

# Calibration of the LIGO detectors for the First LIGO Scientific Run

## LIGO Technical Document T030097-00-D

Rana Adhikari, Peter Fritschel (LIGO-MIT),  
Gabriela González (Louisiana State University),  
Mike Landry, Luca Matone, Hugh Radkins, Akiteru Takamori, (LIGO-LHO),  
Brian O'Reilly (LIGO-LLO)

May 21, 2003

### Abstract

We describe the calibration procedures applied to the data taken by the three LIGO interferometric gravitational wave detectors in the LIGO Science Collaborations First Scientific Run, from August 23 to September 9, 2002. The calibration depends on corrections to feedback servos used in the instrument, and on changing optical gains. We describe the stability of the calibration and detector sensitivity during the 17-day run. We also estimate the errors associated with the calibrations provided to the Upper Limit Data Analysis Groups.

## 1 Introduction

The First LIGO Scientific Run, S1, lasted from 23 August 2002 to 9 September 2002. Figure 1 shows amplitude spectra of equivalent strain noise, typical of the three LIGO interferometers during the S1 run. The strain noise design goal is also indicated for comparison. The differences between the three spectra reflect differences in the operating parameters and hardware implementations of the three instruments; they are in various stages of reaching the final designed configuration. The 17-day run yielded 363 hours of data when at least one interferometer was in stable operation. The three interferometers were operating simultaneously for 95.7 hours.

The calibration of the interferometric signal interpreted as strain is an essential element of the data analysis. We start with a simple model of the interferometer, measurements of different parts of the model and the interpretation of the observed amplitude of calibration lines as model parameters. We describe in this article the procedures used in the LIGO detectors for the S1 run and the results obtained in terms of stability of the detectors during the run.

## 2 A simple model of the detector

The strain we want to measure is the difference of length between the two orthogonal interferometer arms, divided by the arm length:  $s(t) = (L_x(t) - L_y(t))/L_0$ . The length  $L_0$  is 2km for H2 and 4km for H1 and L1. The calibrated strain includes effects of all “external” sources (seismic, electronic noise in coil drivers, thermal noise, gravitational waves); it also translates sensing noise (shot noise, electronics noise in

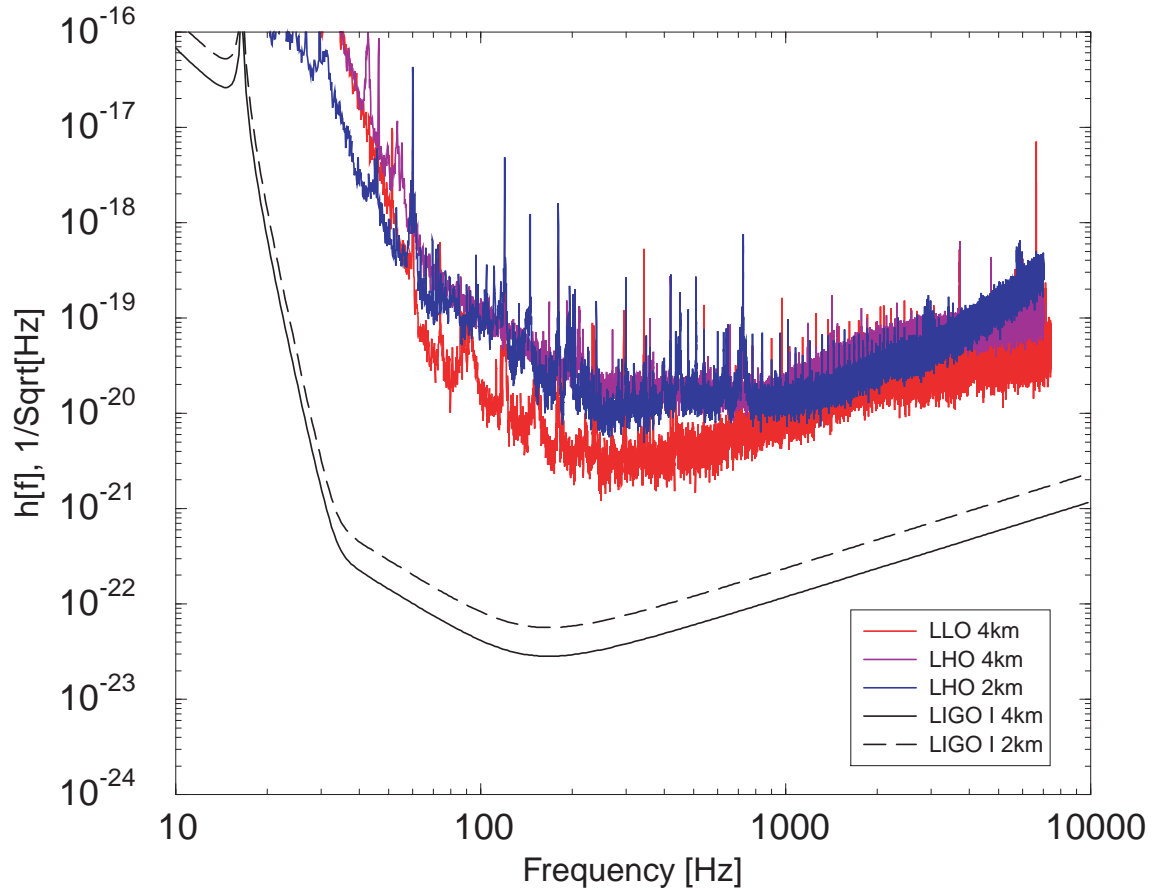


Figure 1: Typical sensitivities of the three LIGO interferometers during the S1 data run, shown as equivalent strain amplitude spectral density  $h(f)$ . These amplitudes were obtained using the calibrations described in this article.

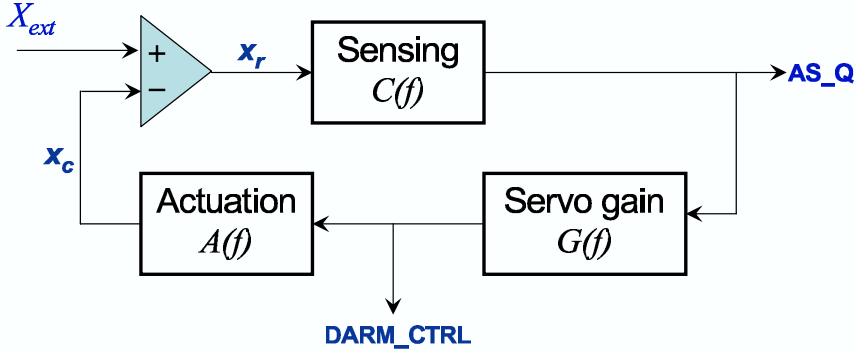


Figure 2: DARM length servo loop

the sensing electronics) into equivalent strain noise. The calibrated strain signal  $s(t)$  does *not* include the motion produced by control forces to keep residual motion small and the optical cavities resonant.

To understand how we estimate the response function from measurements of more fundamental components, we create a simple model of the sensing of the single degree of freedom corresponding to the differential arm signal, or DARM, shown in Fig. 2. In this model, the strain is represented by the motion of a single mass hanging on a pendulum, measured in an inertial frame, and divided by 2km or 4km, depending on the detector.

The actual strain signal  $s(t)$  is derived from the error signal, in counts,  $e(t)=XX:LSC-AS\_Q$  of the DARM feedback loop. The calibration for LIGO in this science run was done in the frequency domain, providing a response function  $R(f)$  that relates the Fourier transforms of the measured error signal  $e(f)$  and the strain signal  $s(f)$ :  $s(f) = R(f)e(f)$ .

The response function can be built from three complex functions in the feedback loop: a sensing function  $C(f)$ , an actuation function  $A(f)$  and a feedback filter function  $G(f)$ . The sensing function, converting the strain into counts in the data, has units of counts/strain; the servo gain function has units of counts/counts (DARM\_CTRL/AS\_Q), and the actuation function (converting the control signal into pendulum motion) has units of strain/count.

The open loop gain, sometimes just called loop gain, is the product of all three functions,  $H = GAC$ .

Given an externally induced change in differential arm distance  $X_{ext} = L_x - L_y$  (by seismic noise, gravity wave or any other external source), the response at the different points is:

$$\begin{aligned}
\text{control length:} & \quad x_c/L_0 = A(f)DARM\_CTRL \\
\text{control signal:} & \quad DARM\_CTRL = G(f)AS\_Q \\
\text{error signal:} & \quad AS\_Q = C(f)x_r/L_0 \\
\text{residual length:} & \quad x_r/L_0 = X_{ext}/L_0 - x_c/L_0 \\
& \quad = X_{ext}/L_0 - A(f)DARM\_CTRL \\
& \quad = X_{ext}/L_0 - A(f)G(f)AS\_Q \\
& \quad = X_{ext}/L_0 - A(f)G(f)C(f)x_r/L_0
\end{aligned}$$

From going around the loop, we deduce

$$\begin{aligned}
x_r &= X_{ext} \frac{1}{1 + H(f)} \\
x_c &= X_{ext} \frac{H(f)}{1 + H(f)} \\
AS\_Q &= \frac{X_{ext}}{L_0} \frac{C(f)}{1 + H(f)} \\
DARM\_CTRL &= \frac{X_{ext}}{L_0} \frac{G(f)C(f)}{1 + H(f)} = \frac{X_{exc}}{L_0} \frac{1}{A(f)} \frac{H(f)}{1 + H(f)}
\end{aligned}$$

At low frequencies where the gain is very high,  $H(f) \gg 1$ , we see that the residual length is much smaller than the exciting motion  $x_r \ll X_{exc}$ , while the control length is very similar to the exciting motion,  $x_c \sim X_{ext}$  (that's how the motion is cancelled, of course). At high frequencies, where the gain is low,  $H \ll 1$ , the opposite is true: the residual length is close to the exciting length,  $x_r \sim X_{exc}$ , while the control length is much smaller,  $x_c \ll X_{exc}$ .

The envelope of the residual and control lengths (the larger of the two) is an approximate measure of the exciting motion, *except* near the unity gain frequency. The residual, control and calibrated motions measured at a time during S1 in LLO are shown in figure 3.

The desired response function, in units of strain/counts, is thus

$$R(f) = \frac{X_{ext}/L_0}{AS\_Q} = \frac{1 + H(f)}{C(f)}$$

In order to get an estimate for the response function during S1, we need estimates for the open loop function  $H$  and the sensing function  $C$ . We know that optical gains change due to alignment, and thus the DC gain of the sensing function changes even within a locked segment; we will assume that the frequency dependence of all three functions A, C and G remain unchanged in time, as well as the DC gains of the G and A functions. If the only change is due to alignment, and the sensing function  $C(f)$  differs from a reference function at some time  $C_0(f)$  by a constant  $C(f, t) = \alpha(t)C_0(f, t_0)$ . We also have a reference open loop gain function  $H_0$ . Then the response function at some time t is given by

$$R(f, t) = \alpha(t) \frac{1 + H_0(f)}{1 + \alpha(t)H_0(f)}$$

We track the changes in  $\alpha$  using calibration lines. If we inject a calibration line at a frequency  $f_i$ , and we know the amplitude  $X_i$  that this excitation would produce if there was no feedback servo, then the actual amplitude of the line in the ASQ spectrum is

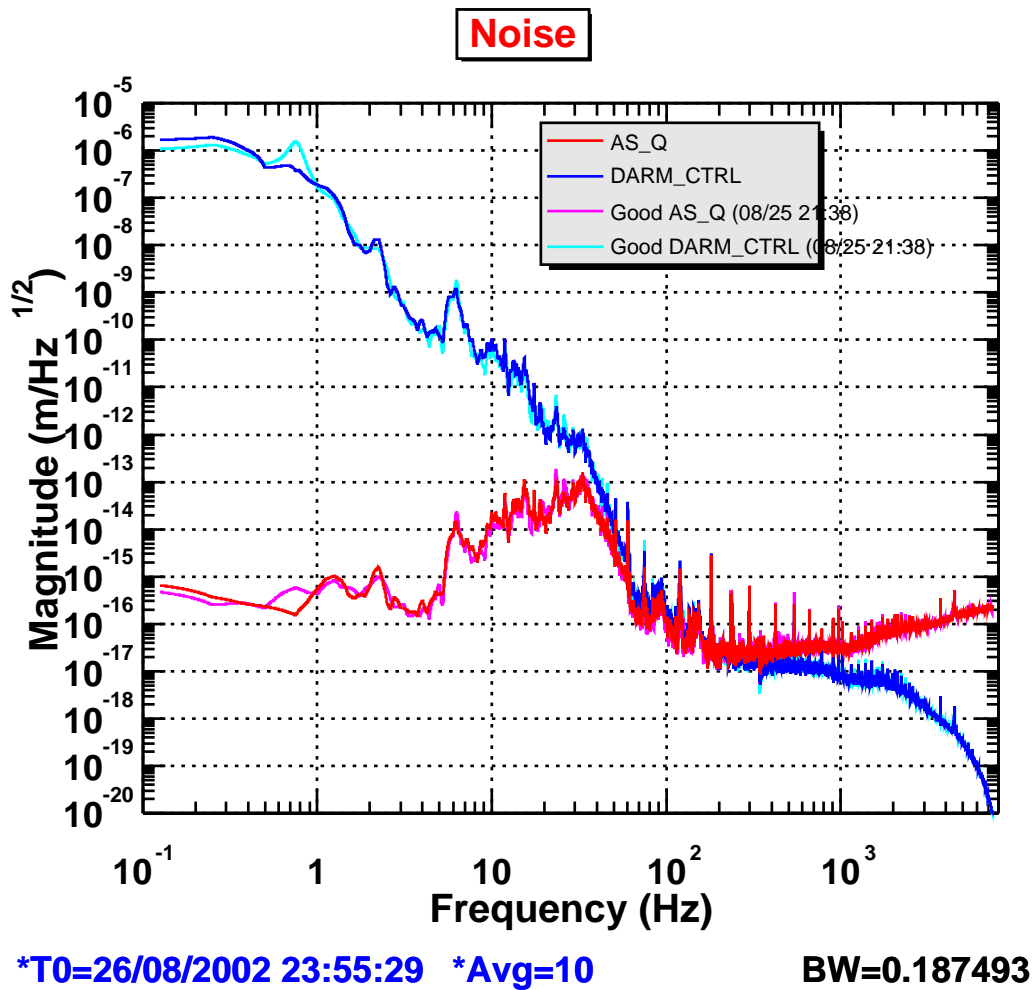


Figure 3: A measurement of input length noise, using AS\_Q and DARM\_CTRL.

$$AS\_Q(f_i, t) = \frac{X_i}{L_0} \frac{\alpha(t)C_0(f_i)}{1 + \alpha(t)H_0(f_i)}$$

If we measure the ratio of the line at any time  $t$  during S1, to the reference time when  $H_0, C_0$  were estimated, we have a function of time:

$$r_i(t) = \frac{AS\_Q(f_i, t)}{AS\_Q(f_i, t_0)} = \frac{\alpha(t)(1 + H_0(f_i))}{1 + \alpha(t)H_0(f_i)}$$

From the value of  $r_i(t)$  and  $H_0(f_i)$ , we can then deduce the value of  $\alpha(t)$ . For calibration lines at frequencies well outside the bandwidth of the feedback loop, where  $|H_0(f_i)| \ll 1$ , we have  $r_i(t) \approx \alpha(t)$ ; but we can use the formula above to get the changes in optical gain for any calibration frequency.

