Some notes on pitch frequency, in particular for two wires from one blade, including torsional effect of blade, and crossing of blades.

Norna A Robertson, 5 Sept 2002

T020209-00-R

General Introduction.

The MATLAB code based on the equations derived in Calum Torrie's thesis does not include several factors which may have an effect on the frequencies of various modes. Considering the longitudinal/pitch model, two factors not included are

a) the torsional motion of the blades, when two wires attached to one blade is used (as in Adv. LIGO design)
b) the effect of having the blades crossed at an angle in the x/y plane (as in Adv. LIGO design).

These effects will be considered in turn for a one stage pendulum.

Section 1. General equations, and introduction of torsional effect of blades.

1.1 Introduction.

Assume the wires are vertical, and assume the blades are not crossed. The appropriate equations from Calum's thesis are 4.17 and 4.19.

$$I_y\ddot{\phi} = \frac{mgd}{l}(x_n - x_0) + \left(-mgd - \frac{mgd^2}{l} - 4k_{tot}s^2\right)\phi \qquad \text{equation 1 (4.17)}$$

$$m\ddot{x}_n = -\frac{mg}{l}(x_n - x_0) + \frac{mgd}{l}\phi \qquad \text{equation 2 (4.19)}$$

where 4.17 has been modified to account for 4 "springs" (wires, blades or combinations), each of spring constant k_{tot}. Other terms are as defined in the thesis.

1.2 Simplification for d ~ 0.

Firstly it can be seen that these equations are coupled through the "*d*" value, which is the vertical separation between the breakoff points for the springs at the mass and a horizontal line through the centre of mass. To consider the simplest situation to start with, assume *d* is small enough that we can ignore the terms in *d*, so that the pitch and longitudinal modes are essentially uncoupled. The equation of motion in pitch in this case is

$$I_y\ddot{\phi} = -4k_{tot}s^2\phi \qquad \text{equation 3}$$

The 4 "springs" each with spring constant k_{tot} are separated in the y direction by 2*s*. I_y is the moment of inertia of the mass with respect to the y-axis (which is directed parallel to the face of the mass). One can see how this equation is derived by considering figure 1 below. Consider that there are 4 springs supporting a mass – two can be seen in the side view as shown, with two more lying behind them. The y-axis is into the page.

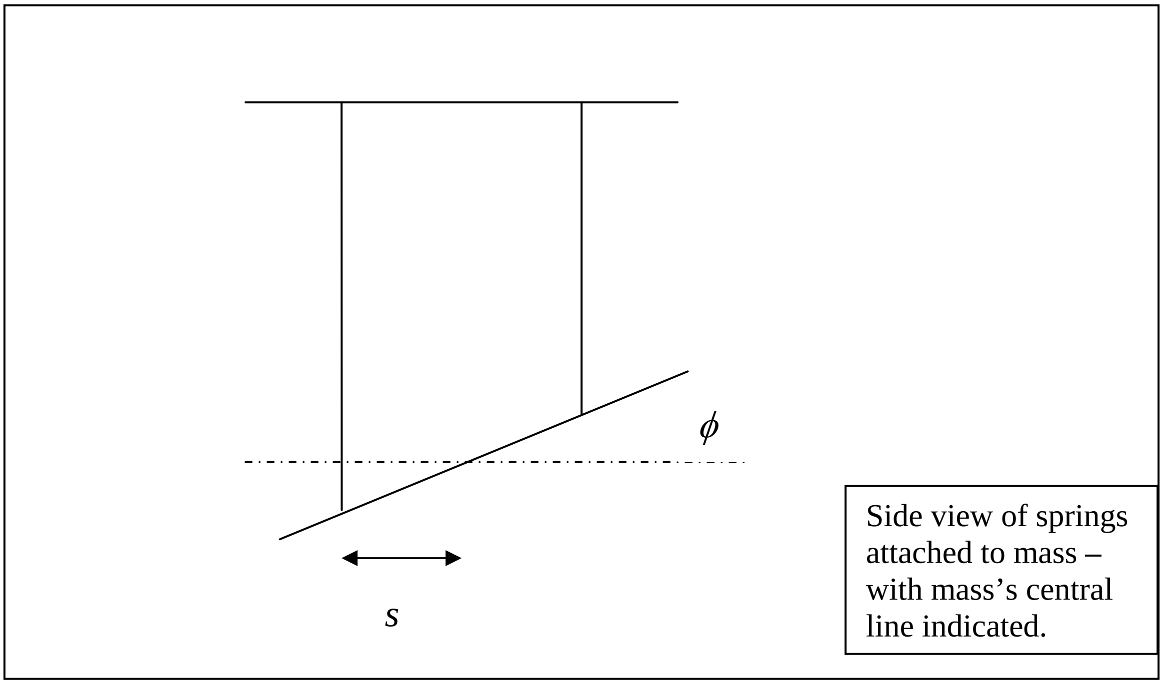


Figure 1.

