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Time-domain implementation of the optimum kernel in a correlation based triggered search

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1 Definitions

The cross correlation $\langle x_1, x_2\rangle$ as defined in [1] can be written in the general form,

$$\langle x_1, x_2\rangle = \int_a^b \int_a^b dt dt' x_1(t) Q(|t-t'|) x_2(t') \ , \tag{1}$$

where the trigger direction dependent time translation has already been applied to $x_2(t)$. The origin $t = 0$ can be identified with the arrival time of the GRB in $x_1(t)$. Both $x_1(t)$ and $x_2(t)$ are assumed to be in units of strain.

The expression for the optimum kernel is given in the Fourier domain by,

$$\widetilde{Q}(f) = \frac{\widetilde{h}_1(f)\widetilde{h}_2^*(f)}{S_1(f)S_2(f)} \ , \tag{2}$$

where $\widetilde{h}_i(f)$, $i = 1, 2$, is the GW signal in the $i^{\rm th}$ IFO and $S_i(f)$ is the PSD of $x_i(t)$.

2 The kernel as a filter

From the above definitions, we can rewrite $\langle x_1, x_2\rangle$ as

$$\langle x_1, x_2\rangle = \int_{-\infty}^{\infty} dt \Theta(t) x_1(t) y_2(t) \ , \tag{3}$$

$$y_2(t) = \int_{-\infty}^{\infty} dt' \Theta(t') Q(|t-t'|) x_2(t') \ , \tag{4}$$

where $\Theta(t) = 1$ for $a \leq t \leq b$ and zero otherwise. Eq. 4 essentially states that $y_2(t)$ can be obtained by boxcar windowing $x_2(t)$ and then using a filter with impulse response $Q(t)$. If the impulse response duration is $\ll b - a$, then apart from ringing at the window edges, $y_2(t)$ is the same as $x_2(t)$ filtered through $Q(t)$,

$$y_2(t) \simeq \Theta(t) \int_{-\infty}^{\infty} dt' Q(|t-t'|) x_2(t') \ . \tag{5}$$

Thus, $\langle x_1, x_2\rangle$ can be obtained by first filtering $x_2(t)$ to get $y_2(t)$ and then taking an ordinary cross-correlation between $x_1(t)$ and $y_2(t)$.

3 Symmetric application of the Kernel Filter

In Section 2, the cross-correlation kernel was interpreted in terms of a time domain filter. Our aim here is to recast the implementation of $\langle x_1, x_2\rangle$ in a form that treats $x_1(t)$ and $x_2(t)$ in a more symmetric manner.

We start by writing $\widetilde{Q}(f)$ as,

$$\widetilde{Q}(f) = \frac{1}{\sqrt{S_1(f)}} \frac{\widetilde{h}_1(f)}{\sqrt{S_1(f)}} \frac{\widetilde{h}_2^*(f)}{\sqrt{S_2(f)}} \frac{1}{\sqrt{S_2(f)}} \ . \tag{6}$$

Thus, the filter corresponding to $\widetilde{Q}(f)$ as its transfer function can be implemented as a chain of two filters $Q_1(t)$ and $Q_2(t)$,

$$\widetilde{Q}_1(f) = \frac{1}{\sqrt{S_1(f)}}\frac{\widetilde{h}_1(f)}{\sqrt{S_1(f)}} , \quad (7)$$

$$\widetilde{Q}_2(f) = \frac{1}{\sqrt{S_2(f)}}\frac{\widetilde{h}_2^*(f)}{\sqrt{S_2(f)}} . \quad (8)$$

Using Eq. 5,

$$y_2(t) = \Theta(t)\int_{-\infty}^{\infty} dt' Q_1(t-t')\int_{-\infty}^{\infty} dt'' Q_2(t'-t'')x_2(t''); \quad (9)$$

Substituting the above expression for $y_2(t)$ in Eq. 3 (and noting that $\Theta^2 = \Theta$), we get

$$\begin{aligned}\langle x_1, x_2\rangle &= \int_{-\infty}^{\infty} dt\Theta(t)x_1(t)\int_{-\infty}^{\infty} dt' Q_1(t-t')\int_{-\infty}^{\infty} dt'' Q_2(t'-t'')x_2(t'')(10)\\ &= \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dt\Theta(t)Q_1(t-t')x_1(t)\int_{-\infty}^{\infty} dt'' Q_2(t'-t'')x_2(t'') ,\\ &\simeq \int_{-\infty}^{\infty} dt'\Theta(t')\int_{-\infty}^{\infty} dt Q_1(t-t')x_1(t)\int_{-\infty}^{\infty} dt'' Q_2(t'-t'')x_2(t'')(11)\end{aligned}$$