At some times during S1, the function filter  $G$  was changed by an overall factor  $\beta$  with respect to its value at the reference time  $t_0$ . Such a change at time  $t$  results in an open loop gain  $H(f, t) = \alpha(t)\beta(t)H_0(f)$ , but it doesn't change the sensing function, which is  $C(f, t) = \alpha(t)C_0(f)$ . The changes in  $\beta$  can easily be tracked in the computer records (conlog), since  $G$  is a digital filter; there were a few discrete values used in L1 during S1. Thus, considering both possible changes, the formula that allows the calculation of  $\alpha(t)$  from the measurement of the amplitude of the calibration lines is

$$r_i(t) = \frac{AS\_Q(f_i, t)}{AS\_Q(f_i, t_0)} = \frac{\alpha(t)(1 + H_0(f_i))}{1 + \alpha(t)\beta(t)H_0(f_i)}$$

### 3 S1 Calibration: procedure and results

The calibration for S1 is available in the web page

[blue.ligo-wa.caltech.edu/scirun/S1/Results/CalibrationStability/finalS1cal.html](http://blue.ligo-wa.caltech.edu/scirun/S1/Results/CalibrationStability/finalS1cal.html).

For each interferometer, we chose a reference time and produced estimates of the open loop gain function  $H_0$ , the sensing function  $C_0$ , a reference calibration function  $R_0$ , and a list of values of  $\alpha(t)$  for each interferometer, for times during science segments in S1, with a sampling time equal to one minute.

In order to obtain the S1 reference functions and  $\alpha$  factors, we followed the following procedure:

- We measured the open loop gain function  $H_0$  at a time chosen as the “reference time”.
- We create models for the filter function  $G$ , and the actuation function  $A$ .
- We estimate the sensing function at the reference time, using the relationship  $C = H/(AG)$ .
- We find out values of  $\beta$  for L1 during S1, and we calculate values of  $\alpha(t)$  from measurements of the amplitude of calibration lines.

We describe in the following sections each of these steps in detail, and estimate associated errors.

### 4 Reference Open Loop Gain Functions

The open loop gain  $H(f)$  can be measured directly, during operation. We made several measurements during the S1 run, which as expected only changed in the overall gain (determined by optical gains in the cavities, and thus alignment). We took one of these measurements as a reference measurement, which was then used

as a starting point to track changes in gain during the run. The reference functions in the gravitational wave band (50 Hz-2000 Hz) are shown in figure 4. The three detectors had slightly different feedback filters, but all loops had a similar bandwidth (or control band): in the reference measurements, the unity gain frequencies were 150 Hz (H1), 200 Hz (H2) and 250 Hz (L1). Of course, the overall gain and therefore the unity gain frequency changed during the S1 run, depending on the fluctuating optical gains.

#### 4.1 LHO reference open loop gains

The reference functions for H1, H2 are direct measurements of the open loop gain. The measurements is a frequency response made with DTT of the response of DARM\_CTRL to ETM\_EXC; the open loop gain is then estimated as  $H = (\text{DARM}/\text{ETM\_EXC}) / (1 - (\text{DARM}/\text{ETM\_EXC}))$ .

In general, the measurement of a transfer function  $T$  has a statistical error  $\delta T$  in amplitude and phase that can be estimated from the coherence of the measurement. The standard deviation of the phase and the relative error in the magnitude of the transfer function are both given by the expression

$$\sigma(\phi) = \frac{\sigma(T)}{T} = \sqrt{1 - \gamma(f)^2} / (\gamma(f) \sqrt{2N_{avg}}),$$

where  $\gamma(f)$  is the coherence measured at each frequency, and  $N_{avg}$  is the number of averages for each point.

This function of coherence for the measurements of DARM/ETM for H1 and H2 is plotted in Fig. 5 for the Sept 04 H1 measurement, and in Fig. 6 for the Aug 29 H2 measurement.

There is a lot of scatter, but it seems that a good estimate of the error for H1 is an error less than 1% between 50 Hz and 100 Hz; above 100 Hz the error is approximate by a power law  $\approx 0.005 \times (f/100\text{Hz})^{1.15}$ . The error is about 1% (or 0.6 deg) at 200 Hz, and 6% (or 3 deg) at 1 kHz. For H2, the error seems to be  $\approx 6\%$  (or 3 deg), constant in frequency.

Since the function  $H$  is calculated from a measurement  $T$  as  $H = T/(1 - T)$ , the error in  $H$  is  $\delta H = \delta T(H/T)^2 = \delta T$ . The relative errors in magnitudes, or errors in phase, are related by  $\delta H/H = (\delta T/T)(1+H)$ ; thus, the error in  $H$  are smaller than the errors of  $T$  near the unity gain frequency, and similar to the errors in  $T$  at high frequencies. The error in  $H$  is larger than the error in  $T$  below the unity gain frequency, but since the measured open loop gain is no larger than 3 above 50 Hz, the errors are still comparable. The functions  $1 + H_0$  for the reference gains in the three detectors is shown in Fig.7.

The errors estimated for  $T$ , described above for H1 and H2, are then good estimates (or over-estimates) above 100 Hz, and slightly underestimates of the error in the open loop gain below 100 Hz. In H1, the error in  $H$  the band 50-100 Hz is then about 2% and 1 degree or smaller; the error in H2 in the same band could be as large as 10%, or 6 degrees.

#### 4.2 L1 reference open loop function: model

For L1, we chose to use as reference open loop gain a function of frequency using models from the known components in the loop, making up the sensing, actuation and filter functions, fitted to match a measured open loop gain. The advantage of this is that we get a smooth function of frequency; the disadvantage is that we are liable to systematic errors arising from incorrect modeling.

The measurement of the open loop gain has statistical errors in the same way we described them for the LHO detectors. We show in Fig. 8 the measurement error derived from the coherence function in the reference Sep 06 measurement: the error is approximately 0.03 for frequencies below 2kHz, and a power law  $\approx 0.03 \times (f/2\text{kHz})^{1.42}$  for higher frequencies. This means we have a relative measurement error of 3%

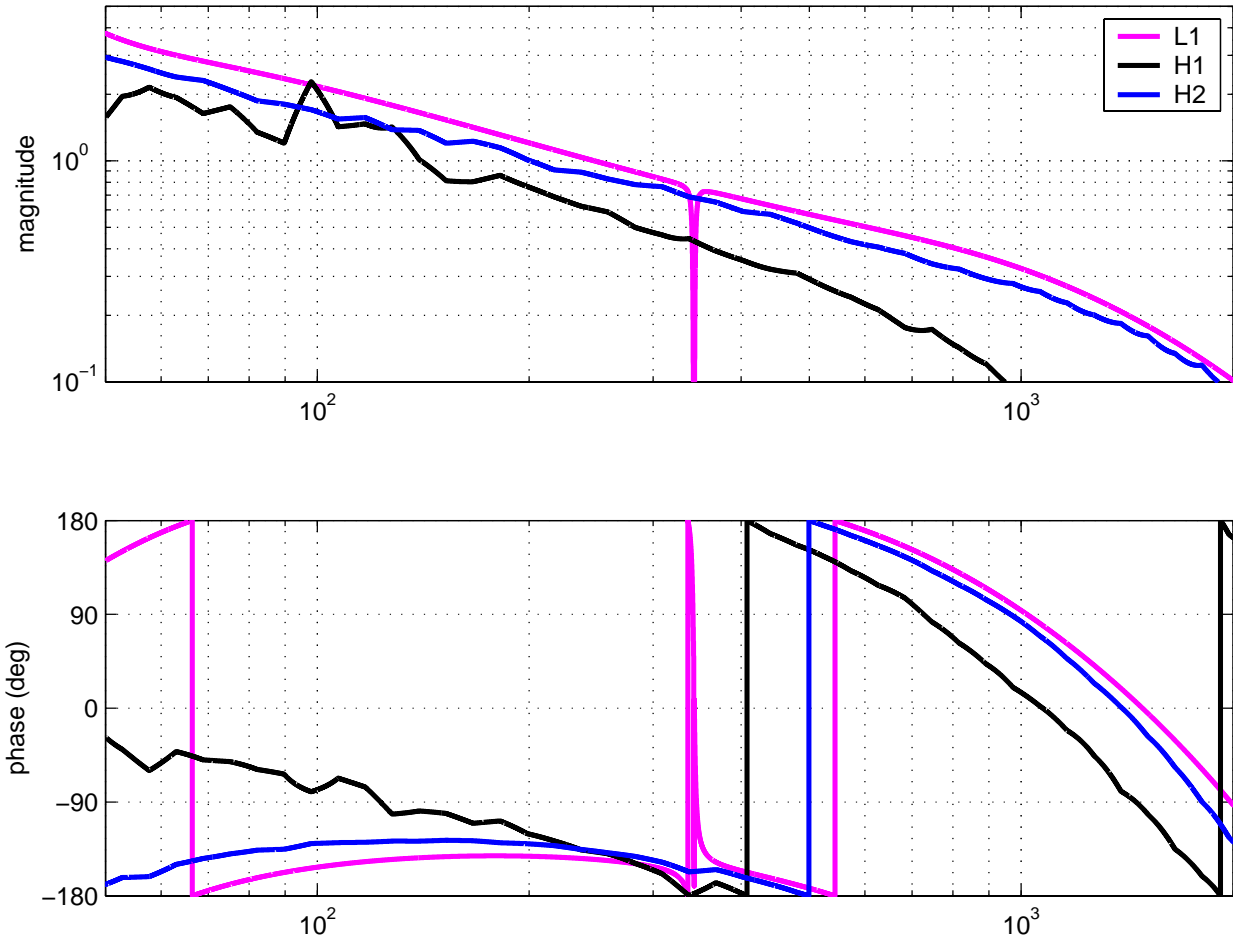


Figure 4: Open loop gains  $H(f)$  taken as reference for the S1 run. The notch in the L1 open loop gain was used to avoid ringing the violin modes of the suspended mirrors. The times for the reference measurements were Sep 06, 2002 23:02 UTC, GPS 715388533 (L1), Sep 04, 2002 06:39, GPS 715156759 (H1) and Aug 29, 2002 08:21, GPS 714644464(H2).



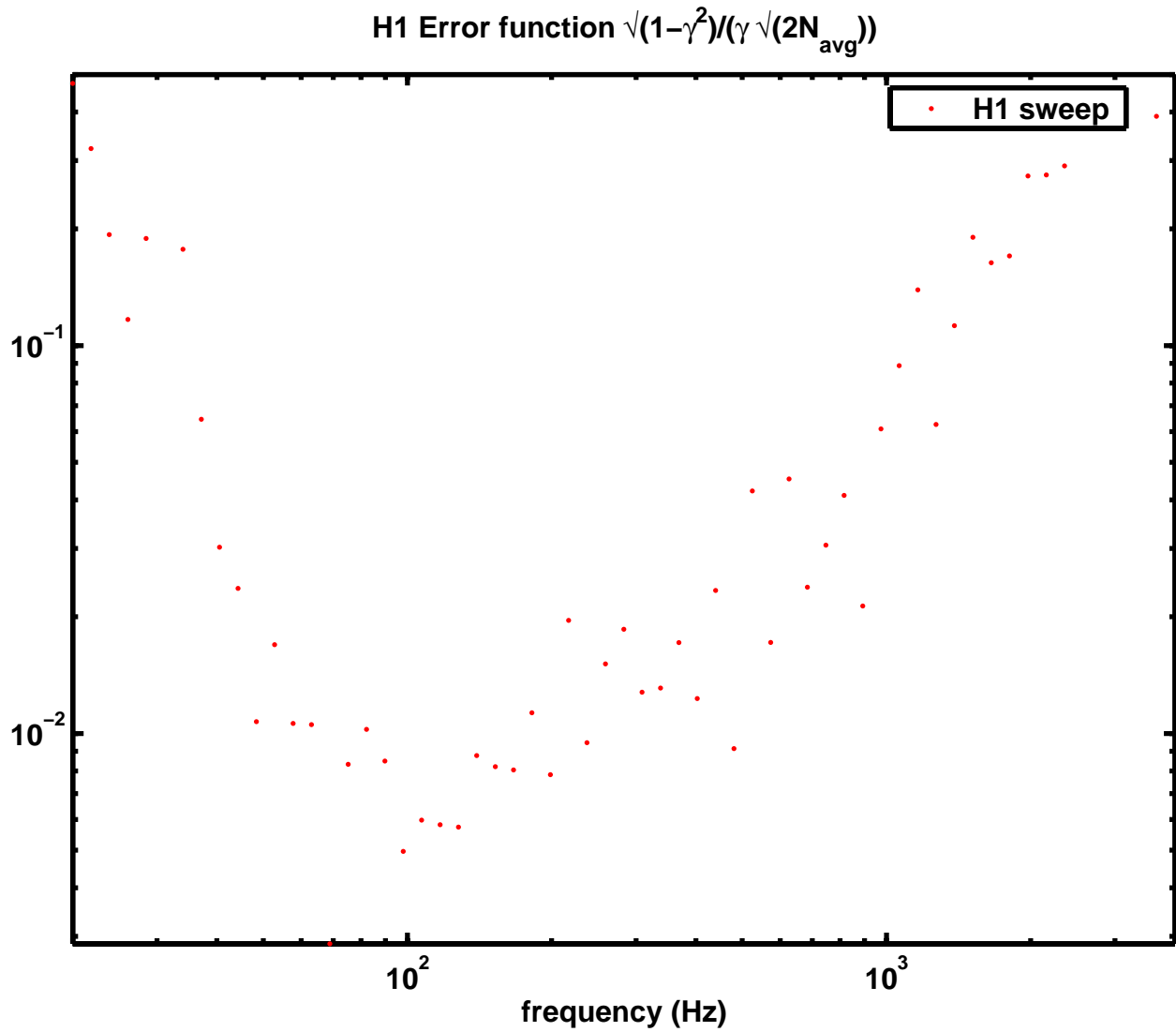


Figure 5: Statistical error function calculated from coherence in the Sept 04 H1 measurement of the open loop gain. This can be interpreted as the relative statistical error in the magnitude of the transfer function, and as the statistical error of the phase of the transfer function, in radians.