Two springs are extended by ~ $s\phi$ for small motions and two are compressed by the same amount. Each spring thus exerts a torque on the mass in the same direction (such as to restore the mass to its equilibrium position). The magnitude of each torque is given by force x distance = $ks\phi$ x $s = ks^2\phi$. Hence with 4 such springs equation 3 is obtained.

1.3 Derivation of k_{tot} for one wire off one blade.

If one wire is attached to one blade, and there are 4 such blade/wire combinations then k_{tot} is given by $1/k_{tot} = 1/k_{blade} + 1/k_{wire}$, where k_{blade} is the spring constant of one blade, and k_{wire} is spring constant of one wire. Note that in practice for typical values, k_{tot} is approximately equal to k_{blade}, since k_{blade} is so much smaller than k_{wire}.

The relationship for "summing" k's can be derived by considering figure 2. Consider force F applied to this system. Spring 1 extends by x_1 and spring 2 by x_2 so that total extension $x = x_1+x_2$. For spring 1, $x_1 = F/k_1$, and for spring $x_2 = F/k_2$. Thus adding these, $x_1+x_2= x = F(1/k_1+1/k_2) = 1/k_{tot}$. Hence result.

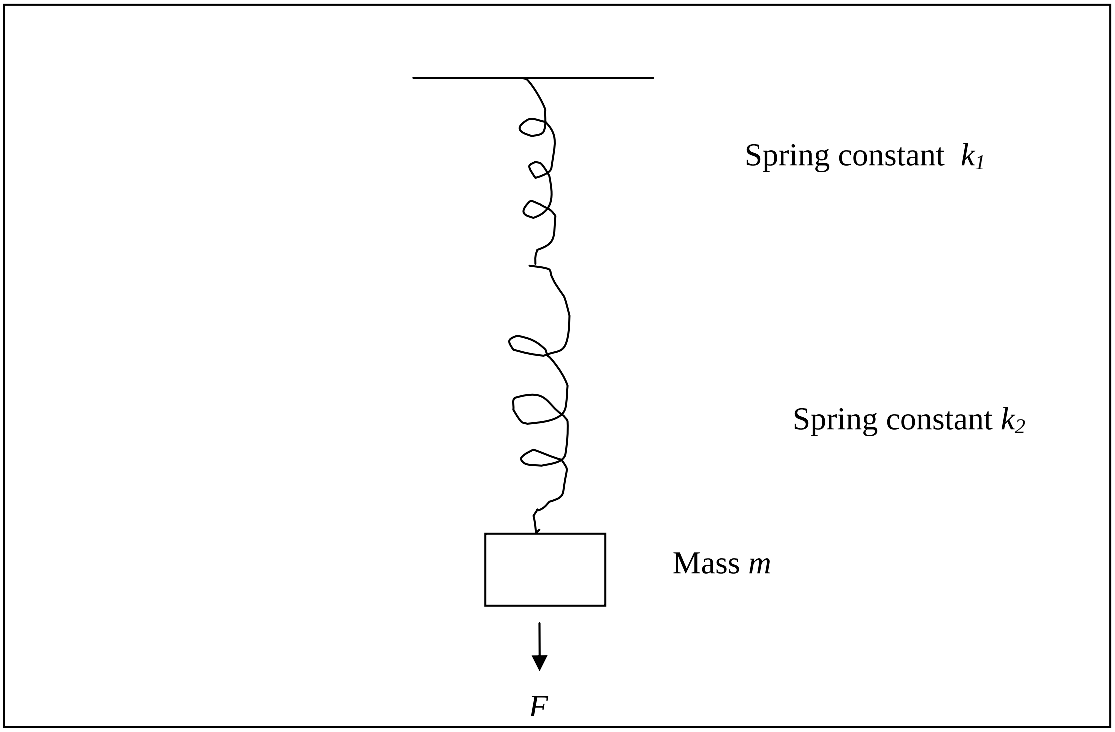


Figure 2.

1.4 Derivation of k_{tot} for two wires from one blade, including torsional effect of blade.

If there are two wires attached to one blade, and two blades in total then firstly ignoring the torsional affect of the blade, (i.e. assuming it is rigid), k_{tot} is simply equal to k_{wire}. This is what is currently in the MATLAB model for the case of two wires off one blade.

However the blade itself will twist as the mass moves in pitch, and a torsional restoring torque will be produced. Considering two blades acting on the mass through rigid wires, the pitch equation of motion is

$$I_y \ddot{\phi} = -2K_{blade}\phi \qquad \text{equation 4}$$

where K_{blade} is the torsional constant for one blade (N.B. K_{blade} and k_{blade} are different quantities!)