Eq. 11 clearly suggests that the cross-correlation $\langle x_1, x_2\rangle$ can be constructed by first pre-filtering each time series $x_1(t)$ and $x_2(t)$ and then taking a kernel-free cross-correlation of the filtered time series.

The result in Eq. 11 is an approximate one. In the next Section, we revisit the problem of finding the optimum kernel by posing the problem in such a way that the final result can be interpreted *exactly* in terms of a symmetric application of filters.

4 Revised derivation of the optimum kernel

Suppose we filter $x_i(t)$, $i = 1, 2$, using a *causal* filter $Q_i(t)$ to get $y_i(t)$ and construct a cross-correlation defined as,

$$\langle x_1, x_2\rangle_w = \int_0^w dt\, y_1(t)y_2(t) . \quad (12)$$

$$y_i(t) = \int_{-\infty}^{\infty} dt' Q_i(t-t')x_i(t') , i = 1, 2 . \quad (13)$$

Assume that the GW signal $h_i(t)$ is fully contained inside the interval $[0, w]$. Let the noise $n_1(t)$ in $x_1(t)$ and $n_2(t)$ in $x_2(t)$ be mutually uncorrelated, $E[\langle n_1, n_2\rangle_w] = 0$. Then what should $Q_i(t)$ be such that the signal to noise ratio (SNR) for a give w is maximised? The SNR is defined as,

$$SNR = \frac{\langle h_1, h_2\rangle_w}{\sigma} \ , \tag{14}$$

$$\sigma^2 = E[\langle n_1, n_2\rangle_w^2] \ . \tag{15}$$

Using Eq. 12, we can express $\langle h_1, h_2\rangle_w$ as,

$$\langle h_1, h_2\rangle_w = \int_0^w dt \int_{-\infty}^{\infty} dt' Q_1(t-t')h_1(t') \int_{-\infty}^{\infty} dt'' Q_2(t-t'')h_2(t'') \ . \tag{16}$$

By switching to the Fourier Transform for every term and doing the integrals over all the time variables, we get

$$\langle h_1, h_2\rangle_w = w \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} df df' \widetilde{Q}_1(f)\widetilde{Q}_2^*(f')\widetilde{h}_1(f)\widetilde{h}_2^*(f')\mathrm{sinc}\left(w(f-f')\right) \ , \tag{17}$$

where $\mathrm{sinc}(x) = \sin(\pi x)/\pi x$. A slightly more tedious calculation yields

$$\sigma^2 = w^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} df df' S_1(f)|\widetilde{Q}_1(f)|^2 S_2(f')|\widetilde{Q}_2^*(f')|^2 \mathrm{sinc}^2\left(w(f-f')\right) \ . \tag{18}$$

Taking the cue from suggestive expressions in the preceding sections, let

$$\widetilde{Q}_i(f) = Z_i(f)\frac{\widetilde{h}_i^*(f)}{\sqrt{w}S_i(f)} \ . \tag{19}$$

Subsituting in Eq. **??** and **??**, we get

$$\langle h_1, h_2\rangle_w = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} df df' \widetilde{Z}_1(f)\widetilde{Z}_2^*(f')\frac{|\widetilde{h}_1(f)|^2|\widetilde{h}_2(f')|^2}{S_1(f)S_2(f')}\mathrm{sinc}\left(w(f-f')\right) \ , \tag{20}$$

$$\sigma^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} df df' |Z_1(f)Z_2^*(f')|^2 \frac{|\widetilde{h}_1(f)|^2|\widetilde{h}_2(f')|^2}{S_1(f)S_2(f')}\mathrm{sinc}^2\left(w(f-f')\right) \tag{21}$$

To be completed ...

References

[1] L. S. Finn, S. D. Mohanty, J. Romano, Phys. Rev. D. (rapid comm) **60**, 121101 (1999).