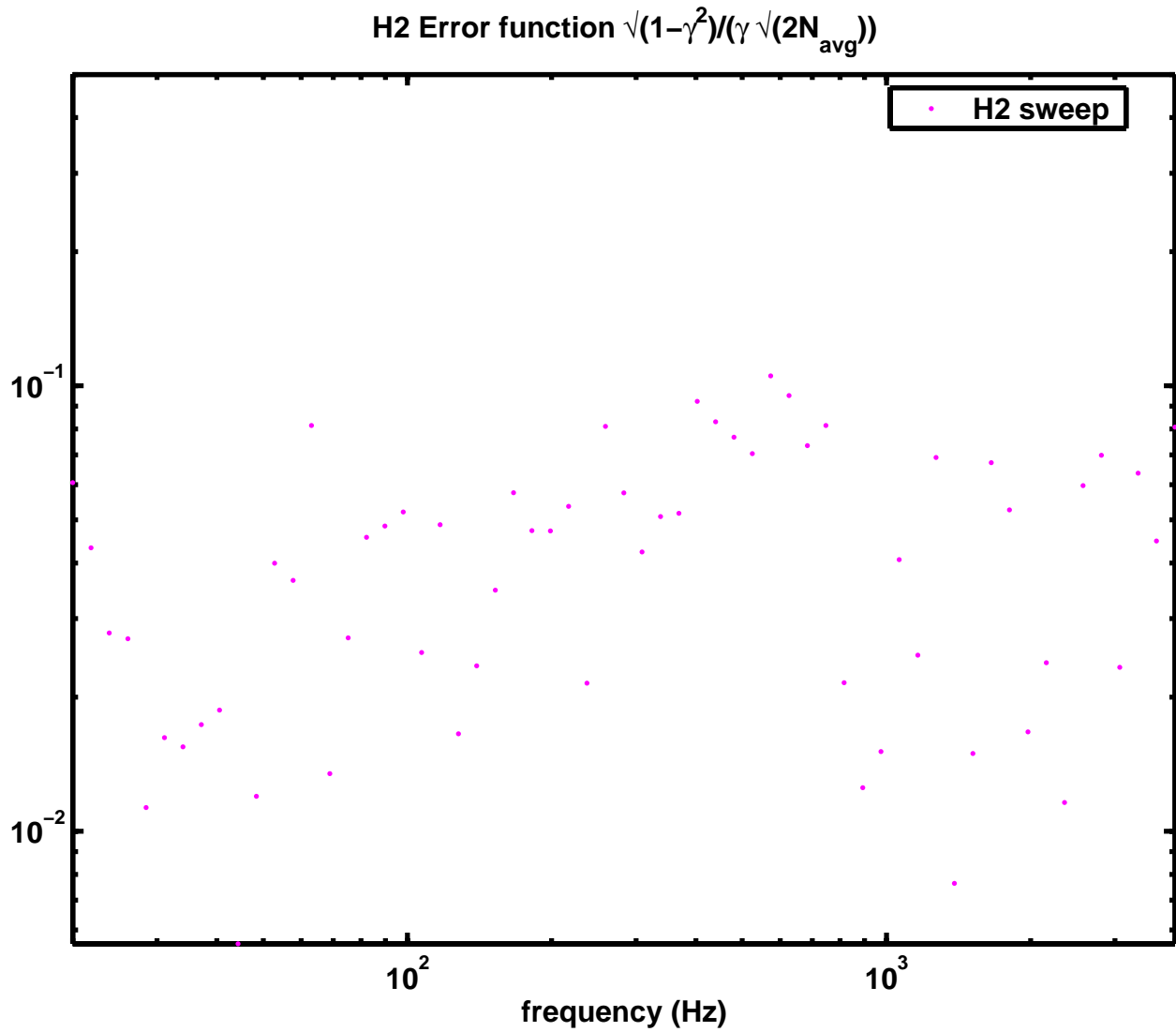


Figure 6: Statistical error function calculated from coherence in the Aug 29 H2 measurement of the open loop gain. This can be interpreted as the relative statistical error in the magnitude of the transfer function, and as the statistical error of the phase of the transfer function, in radians.

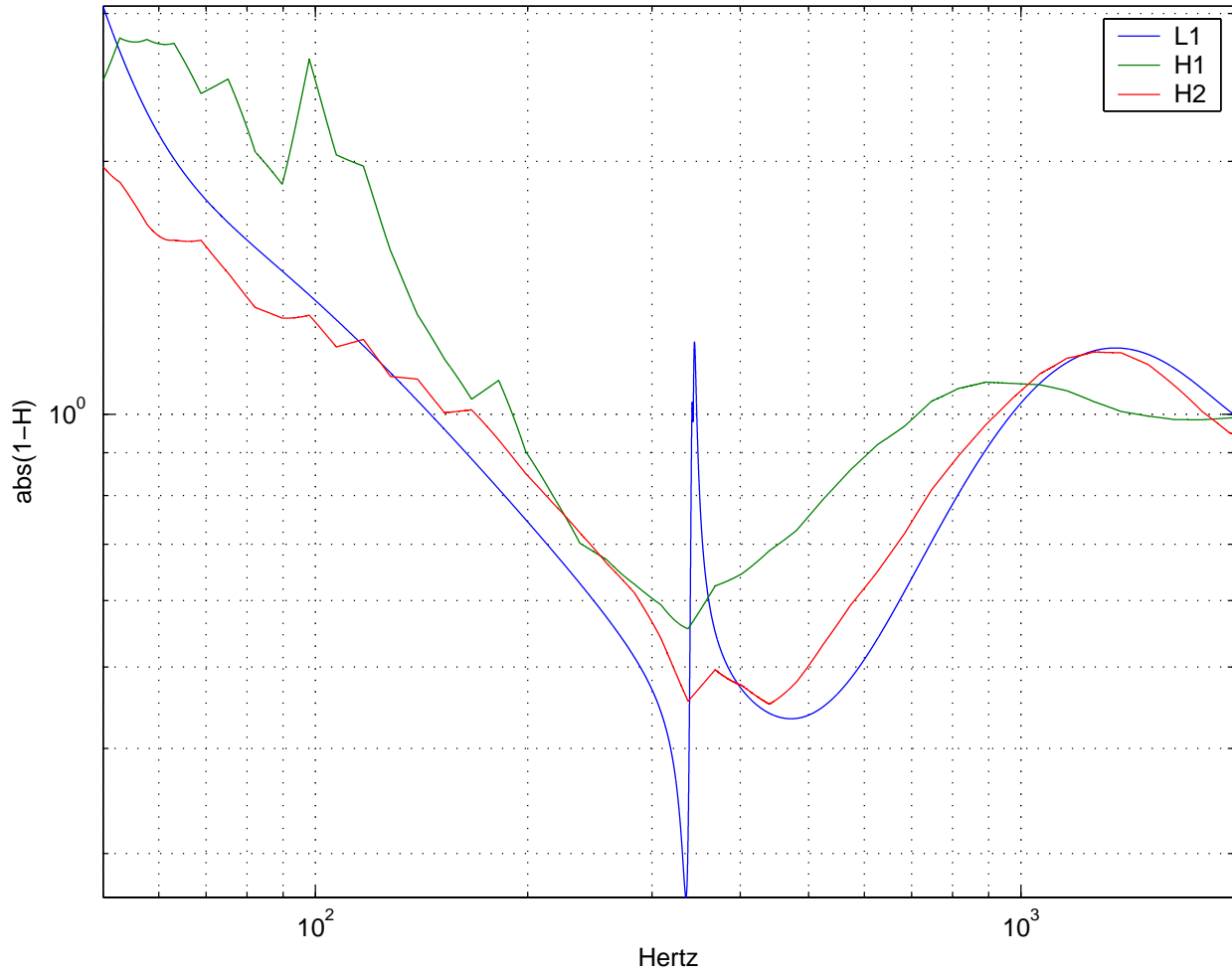


Figure 7: Function  $1 + H_0$  calculated for the detectors' reference open loop gain functions. .

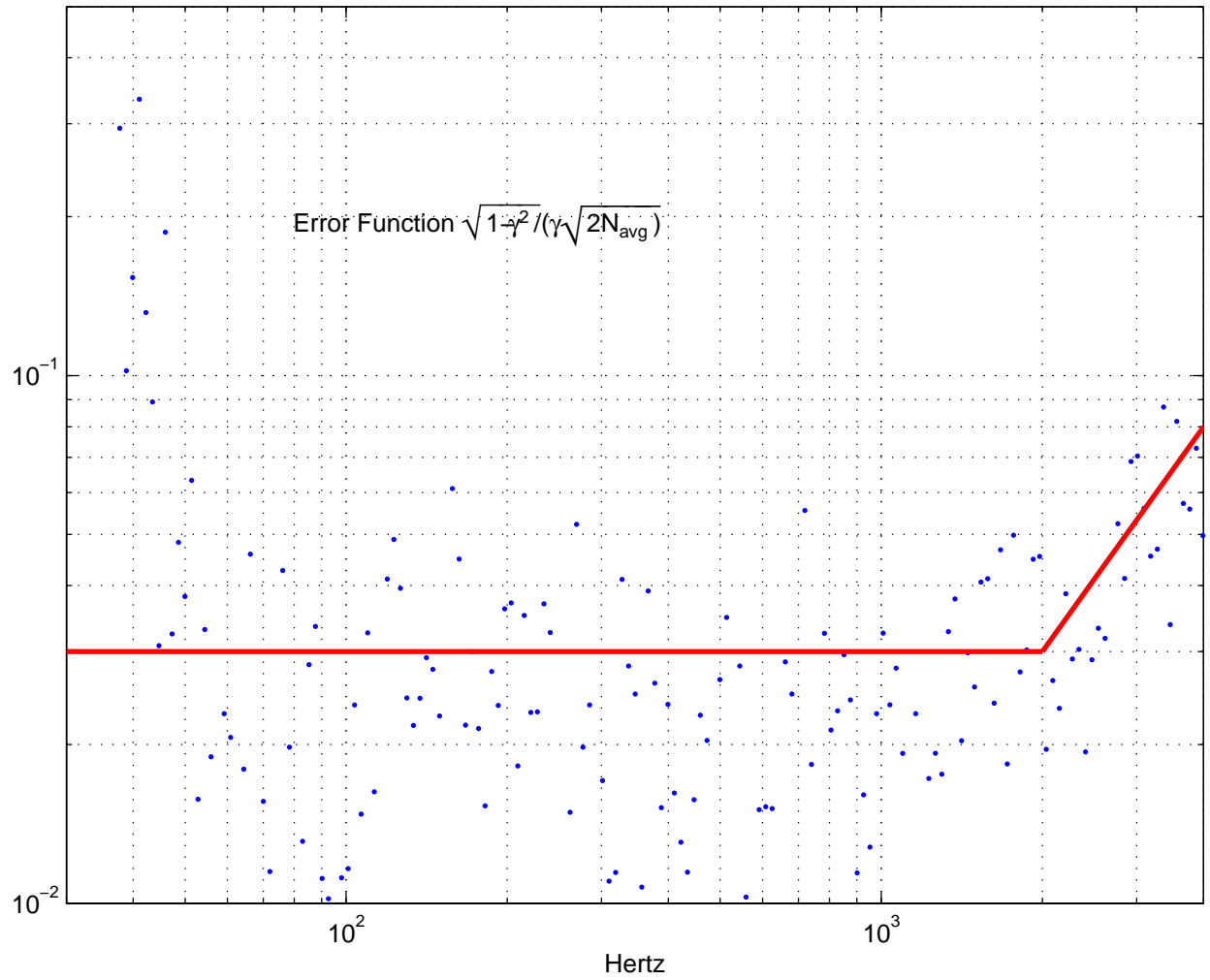


Figure 8: Statistical error function calculated from coherence in the Sept 06 L1 measurement of the open loop gain. This can be interpreted as the relative statistical error in the magnitude of the transfer function, and as the statistical error of the phase of the transfer function, in radians.

in amplitude and a measurement error of 0.03 radians = 1.7 degrees in phase for frequencies below 2 kHz. Again, the actual error in  $H$  is smaller or equal than what the plot shows for frequencies above 100 Hz

We fitted an overall constant to the model, to account for the unknown optical gain at the reference time; and a time delay, to account for the phase measured at high frequencies. We used a Simulink model, shown in Fig. 9. In the model we tried to account to for all known gains in the functions, so that the overall constant due to alignment is close to unity. The components making up the different functions as follow:

- **Sensing function  $C(f)$ :**

The only frequency dependent components in this function are the arm cavity, represented with a single real pole in the Laplace domain; and the anti-aliasing filter. The ADC delay was included in the sensing function, but in fact the the time stamp on the data accounts for this delay, so it should have been included in the actuation function, not the sensing function.

- Arm Cavity `fc = 87.3; % Hz`  
`armTF = 2*pi*fc * tf(1,[1 2*pi*fc]);`
- Optical gain: 0.47e7 Amps/meter. This gain is multiplied by a fitted factor depending on alignment; this factor was 1.2 for the reference measurement on Sept 06, 23:02 UTC.
- LSC Photodiode gain: `pdTF=400*10 V/Amp.`
- Demodulation gain=1/2.8
- Whitening gain:  $10^{18/20}$
- Anti-Aliasing filter: `[z,p,k] = ellip(8,0.035,80,2*pi*7570,'s');` `aaf = zp(z,p,k);`
- Single-to-differential gain: 2
- ADC delay: 50 $\mu$ sec.
- ADC gain: 32768/10 counts/Volt

- **Actuation Function  $A(f)$ :** The frequency dependence is mostly determined by the pendulum suspension, with a pair of complex poles at the pendulum frequency,  $f_p = 0.75Hz$ .

