}{w}\right)^2 + \left(\frac{3\delta_t}{t}\right)^2 + \left(\frac{3\delta_L}{L}\right)^2} \tag{10}$$

It is interesting to note that if each quantity is known to the same accuracy then the errors on the spring thickness and length are the largest contributions. This results from the fact that the stiffness depends cubically on these parameters.

3.1 Notes on Method 1

- The derivation of equation (5) assumed a perfect triangular profile spring where in reality the design may include an extended foot where the load is applied. As a result it would probably be wise to use Finite Element Analysis (FEA) to compare the theoretical deflection of the spring to the measurements. Furthermore the FEA could start with a curved spring onto which the load is applied. In this scenario the load acts to flatten the spring, which is identical to the measurements. FEA could also ensure that P=ky is a suitable approximation (used to derive equation (6)).
- It would also be interesting to apply larger and smaller loads each side of the "flat load" in order to ensure that the spring behaves linearly. For example one could consider applying loads of $P_{flat} \pm \delta P$, in which case it would be expected that the deflection would be $y_{flat} \pm \delta y$ and the stiffness

$$k = \frac{\left(P_{flat} \pm \delta P\right)}{\left(y_{flat} \pm \delta y\right)} \qquad (11)$$

where it is assumed that a positive load causes a downward, or positive, increase in the deflection. In this expression y is measured at the position where the load is applied. By plotting the load versus the deflection the linearity of the spring can be assessed (P=ky) while the gradient of the resulting line would give the stiffness.

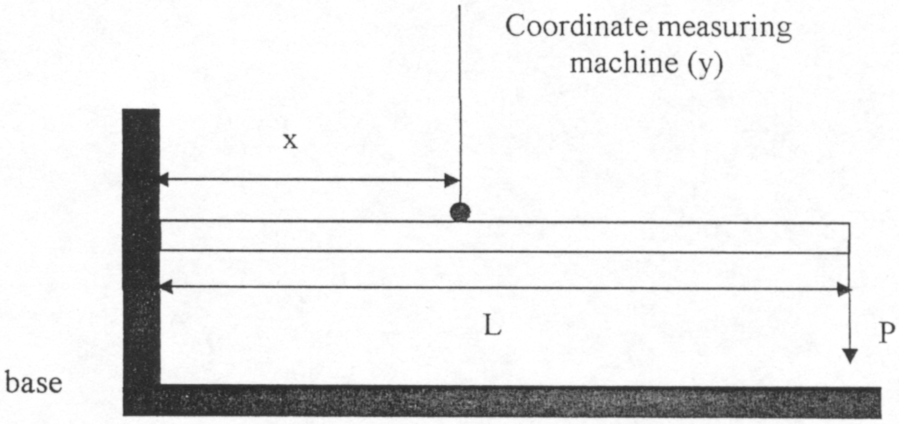

Figure 2. Static deflection method utilising a coordinate measuring machine

4. Method 2: Dynamic Response

An estimate for the stiffness of the cantilever spring may also be obtained from a dynamic measurement. Consider that the spring is loaded with the "flat load", P_{flat}, and that the fundamental cantilever mode is excited. The frequency of the oscillation is given by

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{kg}{P}} \qquad (12)$$

where P=mg. The measurement may be performed using the setup shown in figure 3. In this figure the cantilever mode is excited with a small tap on the end of the spring and the resulting oscillation is measured by a suitable sensor. For example this could be an inductive proximity sensor that senses the maraging steel foot of the spring. It should be noted that it is not critical to accurately position the sensor as it is only the frequency of oscillation that is of interest. The frequency can be obtained by measuring the voltage output of the sensor as a function of time $V^{meas}(t)$. It is convenient to store the voltage measurements on a storage oscilloscope or a spectrum analyser. Once a set of voltage versus time measurements have been obtained it is possible to find the frequency by assuming that the spring oscillates sinusoidally

$$V_{theory}(t) = V_{OUT}\sin(2\pi f t) \qquad (13)$$

where V_{OUT} is the amplitude of the oscillation and it is assumed that damping in the spring is small (it can be included if needed). We can utilise the method of residuals to find the best frequency as follows

$$R^{S}_{dynamic} = \sum_{i=1}^{N}\left(V_i^{meas} - V_i^{theory}\right)^2 \qquad (14)$$

and the standard deviation is

$$\sigma_{dynamic} = \sqrt{\frac{R^{S}_{dynamic}}{N-1}} \qquad (15)$$

where we have used the same notation as the static deflection method of residuals. The frequency can then be adjusted until the standard deviation is minimised.