- Output matrix `darm2lm=-0.33` (from output matrix to ETMs, from conlog).
- ETM filters: there were steep stopbands near the internal modes (near 7 kHz) and a violin mode notch (near 345 Hz) used for the ETM drive. We could not include the stopband filters without running into what looked like Simulink numerical problems. Since they are outside the gw band, we left them out. Thus, we use just the violin notch for the test mass filter:  
`violin = zp([0+i*2152.47;0-i*2152.47;0+i*2156.2;0-i*2156.2], [-10.7623+i*2152.44;-10.7623-i*2152.44;-10.781-i*2156.18],1);`  
`ETMDF=violin;`
- DAC gain= (5/32768) Volt/count
- DAC delay: 75 $\mu$ sec. This delay was found to get a good fit to the loop gains.
- Anti-imaging filter: `[z,p,k] = ellip(4,4,60,2*pi*7570,'s');` `aif = zp(z,p,k*104/20);`
- Dewhitening gain=3
- Voltage to current gain: 1/30 Amp/Volt

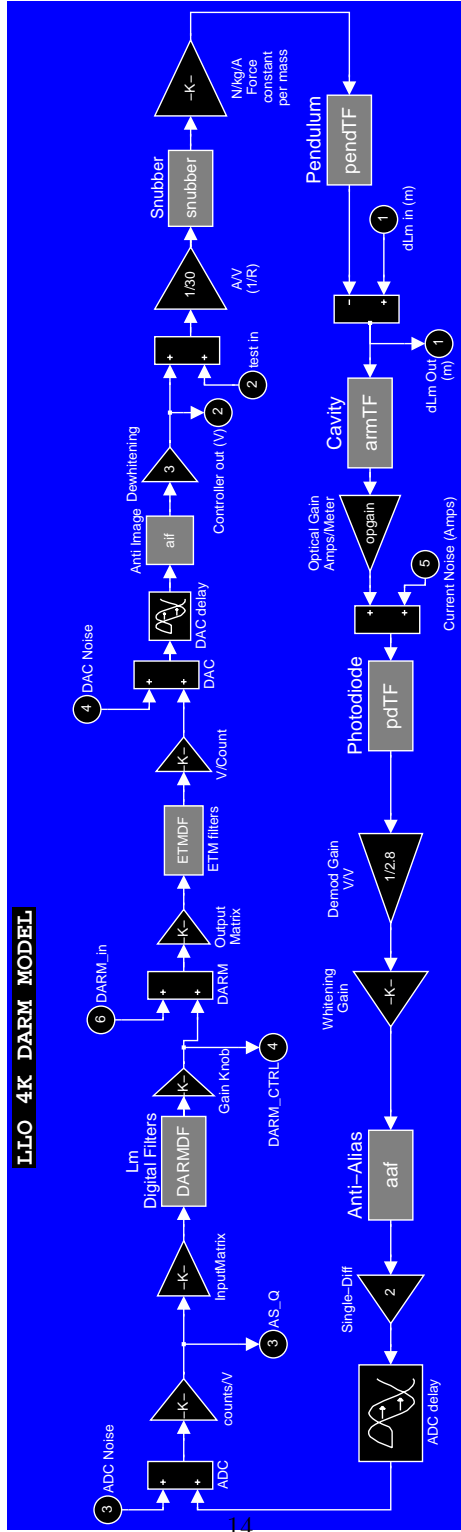


Figure 9: Simulink model S1darm.08.mdl

- Snubber =0.63. This was used a place holder for a constant factor, fitted to get the measured DC drive of 1.72nm/ct for DARM\_CTRL. However, there was an actual snubber filter which was not included in the S1 model, with a unity DC gain.
- Coil/pendulum gain: 0.064/10.3 N/kg/Amp
- Pendulum transfer function (m/N/kg)

$$Q = 10;$$

$$w0 = 2*pi*0.76;$$

$$pendTF = 2 * tf(1,[1 w0/Q w0^2]);$$

(The factor of 2 accounts for Ly-Lx.)

The measured DC calibration for DARM\_CTRL into effective mirror motion (for (Ly-Lx)) was 1.72nm/count. In this model, that gain is made up of the output matrix (0.33), times the DC gain of the ETM filters (1.0), times the DAC gain (5/32768 Volts/counts), past the DAC delay (unity gain), the antiimaging filter (unity gain), times the dewhitening gain (3.0), times the voltage-to-current gain (1/30 Amp/Volt), times the force constant (0.064/10.3 N/m/Amp), times the DC pendulum gain (1/w0<sup>2</sup>=0.044 m/N/kg). This product is:

$$x = 0.33*(5/32768 \text{ V/ct})*3*(1/30 \text{ A/V})*(0.064/10.3 \text{ N/kg/A})*(0.044\text{m/M/kg})*\text{DARM[ct]} = 2.74 \text{ (nm/ct)}*\text{DARM[ct]}$$

Since the measured value is 1.72nm/count, we use a “correction factor” of 1.72/2.74=0.63, included as a “snubber” gain. The difference between this factor and unity is probably due to the coefficients in the output matrix for the length to each coil drive, which average 85% of the full value (for minimizing length to angle drive at DC).

- **Feedback Filter Function G(f):** The function has two gain constants: the input matrix element, and the loop gain GW\_K. Although these had a set of different values during the run in L1, they are known, so they have no error associated with them.

The function has several LSC digital filters used to shape the servo loop. We used “Foton” to translate the numbers in the coefficient filters file (/cvs/cds/llo/chans/L1.txt) used during S1, to zeros and poles to use in LTI Matlab models. The filters are digital filters, but we use them in Matlab/Simulink as analog filters. This introduces errors, especially at frequencies close to the Nyquist frequency, 8kHz. We associate with the function *G* an error as a function of frequency shown in Figs10, estimated as the difference between our Matlab model and the measured filter.

- Input matrix constant gain asq2darm=0.014 (from conlog).
- LSC Digital Filters
  - FM1=zpk([-75.757+i\*153.31;-75.757-i\*153.31],[0;0],1.99078);
  - FM2=zpk([-62.8318;-628.241],[-4389.6-i\*4389.6;-4389.6+i\*4389.6],976.28);
  - FM3=zpk([],[-8485+i\*8485;-8485-i\*8485],1.43991e+08);
  - FM4=zpk([-125.663],[-6.28319],0.996375);
  - FM8 = zpk([-6207.83;-6207.84],...  
[-25045.8+i\*0.00338783;-25045.8-i\*0.00338783],16.2775);

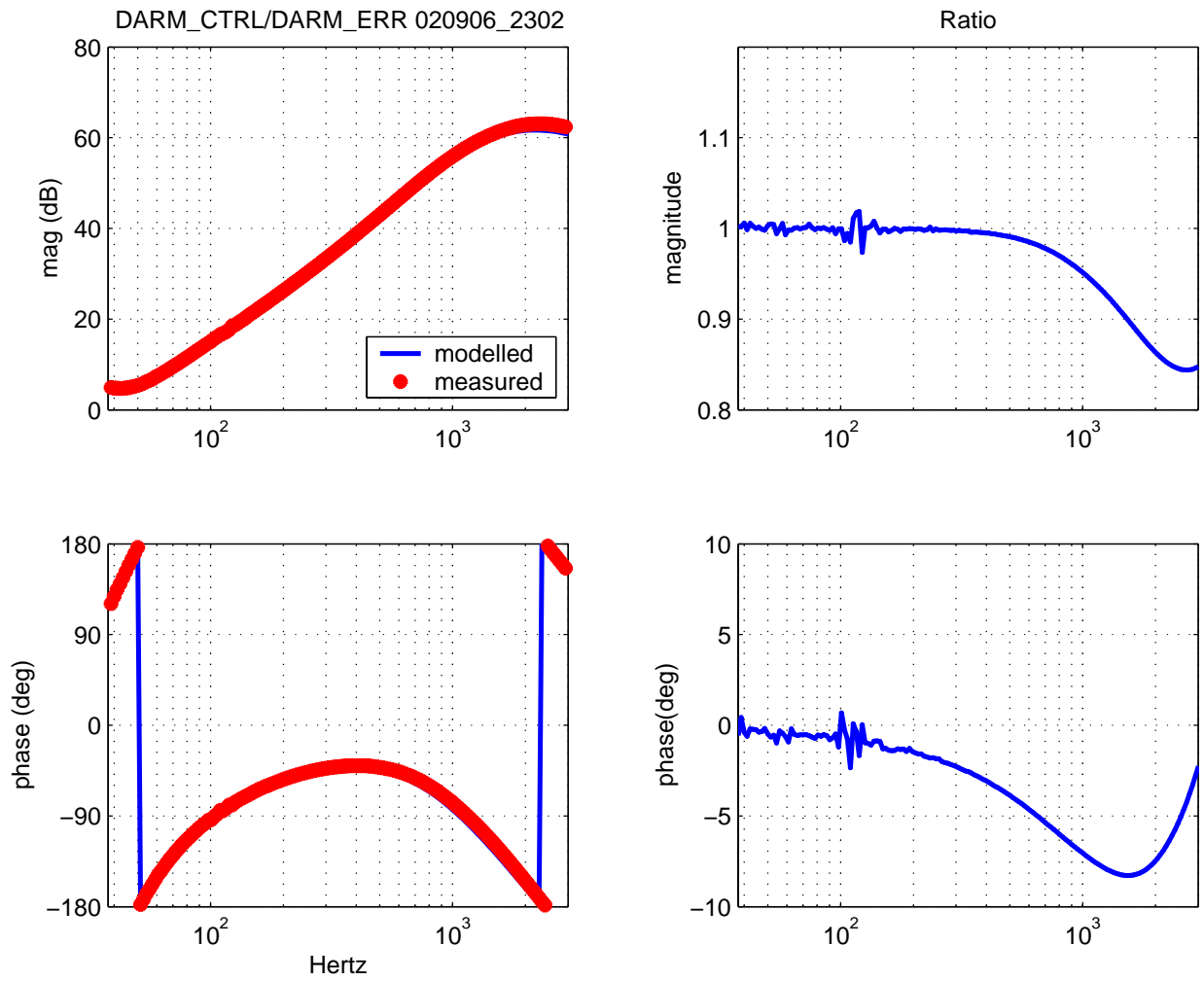


Figure 10: Measured and model transfer function between  $\text{DARM\_ERR}=\text{AS\_Q}/\text{asq2darm}$  and  $\text{DARM\_CTRL}$ .



```

FM9=zpk([-126.26+i*255.513;-126.26-i*255.513],...
[-12.6263+i*25.552;-12.6263-i*25.552],1.00005);
DARMDF= FM1 * FM2 * FM3 * FM8 * FM9;

```

- Gain knob: GW\_K. This gain is negative in the LSC code, but we need a negative sign somewhere to make the Simulink loop stable, as the real loop was (this was probably a minus sign in some whitening/dewhitening board).

This gain was changed a few times during S1. At the reference time Sept 06, 23:02, the gain was -0.231601. From conlog queries, L1:LSC-DARM\_ERR\_K in science segments had the following values:

```

714179393-714212083 -0.187838
Aug 23, 2002 23:09:40 UTC - Aug 24, 2002 08:14:30 UTC
714215476-714250468 -0.249838
Aug 24, 2002 09:11:03 UTC - Aug 24, 2002 18:54:15 UTC
714256623-714340429 -0.178133
Aug 24, 2002 20:36:50 UTC - Aug 25, 2002 19:53:36 UTC
714345572-715395019 -0.231601
Aug 25, 2002 21:19:19 UTC - Sep 07, 2002 00:50:06 UTC
715406173-715406478 -0.468361
Sep 07, 2002 03:56:00 UTC - Sep 07, 2002 04:01:05 UTC
715408880-715497528 -0.327985
Sep 07, 2002 04:41:07 UTC - Sep 08, 2002 05:18:35 UTC
715498597-715527400 -0.231601
Sep 08, 2002 05:36:24 UTC- Sep 08, 2002 13:36:27 UTC
715564704-715580548 -0.327985
Sep 08, 2002 23:58:11 UTC - Sep 09, 2002 04:22:15 UTC

```

- **Open Loop Gain function  $H(f)$**

From the model described, we construct a smooth function for the open loop gain, and fit an overall factor to match the measurement at the reference time. The agreement between model and measurement, as well as the difference, is shown in Fig. 11. The error in amplitude can be parameterized as 3% below 500 Hz, and a power law  $.03(f/500)^{1.4}$  for higher frequencies. The error in amplitude gets as large as 8% at 1kHz, and 20% at 2 kHz. The error in phase is not easily parameterized, but it is smaller than 5 degrees for frequencies below 2kHz. Below 500 Hz, the error is consistent with the statistical error in the measurement of the open loop gain; at higher frequencies there is a dominant systematic error that is likely due to the shortcomings of the model.

## 5 Reference Sensing Functions

The sensing functions  $C_0(f)$  can be estimated from independent estimates for the open loop function  $H$  and for the actuation and feedback filter functions,  $C = H/(AG)$ . We did the estimate of  $C(f)$  in different ways for LLO and LHO. The reference sensing functions are shown in Fig. 12. **Notice that as explained in section 6, the reference sensing function for H1 is 28% higher than the actual sensing function.**

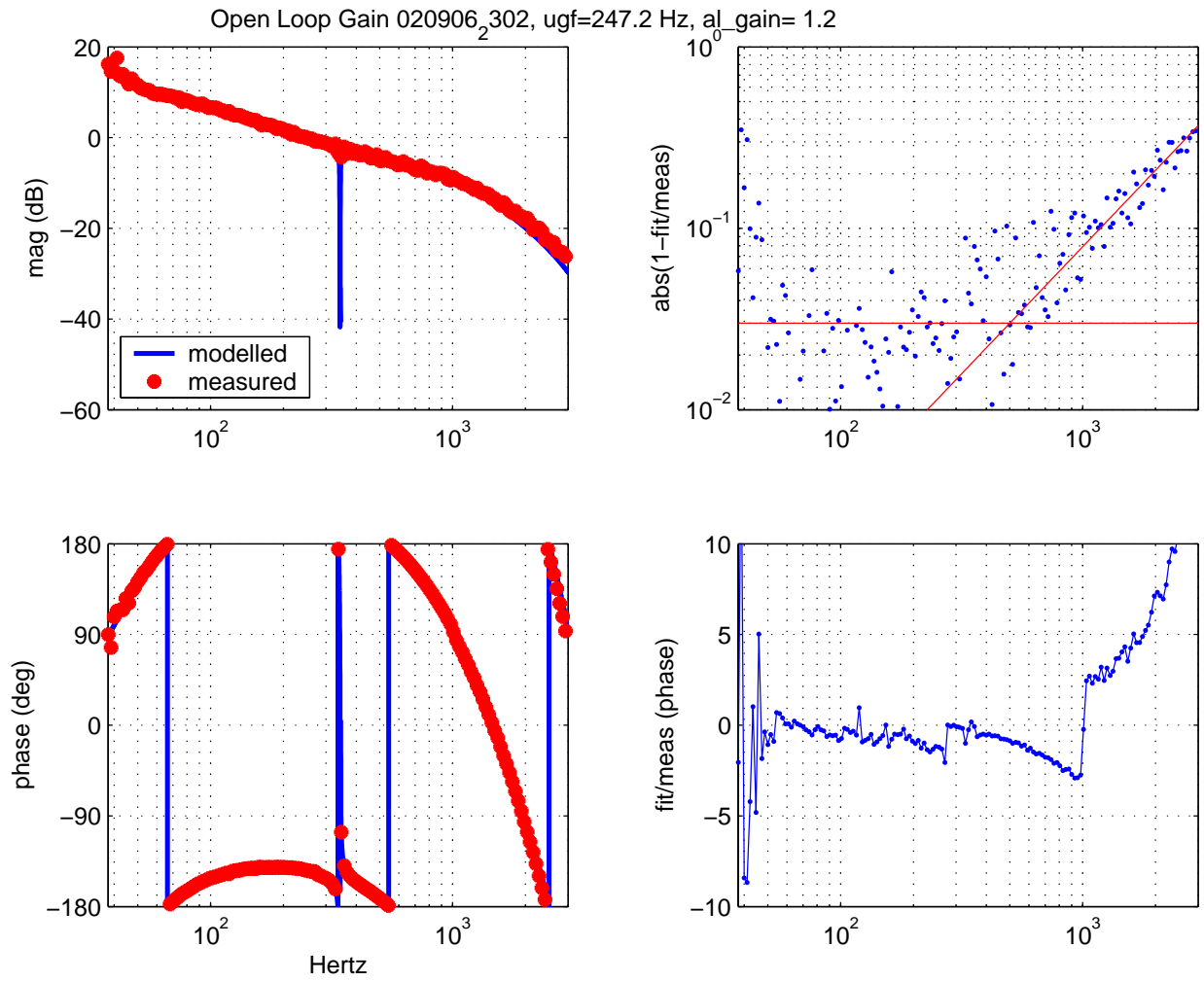


Figure 11: Measured and modeled open loop gain. The errors in magnitude and phase on the right panels are the errors (statistical and systematic) in the reference open loop function for L1.

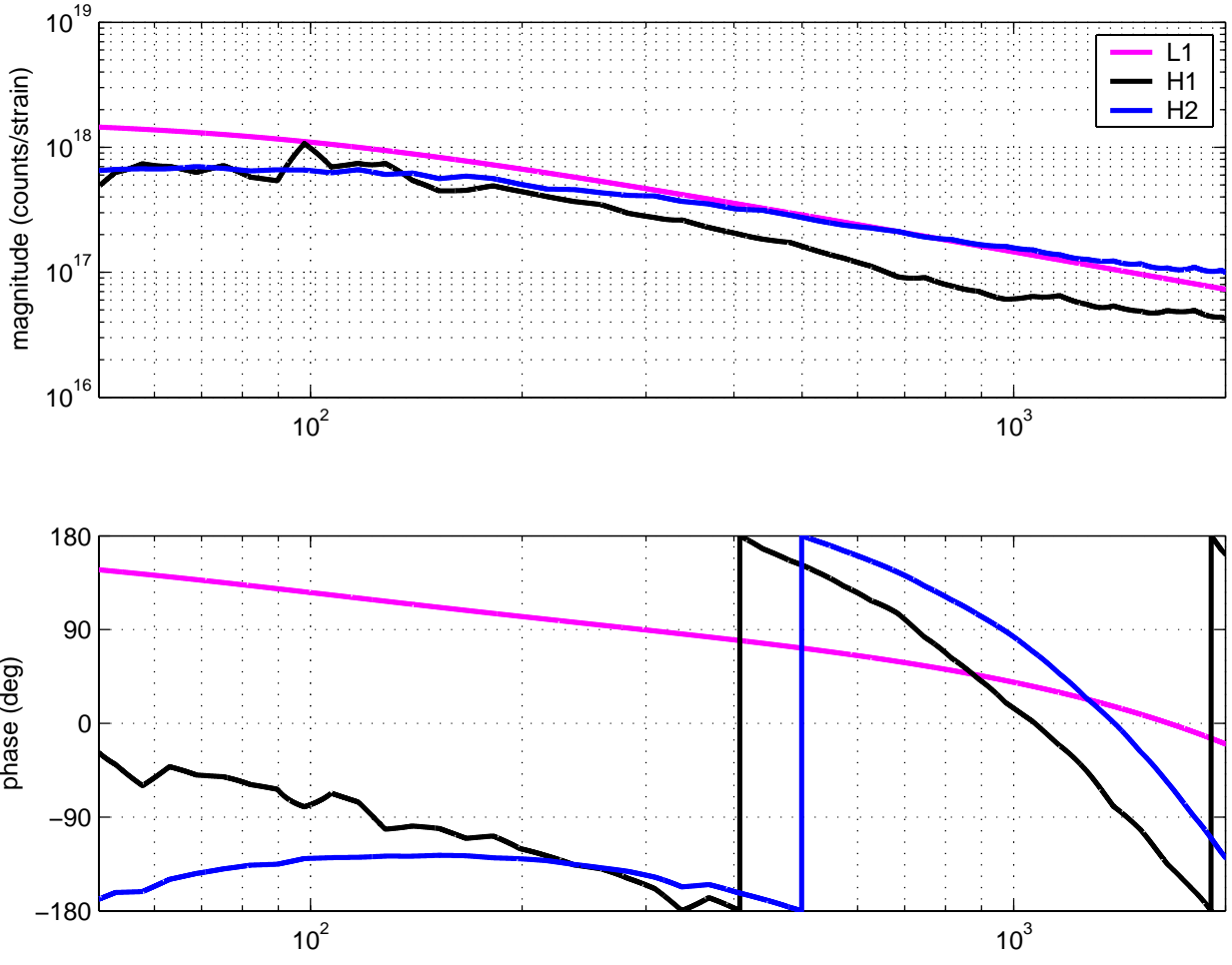


Figure 12: Sensing functions  $C_0(f)$  for the three LIGO detectors.

## 5.1 LHO detectors

For LHO detectors, we assumed the actuation function was just a pendulum transfer function, with a DC gain which was estimated according to procedures detailed in section 6.

The feedback filter function  $G$  was interpolated from a measurement of the transfer function  $DARM\_CTRL/ASQ$ , this measurement has negligible error since it is a digital filter. The reference function  $C_0$  is then calculated as  $H_0/(AG)$ . The error as a function of frequency is at least the error in  $H_0$ , plus the error in  $A$ . There is a systematic error in  $A$ , due to the absence of anti-imaging filters, snubbers, whitening filters, etc; there is also an error due to the absence of the DAC/ADC time delay. During S2 investigations, comparisons of results from the ‘‘autocalibrator’’ (which used the same method described here), and the official calibration (which took into account these filters and time delays in a more complete model)<sup>1</sup> showed good agreement in magnitude below 1kHz, but growing discrepancies of up to 30-40% at 2 kHz; the phase showed large disagreements that were consistent with time delays of up to 150  $\mu$ seconds. We thus estimate a statistical error in  $C(f)$  for H1 and H2 equal to that of  $H(f)$  (shown in Figs. 5, 6) below 1 kHz; and a larger systematic error in magnitude, of the form  $0.35(f/2kHz)$ , which dominates above 1 kHz. The anti-aliasing filter used in LLO’s model has -10 degrees at 300 Hz, and -40 degrees of phase at 1kHz: these are then estimates of the systematic error in phase of  $A$ , and  $C$ . There is also an overall error in the gain of  $C$ , due to the error in the DC gain of  $A$ , this is estimated to be 7%.

Summarizing, the LHO reference sensing functions have a statistical error plotted in Figs. 5,6 (plus 7% added in quadrature); systematic errors in magnitude given by  $0.35(f/2kHz)$ ; systematic errors in phase, less than 10 degrees below 300 Hz, growing to -40 degrees at 1 kHz; and a time delay of up to 150  $\mu$ s.

## 5.2 LLO Reference Sensing Function

For the L1 detector, we chose to use for the reference sensing function the one resulting from the model described in Section 4.2, including the arm cavity pole, the anti-aliasing filter, an ADC time delay, and an overall gain obtained fitting the overall gain of the open loop gain to match the measured reference open loop gain. The frequency dependence then does not have statistical errors, since it is not derived from a measured function; it has, however, systematic errors associated with the assumptions in the model. We can estimate these errors from the deviations of the open loop gain  $H(f) = A(f)C(f)G(f)$  measurement and its model; these are shown in Fig. 11. However, some of these deviations are due to the failures of the model for  $G(f)$ , shown in Fig. 10. We show in Fig. 13 the deviations of measurement and model for both  $H$  and  $G$ . It seems that most of the deviations in the magnitude of  $H$  below 2 kHz can be explained by the deviations in the model for  $G$ : we estimate the residual systematic error in the product of  $AC$ , equal to  $H/G$ , to be then less than 10%. In the phase, however, we have the opposite case: it looks like the error in  $AC = H/G$  is larger than the error in  $G$ . In hindsight, the time delay of chosen to fit the phase of model and measurement of  $H$  was wrongly compensating for the error in  $G$ . The systematic phase error in  $AC$  is then less than 5 degrees below 1 kHz, and between 10 and 15 degrees for frequencies between 1 kHz and 2 kHz. If we had chosen a time delay of 140  $\mu$ seconds instead of 125  $\mu$ seconds, the systematic error in  $AC$  would be less than 5 degrees at all frequencies below 2 kHz.

If we assume the error in the model for  $AC$  is all due to error in  $C$ , and that we do not have correlation (or anti-correlation) in the models for  $A$  and for  $C$ , then we can have an estimate for the error in the reference sensing function  $C$  for L1: there is only systematic error; the error in magnitude is  $\approx 5\%$  below 2 kHz; in phase it is less than 5 degrees below 1 kHz, and 15 degrees between 1 kHz and 2 kHz. The error in the magnitude of  $C$  is correlated with the systematic error in the magnitude of  $H$ .

---

<sup>1</sup>See Patrick Sutton, LIGO-G0030093

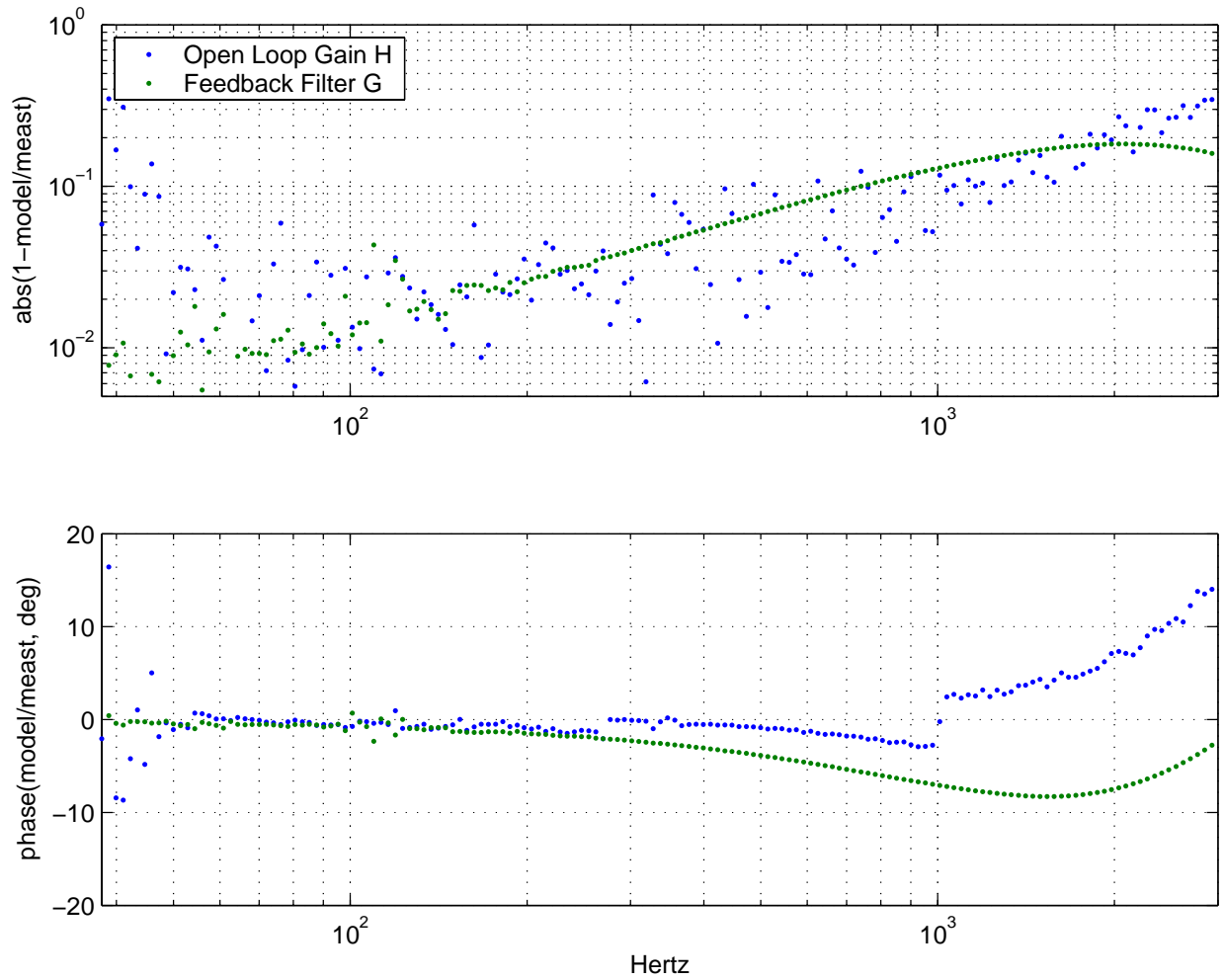


Figure 13: Systematic error in  $G$ ; and statistical and systematic error in  $H$ . The systematic error in the amplitude of  $H$  can be mostly accounted by the error in  $G$ ; the systematic errors in phase for  $H$  are smaller than in  $G$ , implicating the systematic errors in the phase of  $H/G$  are similar (but opposite in sign) to the ones in  $G$ .