The error on the measurement can be obtained from equation (9) as

$$\frac{\delta_k}{k} = \sqrt{\left(\frac{2\delta_f}{f}\right)^2 + \left(\frac{\delta_P}{P}\right)^2} \qquad (16)$$

4.1 Notes on Method 2

- It would also be interesting to apply larger and smaller loads each side of the "flat load" in order to measure the frequency response. For example one could consider applying loads of $P_{flat}\pm\delta P$, in which case it would be expected that the incremental frequency change would be

$$f_{flat} \mp \delta f = \frac{1}{2\pi}\sqrt{\frac{kg}{P_{flat} \pm \delta P}} \qquad (17)$$

where it is assumed that a positive load causes a reduction in the frequency. By plotting the frequency squared versus the inverse of the load the gradient, β, of the plot would be

$$\beta = \frac{kg}{4\pi^2} \qquad (18)$$

from which the stiffness could be obtained.

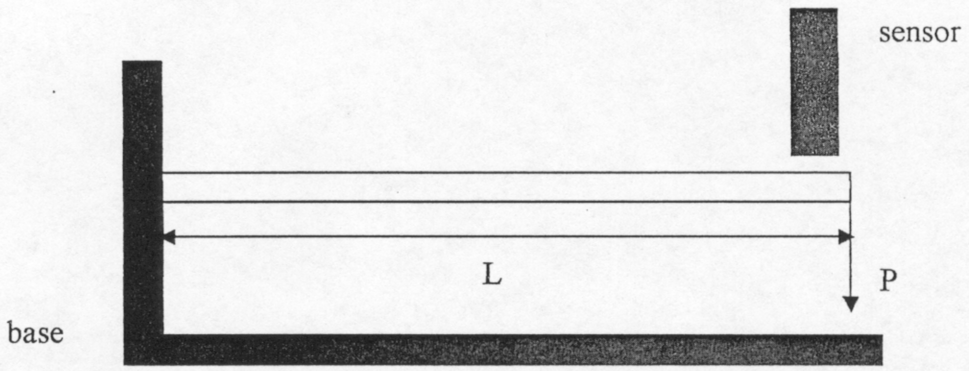

Figure 3. Dynamic response method for measuring the spring constant

5. References

[1]Young, W.C., Roark's Formulas for Stress and Strain, 6th Edition, McGraw-Hill International Series, p.95 (1989)

[2]Lyon, L., A Practical Guide to Data Analysis for Physical Science Students, Cambridge University Press, p.26 (1991)

6. Appendix A

The deflection of the leaf spring is given by equation (5) as

$$y = \gamma x^2 \qquad (5)$$

where the integration constant B has been set to zero and $\gamma = 6PL/Ewt^3$. Typically, for manufacturing purposes, the profile is approximated by a single radius of curvature and this can be represented by the equation for a circle

$$y = R - \sqrt{R^2 - x^2} = R - R\sqrt{1 - \left(\frac{x}{R}\right)^2} \qquad \text{(A1)}$$

where R is the radius of curvature and the first term on the left hand side assures that y=0 at x=0. If $x \ll R$ (large radius of curvature or small deflection), then equation (A1) can be expanded to first order in $(x/R)^2$ to give

$$y = R - R\left[1 - \frac{1}{2}\left(\frac{x}{R}\right)^2\right] = \frac{1}{2}\frac{x^2}{R} \qquad \text{(A2)}$$

thus $y = \gamma x^2$ where $\gamma = 1/2R$. In this case a single radius of curvature can be used to give a good approximation to the required parabolic profile for the spring. An example is shown in the attached EXCEL file (SPRING.XLS) under the heading spring 1. For spring 1 $x/R \ll 1$ and the expression $\gamma = 1/2R$ holds quite well. For smaller radii of curvature where x/R is no longer much less than 1 the expression $\gamma = 1/2R$ becomes more approximate. This is shown for spring 2 where the approximation to the parabolic profile is not as good. In each example the squared residuals and the standard deviation are used to estimate the best fit to the theoretical profile. It is evident that the best fit for spring 2 is not when the tip of the springs are coincident and this would presumably lead to an error in the load required to flatten the spring.