## 6 Calibration methods of the actuation function

As we mentioned earlier, the calibration of the actuation function is critical to the whole process of reconstructing the function  $R(f)$ . The procedure involves several steps, starting with the laser wavelength ( $\lambda = 1064$  nm) as the length standard. Using only the beamsplitter and the input mirrors of the arm cavities, a simple Michelson loop signal is read at the antisymmetric port, and a feedback loop is closed acting on only one of the mirrors. Two general class of methods are used to calibrate the actuation function to the desired input test mass, using either the error signal or the control signal of the Michelson loop (see T020141-00-D for more details). If there is no feedback control, and the mirrors move with a displacement function  $x(t)$ , the sensor signal at the dark port is  $e(t) = S_0 \sin(2\pi x(t)/\lambda)$ . The power at the antisymmetric port is  $p(t) = P_0 \sin(4\pi x(t)/\lambda)$ . We can easily obtain the peak value  $S_0$  and  $P_0$  in volts, or counts of the signal that is read, by driving the mirrors to swing through full fringes.

If external disturbances are small enough, a mirror can be driven at a known, slow frequency, that sweeps through several fringes before turning back. The observed signals for power and sensor signal can be then fitted to the motion, and the known wavelength can be used to calibrate the motion and thus the actuation function.

If the feedback loop is closed, the sign of the loop can be toggled to produce a dark or a bright port at the sensed output. With high gain, the step produced when toggling the sign is an odd integer number of quarter-wavelengths  $\lambda/4$ , and in general is only  $\lambda/4$  or  $3\lambda/4$ . By repeating the procedure many times, measuring the control signal needed to move the mass each time, and taking differences of the consecutive signals, a calibration of the actuation can again be obtained (and the predicted quantization can be confirmed).

Alternatively, if the peak amplitude  $S_0$  is measured, when the loop is closed, the error signal near the null point is calibrated:  $k = de/dx|_{x=0} = S_0(2\pi/\lambda)$ , and  $e(t) = kx(t)$ . We then drive the mirror at different frequencies with a swept sine function, and measure the motion produced with the calibrated error signal. The advantage of this method is that we can measure the response at the frequencies in the gravitational wave band.

Having calibrated the actuation function on the input test masses with one or more of the methods described, we align an input test mass and the corresponding test mass at the end of the arm. We close a feedback loop on the resonant optical cavity of the single arm by actuating on the end test mass. We can then drive the calibrated input test mass with a sine wave of know amplitude, and measure the response of the end test mass; assuming the measurement is done at frequencies with high loop gains, we can infer the motion of the end test mass. Alternatively, we can calibrate the error signal of the feedback loop using the calibrated drive to the input test mass, and calibrate the actuation on the end test mass at different frequencies, independently of the magnitude of open loop gain.

The measurements are reported as “DC gains” of pendulum transfer functions for each mass, in meters/count. The details of procedures used, results for the different detectors, and error estimates follow:

The “official” calibrations obtained for S1, and used for obtaining the reference sensing functions were:

### 6.1 H1 detector: error estimates

The reference DC calibration used for the H1 actuation function (from which the gain for the reference sensing function is derived) is 1.11 nm/count. Two different measurements, and several follow-up corrections were done, as follows:

- June 25-26th, 2002 ITM/ETM transfer measurements (at 4Hz, derived from sign toggling)<sup>2</sup>: H1X:

---

<sup>2</sup>LHO elog Jun 27, by landry\_m

Table 1: DC Calibrations for end test masses during the S1 run.

Detector	Mirror	DC calibration (nm/count)
<b>L1</b>	ETMX	2.53
<b>L1</b>	ETMY	2.67
<b>H1</b>	ETMX	1.08
<b>H1</b>	ETMY	1.11
<b>H2</b>	ETMX	1.37
<b>H2</b>	ETMY	1.45

1.05nm/ct; H1Y: 1.20nm/ct. The ETM mirror controllers were modified on July 16, 2002<sup>3</sup>, resulting in a scaling of 1.0687, or

H1X: 1.122nm/ct and

H1Y: 1.283nm/ct.

The statistical errors associated with this method are approximately 2% (from method description in T020141-00-D).

- PZT fine-actuator methods for DC cal (LHO elog, Aug 20 2002, by hugh): H1X: 1.08nm/ct; H1Y: 1.11nm/ct Scaled by 0.95 to account for "Josh box" filtering<sup>4</sup>, resulting in H1X: 1.026nm/ct, and H1Y: 1.055nm/ct Scaled by 0.83 for coil driver filtering<sup>5</sup>:

H1X: 0.848nm/ct

H1Y: 0.872nm/ct

The statistical error associated with these results is 7%<sup>6</sup>.

We studied (a long time after S1) the possible systematic effects in each method, and although we don't understand the difference between the answers, we found strong evidence supporting the answer obtained with the PZT method, with the corrections mentioned. The point calibration for H1 (measured open loop function) used the Y-arm, so the DC calibration to use for the actuation function, in order to obtain the reference sensing function, is 0.87 nm/ct, with a 7% statistical error.

The official S1 point calibration for H1 employed a Y-arm transfer function, and used the original PZT value (without corrections) of 1.11nm/ct was the original number used. **Thus, the sensing function for H1 has a gain 28% higher than the official one used for data analysis group at this time (May 20, 2003).**

## 6.2 H2 DC Calibration

The calibration for H2 was done using only the PZT method (LHO elog entry on Jan 27, 2002, by hugh). Again, the estimated statistical error is 7%. The official numbers in Table 1 need to be corrected by a factor

<sup>3</sup>e-log entry at LHO 07/16/02 by richardm

<sup>4</sup>elog 04/14/2003, by hugh

<sup>5</sup>elog 04/27/2003, by peterF

<sup>6</sup>see details in [www.ligo-wa.caltech.edu/~hradkins/DCcal.html](http://www.ligo-wa.caltech.edu/~hradkins/DCcal.html)

Table 2: Calibration lines injected in the LIGO detectors during the S1 run.

Detector	Frequency (Hz)	Amplitude (m)	Amplitude (strain)
<b>L1</b>	51.3	$2.2 \times 10^{-14}$	$5.5 \times 10^{-18}$
<b>L1</b>	972.8	$3.8 \times 10^{-16}$	$9.5 \times 10^{-20}$
<b>H1</b>	37.25	$1.3 \times 10^{-12}$	$3.3 \times 10^{-16}$
<b>H1</b>	973.3	$6.4 \times 10^{-16}$	$1.6 \times 10^{-19}$
<b>H2</b>	37.75	$1.35 \times 10^{-13}$	$6.8 \times 10^{-17}$
<b>H2</b>	973.8	$8.1 \times 10^{-16}$	$4.1 \times 10^{-19}$

of 0.99 due to the mentioned RC filtering in the drive to the actuators; this then represents a systematic 1% error.

### 6.3 L1 DC Calibration

The measured number 1.72 nm/ct for (Ly-Lx)/DARM\_CTRL is from an LLO e-log entry by rana on August 8, 2002. The calibrations were “AC” calibrations done comparing responses of calibrated ITMs and ETMs in as measured by the error signal in a single arm cavity. The method is describe in a long LLO e-log entry for E7 calibrations, on Dec 31 2001, by rana.

No errors are mentioned in the S1 entry on Aug 8. For S2 calibrations logged on LLO’s e-log on February 1, 2003, the errors associated with ETMcalibrations done with the same method, are 5% and 8% for ETMX and ETMy respectively. We will then assume a conservative 8% error for the constant used in the S1 calibration.

## 7 Use of calibration lines

Since it was recognized that the calibration changed considerably with a time scale of tens of minutes (or faster), we introduced calibration lines to track the changes in gains. All detectors had two calibration lines at different frequencies, one below the unity gain and one near 1kHz, where the gain was very low. The excitation acted on the mirror at the end of the X-arm of each detector. In the absence of a feedback loop, the excitations produced a sine wave of peak amplitude shown in Table 2, clearly visible in the spectrum of the error signal  $e(f)$ . The low frequency line is suppressed by the loop gain, so it appears smaller than the magnitudes shown in Table 2, while the observed high frequency line amplitude was very close to the one presented in the same table.

The amplitudes of the lines in the spectrum were followed using a software tool that wrote an estimate using a minute averaging time<sup>7</sup>. At the reference time when the open loop gain  $H_0$  was measured, the amplitude of a calibration line with amplitude  $S_0$ , at a frequency  $f_{cal}$  in the error signal is given by  $e_0(f_{cal}) = S_0 C_0(f_{cal}) / (1 + H_0(f_{cal}))$ . The sensing function  $C_0$  can be calculated from the measured  $H_0$  and the modeled actuation and feedback filter functions, which have fixed gains:  $C_0 = H_0 / (AG)$ .

Assuming the sensing function fluctuations (and therefore those of the open loop gain) can be parameterized by an overall multiplicative factor  $\alpha(t)$ , the amplitude of the calibration lines at any time during S1

<sup>7</sup>see S.Klimenko, G.Mitselmakher, A.Sazonov, E.Daw, J.Castiglione, “Line Monitor”, LIGO Technical document LIGO-T010125-01-D



will be

$$e(f_{cal}, t) = \alpha(t) \frac{C_0(f_{cal})}{1 + \alpha(t)H_0(f_{cal})} = e_0(f_{cal})\alpha \frac{1 + H_0(f_{cal})}{1 + \alpha(t)H_0(f_{cal})}.$$

From the ratio of the measured amplitude  $e(f_{cal}, t)$  and the reference amplitude  $e_0(f_{cal})$ , we get the amplitude of the function  $\alpha(t)$ .

The feedback filter  $G$  was changed a few times during S1 in L1. If we consider  $G$  changing by a known ratio  $\beta$  from its reference value, and the sensing function changing by the ratio  $\alpha$  considered above then the open loop gain will change by a factor  $\alpha\beta$  and the measured line amplitude will be

$$e(f_{cal}, t) = \alpha(t) \frac{C_0(f_{cal})}{1 + \alpha(t)\beta H_0(f_{cal})} = e_0(f_{cal})\alpha \frac{1 + H_0(f_{cal})}{1 + \alpha(t)\beta H_0(f_{cal})}.$$

The graphs used to deduce the value of  $\alpha(t)$  from the measured ratios  $e(f_{cal}, t)/e_0(f_{cal})$  in the L1 detector, for all values of  $\beta$  used during S1, are shown in Figure 14. Since the loop gain at the high frequency line is small ( $\approx 0.3$ ), the amplitude of the line is approximately linearly dependent on the value of  $\alpha$ ; while the amplitude of the low frequency line, where the loop gain is high ( $\approx 4$ ), has a steep, nonlinear dependence on the value of  $\alpha$ . For this reason, we used the high frequency line to derive our estimate of  $\alpha$ , and only used the lower frequency line as a consistency check.

Note that there is only a finite possible range for the product  $\alpha\beta$  that makes the loop stable: if the gain drifts too high or too low, the feedback loop will become unstable and the detector will not be operational. For example, at L1 this range is  $0.4 \leq \alpha\beta \leq 1.4$ .

The slope  $d\alpha/dR$ , where  $R$  is the ratio of the calibration line amplitude to the reference line amplitude, is approximately one; this means that the relative errors in the estimate for *alpha* are equal to the relative errors in the estimate of the line amplitude.

The measured gain factor  $\alpha\beta$  as a function of time is shown in Figure 15 for the three detectors. The breaks in the plot are due to the fact that the interferometers were not in continuous operation for more than a few hours at a time. The duty cycle varied between 42% (for L1) to 73% (for H2). There is a significant scatter, even within a continuous segment when the detector is in operation, due to fluctuations in alignment. These fluctuations will ultimately be controlled by angular control feedback loops, which were only partially commissioned at the time of S1. The fluctuations are not random, and the distribution of values is not Gaussian, as shown in the histograms of all values, in Figure 16.

There is also some random scatter due to the statistical error in the estimation of the line amplitude. If we take the difference of consecutive measurements of the line amplitude, we see that the distribution of the differences is now Gaussian, which is more typical of statistical errors in the amplitude spectral estimate than of alignment fluctuations with a minute time scale. We can use the width of these distributions to assign an error to any single estimate of  $\alpha(t)$ . These error estimates are 2% (for L1), 4% (for H1), and 7% (for H2); they are mostly dependent on the signal to noise of the calibration lines.

It is known that there are alignment fluctuations with several time scales shorter than 60 seconds, in particular at the microseismic peak (6 sec), stack resonances (1-2Hz), mirror vertical bounce modes (12-18 Hz), etc. Some estimates for the fluctuations observed in the L1 detector before S2 are detailed in an LLO e-log entry on Feb 8, 2003, by gaby. This is probably a worst case scenario since the LHo detectors are in general quieter; the estimated relative fluctuations within 60 seconds were 8% in sideband power and 4% in arm power. This means that if we were to use an estimate for  $\alpha$  for analysis on data shorter than 60 seconds, there is an additional statistical error of up to 9%.

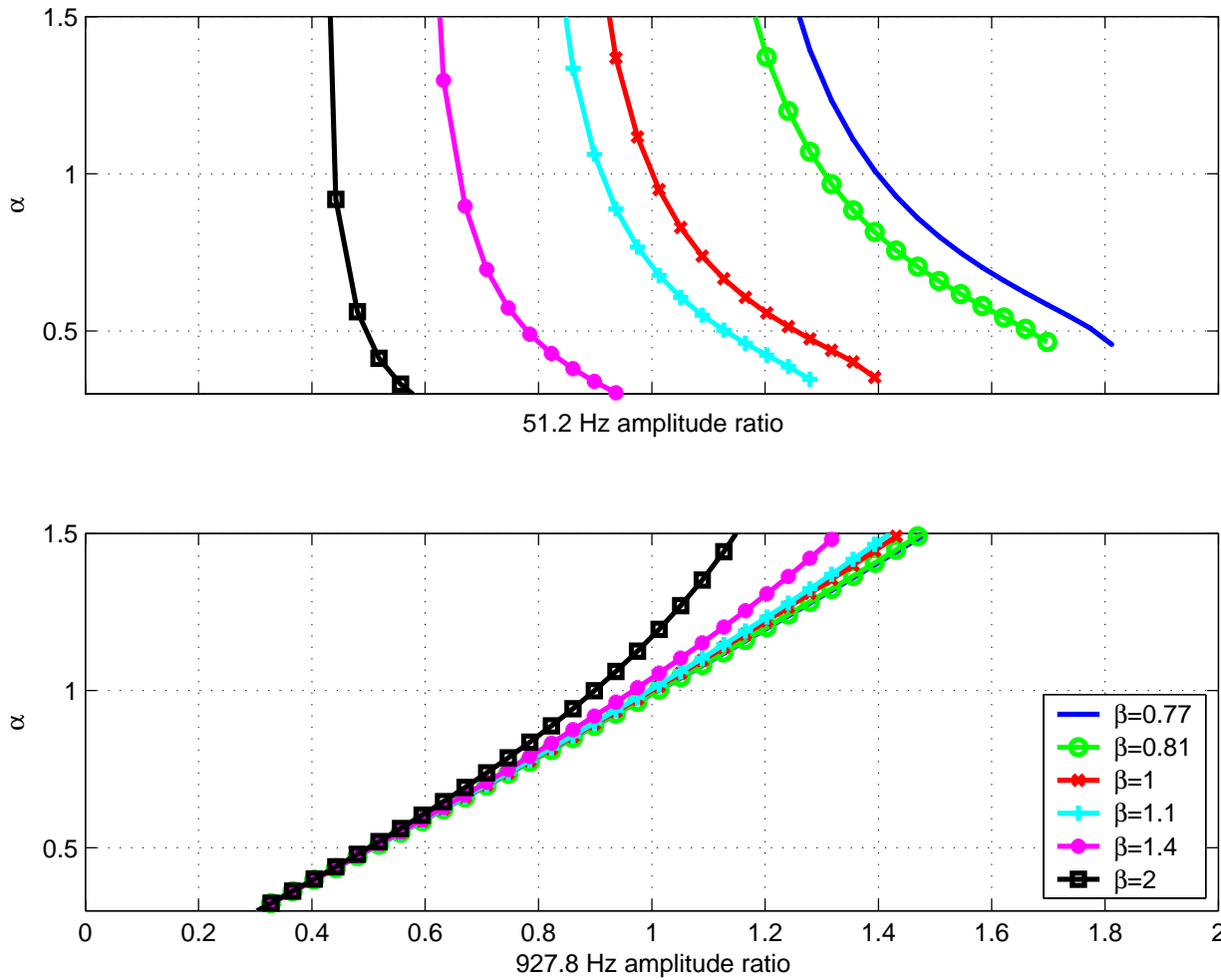


Figure 14: The inferred value of  $\alpha$  from the measured ratio of the amplitude of the calibration lines at any given time, to the amplitude of the same line at the reference time for L1, Sep 06, 2002 23:02 UTC. Since there were several values of gains used for the feedback filter function  $G(f)$ , we have one curve for each  $\beta$ .

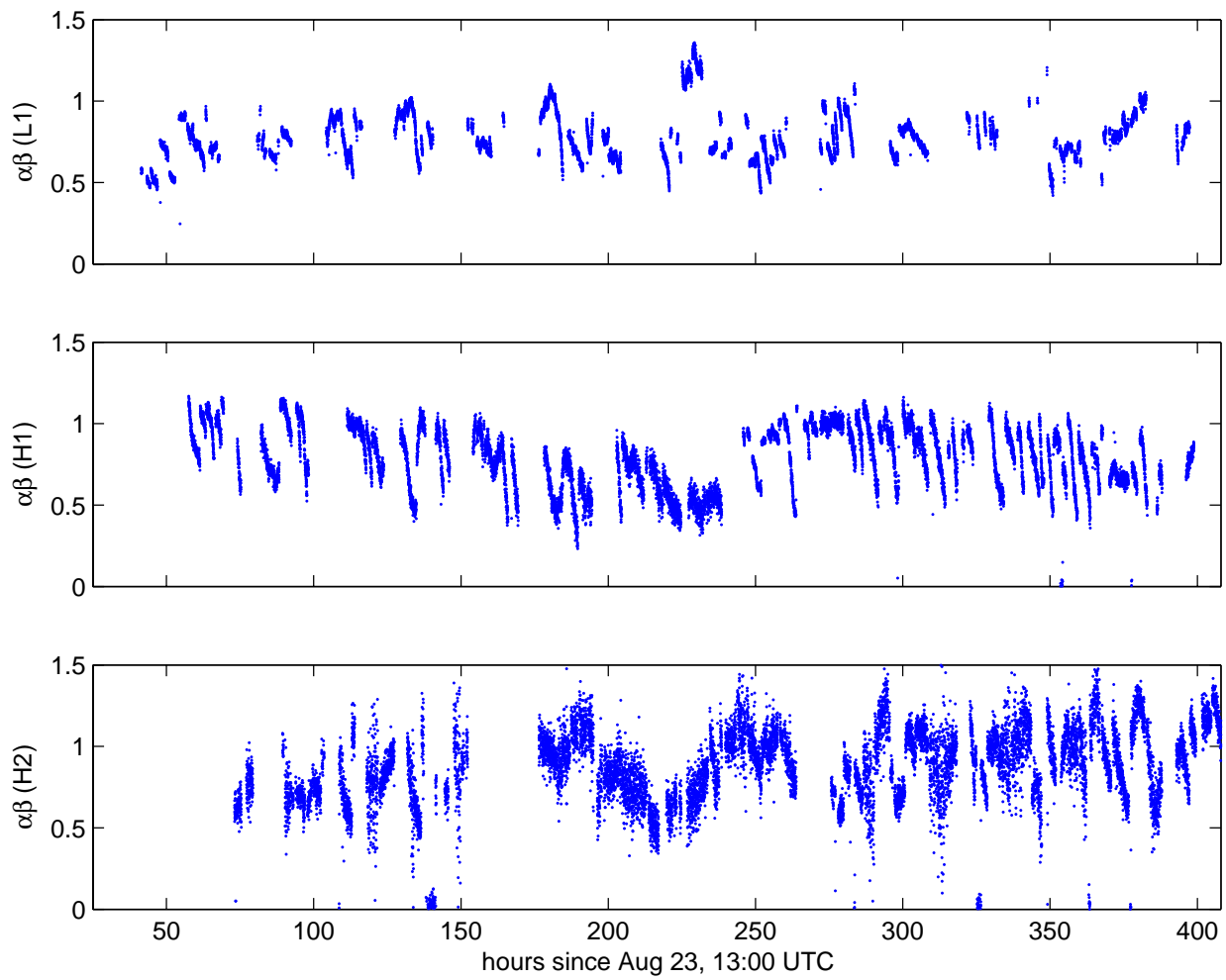


Figure 15: Open loop gain ratio  $\alpha\beta$  to the reference times during S1 times when data was available for data analysis. The first day was dedicated to calibration, so it does not include available data.

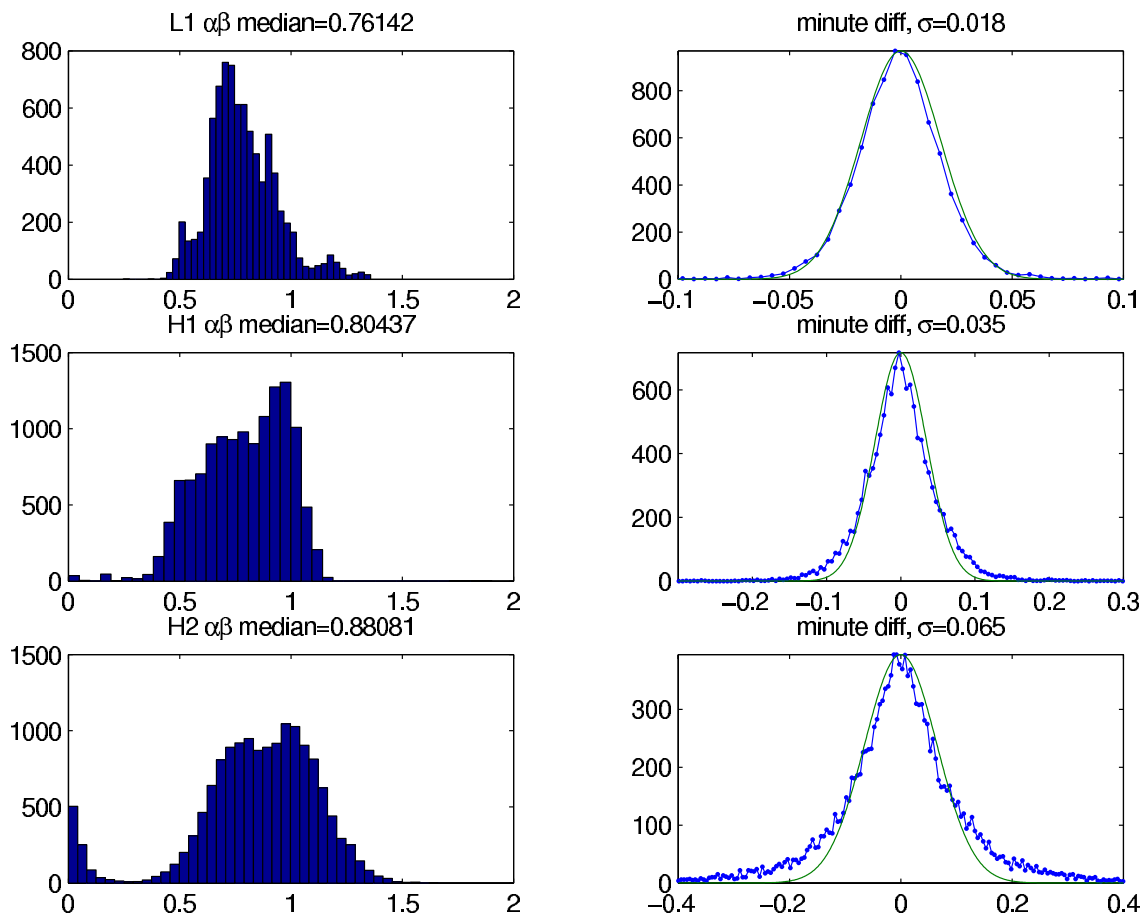


Figure 16: On the left side, the histogram of the values  $\alpha\beta$  in the three detectors. The distributions are not well random, indicating systematic variations due to alignment fluctuations. On the right side, we plot the histogram of the difference between consecutive values. These values are better approximated by Gaussian distributions, representing the statistical error in the estimate of  $\alpha\beta$  from the amplitude of calibration lines.

## 8 Conclusions: S1 Response Functions

Once we know the value of  $\alpha$  (and  $\beta$ ) at any given time, we can deduce the response function:

$$R(f, t) = \frac{\alpha(t)C_0(f)}{1 + \alpha(t)\beta H_0(f)}. \quad (1)$$

In general this will not be a simple scaling of the the response function at the reference time. The changes can be very significant, as shown in Figure 17, where the reference calibration for L1 is shown, together with the calibrations obtained for gains close to maximum and minimum values of the product  $\alpha\beta$ .

We took as reference functions the best measurements of loop gains during S1; the corresponding reference response functions are shown in Figure 18. This response is then recalculated using Equation 1 for every minute of data during S1 when the interferometers were in operation, and an estimation of  $\alpha$  was available. **Notice that the H1 response is acutally 22% smaller than the reference one shown in 18, as explained in Section6.**

The error in each of the calculated response functions  $R(f, t)$  comes from three sources: the error in the reference functions  $C_0$ ,  $H_0$ , and the error in  $\alpha$ .

The reference functions  $H_0$  and  $C_0$ , in turn, are complex:  $H_0 = |H_0|e^{i\theta_H}$ ,  $C_0 = |C_0|e^{i\theta_C}$ , and we have estimated errors in amplitude and phase for each. In order to propagate these errors, and errors in  $\alpha$ , into errors for magnitude and phase of the response function, we need a bit of algebra and calculus. We present in the following table the weights associated for a relative error in magnitude  $\delta|R|/|R|$ , and in phase  $\delta arg(R)$ , for each of the errors in magnitude and phase  $\delta|H_0|/|H_0|$ ,  $\delta|C_0|/|C_0|$ ,  $\delta\theta_H$ ,  $\delta\theta_C$ , and  $\delta\alpha$ . We evaluate the formulas for  $\alpha$  equal to the median value  $\alpha_0$  for each detector.

$$\begin{aligned} \frac{\delta|R|}{|R|} &= \frac{1 + \alpha_0|H_0| \cos(\theta_H)}{|1 + \alpha_0 H_0|^2} \frac{\delta\alpha}{\alpha} + \frac{\delta|C_0|}{|C_0|} \\ &+ \alpha_0|H_0| \frac{\alpha_0|H_0| + \cos(\theta_H)}{|1 + \alpha_0 H_0|^2} \frac{\delta|H_0|}{|H_0|} \\ &+ \frac{\alpha_0|H_0| \sin(\theta_H)}{|1 + \alpha_0 H_0|^2} \delta\theta_H \\ \delta\theta_R &= \frac{\alpha_0|H_0| \sin(\theta_H)}{|1 + \alpha_0 H_0|^2} \frac{\delta\alpha}{\alpha} + \delta\theta_C \\ &+ \frac{\alpha_0|H_0| \sin(\theta_H)}{|1 + \alpha_0 H_0|^2} \frac{\delta|H_0|}{|H_0|} \\ &+ \frac{\alpha_0|H_0|(\alpha_0|H_0| + \cos(\theta_H))}{|1 + \alpha_0 H_0|^2} \delta\theta_H \end{aligned}$$

Of course, we take only the magnitudes of the error weights, and add them in quadrature. The formulas seem complicated for all but the errors in the sensing function, which get weighted by unity for relative magnitude and phase. However, a plot of the relevant functions for the L1 detector shown in Fig. 19 reveal that all factors are less than two, and in fact less than unity for most frequencies. Collecting all the information on the individual errors and using the appropriate weights, we can then estimate an error for the response functions in each of the detectors as follows:

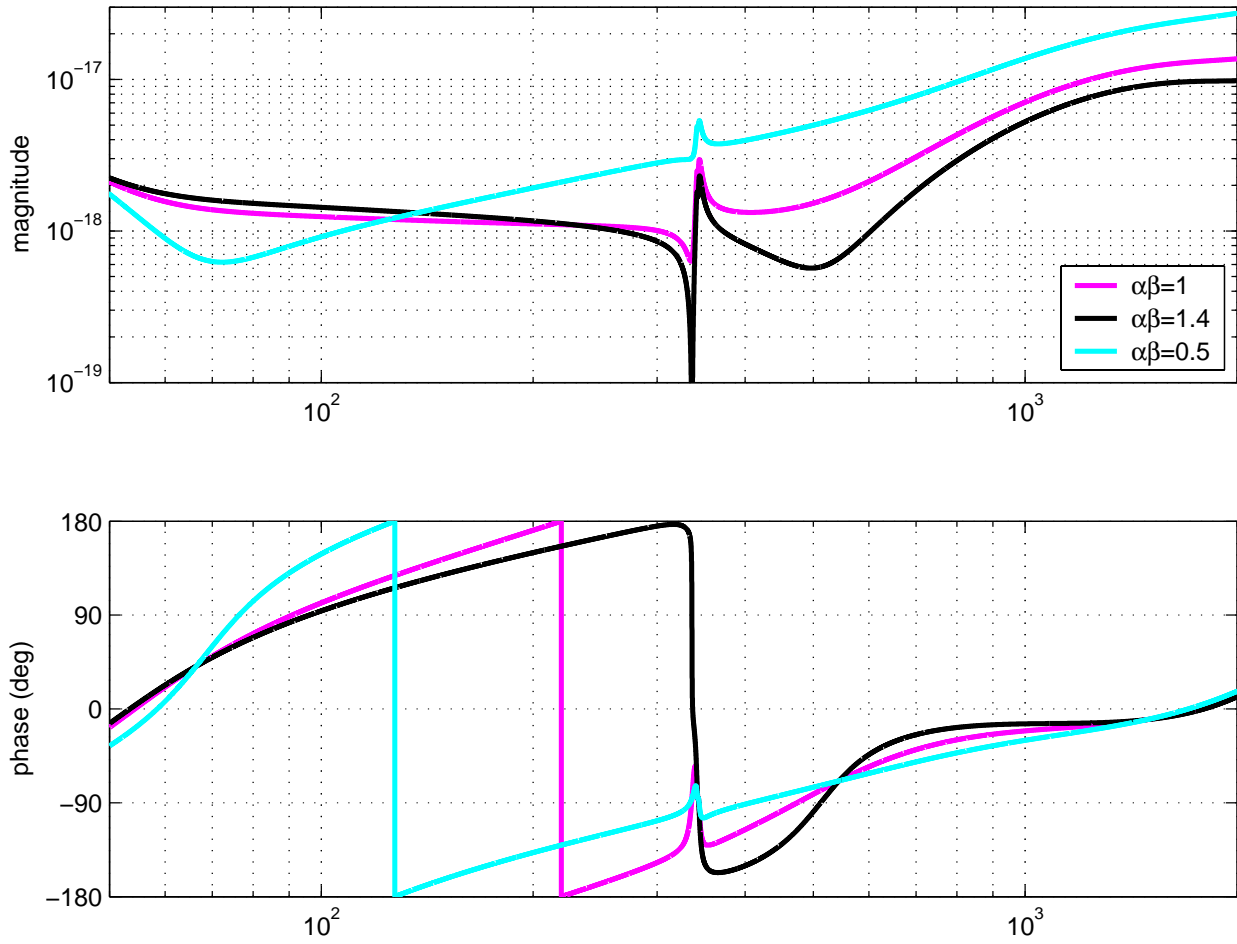


Figure 17: Response function  $R_0(f)$  for L1 at reference time Sep 06, 2002 23:02 UTC, when  $\alpha\beta = 1$ , and response functions for extreme values of the open loop gain scale  $\alpha\beta=0.5$  and  $1.4$ . These two latter curves should be taken as an envelope of possible response functions during S1. The values of  $\alpha\beta$  for L1 had a median of 0.76, and were usually between 0.5 and 1.0.

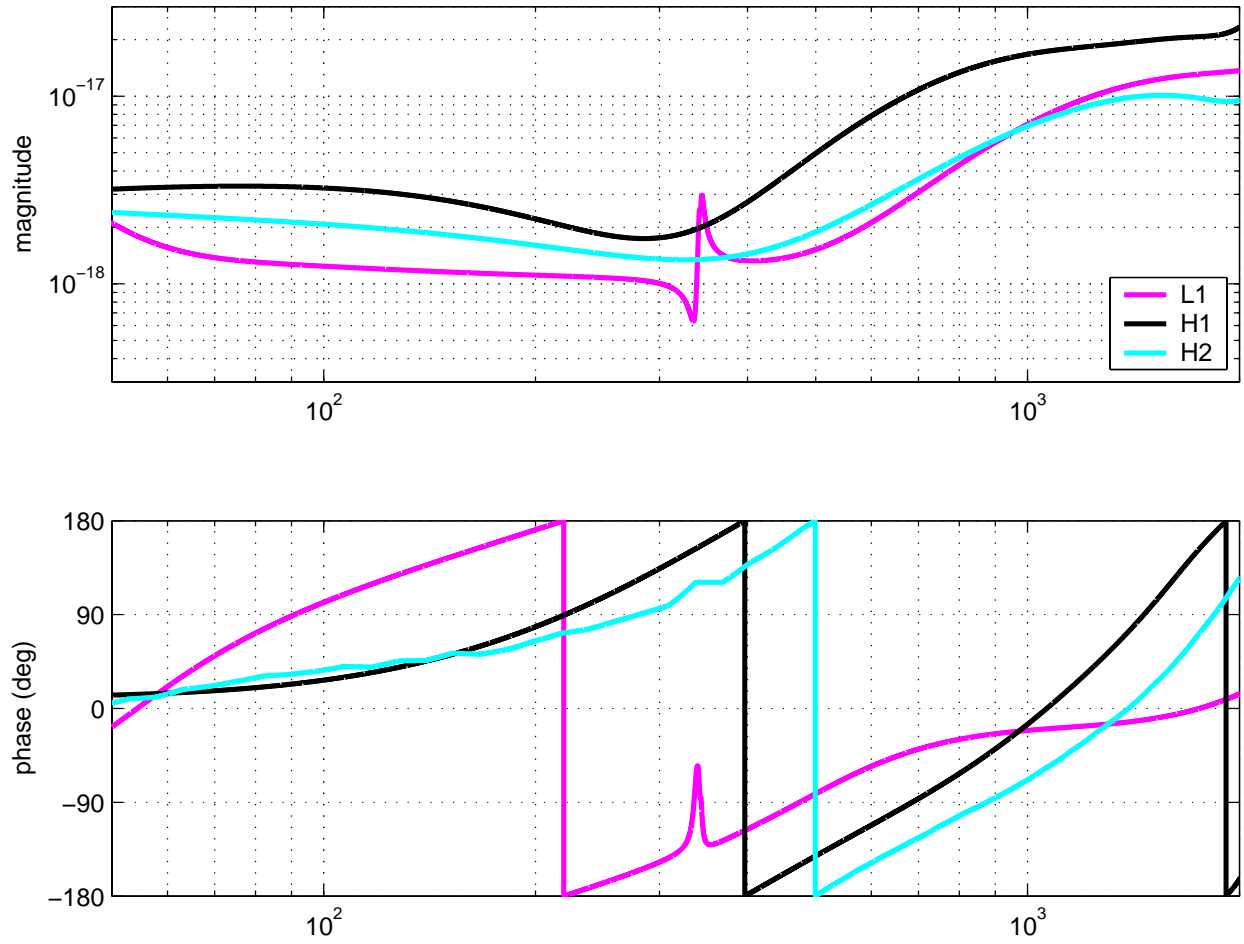


Figure 18: Response Functions calculated from the reference open loop gain and sensing functions.

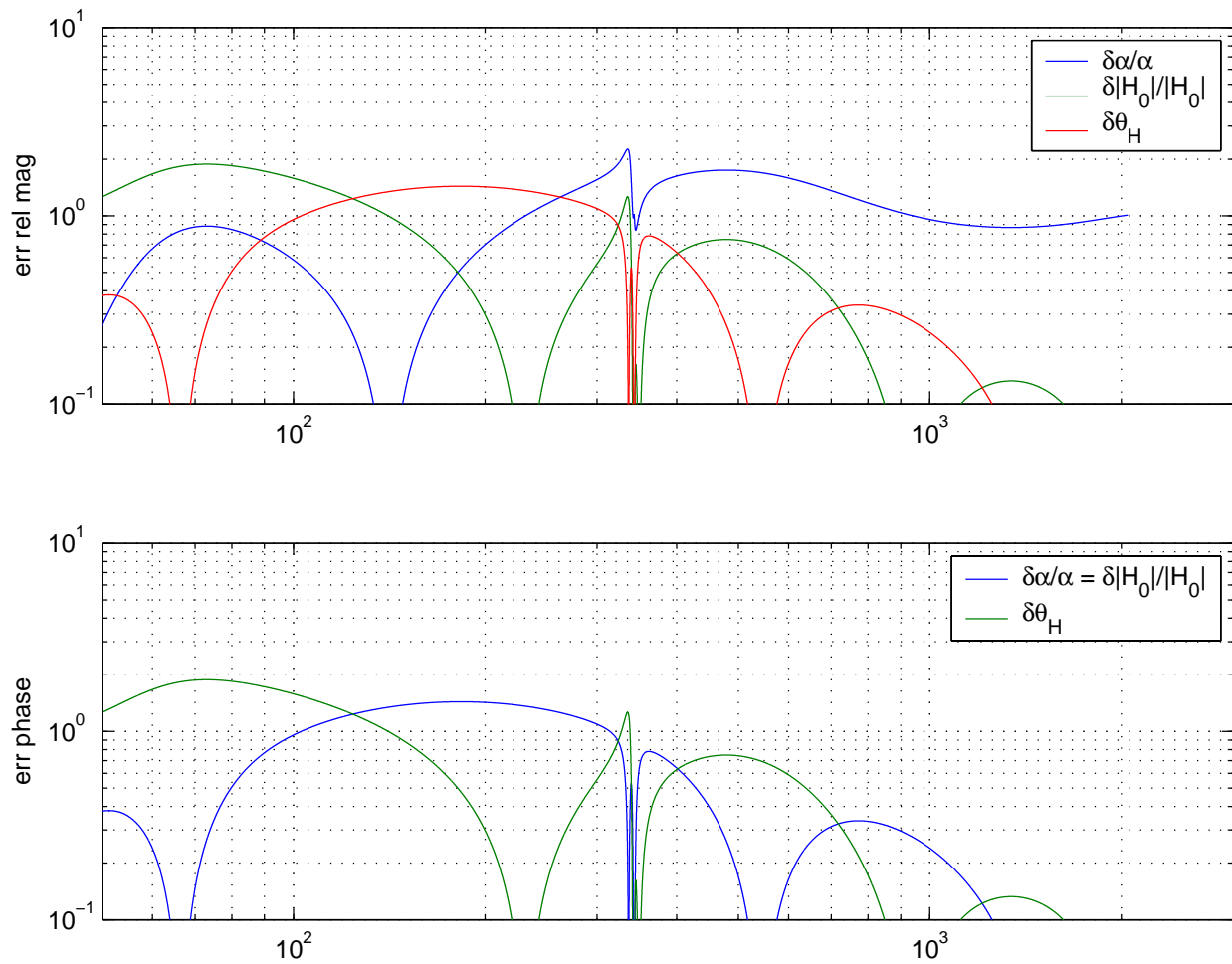


Figure 19: Weights for the different errors contributing to the relative error in magnitude and the error in phase of the L1 response function (other than the contributions of the errors in the sensing function, which have unity weighting gfactor).



## 8.1 L1 detector

Assuming the following for statistical errors:

- $\delta\alpha/\alpha_0=2\%$  (Fig. 16)
- $\delta|H_0|/|H_0| = \delta\theta_H=3\% \times (f/2\text{kHz})^{1.4}$  (Fig. 8)
- $\delta|C_0|/|C_0|=10\%$  (from DC calibration, and overall gain fit of  $H(f)$  measurement and model);  $\delta\theta_C=0$  (since all the error is systematic);

then the statistical error in the relative magnitude of the response function is dominated by the error in the relative magnitude of the sensing function, 10%; and the statistical error in phase is less than 3 degrees at all frequencies.

If the response function is used to calibrate events shorter than 60 seconds, the error in  $\alpha$  is 9%, and thus the error in the relative magnitude of the response function 200 Hz and 800 Hz can be as large as 20%.

The systematic errors in  $H$  and  $C$  (in Figs. 11,13), are larger than the statistical ones (especially in phase), for frequencies above 300 Hz; however the error in  $H$  is suppressed at those frequencies by its weighting function, so it is the error in  $C$  which will dominate. For the magnitude of the the response function, the systematic error is then less than 3% below 500 Hz, less than 8% below 1kHz, and less than 10 % below 2 kHz. The systematic error in the response function above 300 Hz is 5 degrees, jumping to 15 degrees between 1kHz and 2kHz.

## 8.2 H1 detector

Assuming the following for statistical errors:

- $\delta\alpha/\alpha_0=4\%$  (Fig. 16)
- $\delta|H_0|/|H_0| = \delta\theta_H = 0.005 \times (f/100\text{Hz})^{1.15}$  (Fig. 5)
- $\delta|C_0|/|C_0|=7\%$  (from DC calibration) +  $0.5\% \times (f/100\text{Hz})^{1.15}$  (from H measurement);  $\delta\theta_C = 0.005 \times (f/100\text{Hz})^{1.15}$ ;

then the statistical error in the relative magnitude of the response function is less than 8%, in and the statistical error in phase is less than 2 degrees at all frequencies.

The systematic errors in phase of the response function are less than 10 degrees below 300 Hz, growing to -40 degrees at 1 kHz; there may also be a time delay of up to 150  $\mu\text{s}$ .

**Notice that the actual sensing function whose errors are estimated here is  $1.11/0.87=28\%$  higher in magnitude than the reference sensing function; thus the response function is  $22\%$  smaller in magnitude from the reference response function. This makes the H1 detector  $22\%$  more sensitive than estimated by the analysis done with the reference sensing functions propagated in time.**

## 8.3 H2 detector

Assuming the following for statistical errors:

- $\delta\alpha/\alpha_0=7\%$  (Fig. 16)

- $\delta|H_0|/|H_0| = \delta\theta_H = 0.06$  (Fig. 6)
- $\delta|C_0|/|C_0| = 7\%$  (from DC calibration) +  $6\%$  (from H measurement) =  $9\%$ ;  $\delta\theta_C = 0.06$ ;

then the statistical relative error in the magnitude of the response function is 11%, and the statistical error in phase is 6 degrees below 400 Hz (where errors in  $\alpha$ ,  $\delta|H_0|/|H_0|$  and  $\delta\theta_C$  all contribute) and 3 degrees above 400 Hz.

The systematic error in the response function is 1% in magnitude; in phase is less than 10 degrees below 300 Hz, growing to -40 degrees at 1 kHz; there may also be a time delay of up to 150  $\mu$ s.