By analogy to the case of restoring force when two springs are in series (as in figure 2), we can "sum" the torsional effects of the blade and the wires, with a resulting equation

$$I_y \ddot{\phi} = -2K_{tot}\phi \qquad \text{equation 5}$$

Here K_{tot} is the total torsional constant and it is given by

$$\frac{1}{K_{tot}} = \frac{1}{K_{blade}} + \frac{1}{2k_{wire}s^2} \qquad \text{equation 6}$$

Note that if K_{blade} tends to infinity, K_{tot} is simply equal to 2 $k_{wire}s^2$, and equation 5 simplifies to equation 3.

1.5 Estimation of torsional constant for blades.

In the summer of 2000 a student visiting Glasgow from Germany, Maike Keuntje, made a series of experimental investigations of pitch frequency to look at the torsional effect of the blade. She directly measured the torsional constant for a blade, and compared the experimental pitch frequencies with theoretical ones, assuming a simple model as given above for incorporating the blade. These results when fully analysed supported the model presented above to an accuracy of ~7% or better.

To be able to extend this modeling to any arbitrary blade it is necessary to be able to estimate from the blade parameters what its torsional constant will be.

From a web site http://www.aoe.vt.edu/~johnson/AOE3024/Chapter7.pdf I found the following equation (7.15) for a rectangular bar

$$\frac{d\theta_z}{dz} = \frac{T}{G\left(\tfrac{1}{3}bt^3\right)} \qquad \text{equation 7}$$

where b = width of bar, t = thickness, G = shear modulus , T = torque, and bar lies along z direction.
G = = E/(2(1+ν)), where E = Young's modulus and ν = Poisson's ratio

Thus for a simple rectangular bar length L, rearranging eqn 7 as $T = K\theta$
we can identify the torsional constant K as

$$K = \frac{Gbt^3}{3L} \qquad \text{equation 8}$$

For a more complicated shape, where b is a function of z, rearranging equation 7 we have

$$d\theta = \frac{Tdz}{G\,\tfrac{1}{3}b(z)t^3} \qquad \text{equation 9}$$

Thus

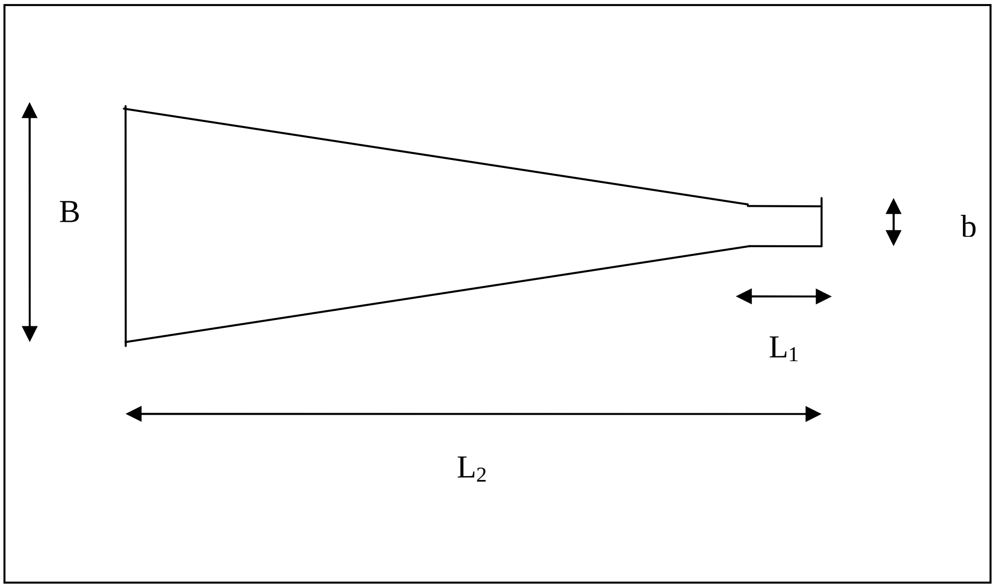
$$\theta = \frac{3T}{Gt^3}\int\frac{dz}{b(z)} \qquad \text{equation 10}$$

and torsional constant K is given by

$$K = \frac{Gt^3}{3\int\frac{dz}{b(z)}} \qquad \text{equation 11}$$

Equation 11 can be used to compute a value for K for any arbitrary blade shape.

Consider a blade shaped as below.



It can be shown that

$$\int \frac{dz}{b(z)} = \frac{L_1}{b} + \frac{L_2}{B} \ln\left(\frac{L_2}{L_1}\right) \qquad \text{equation 12}$$

Section 2. Effect of crossing blades in x/y plane.

The above analysis assumes the blades are arranged (looking from above) as in figure 3. The wires are separated in the x direction by 2*s* as before.

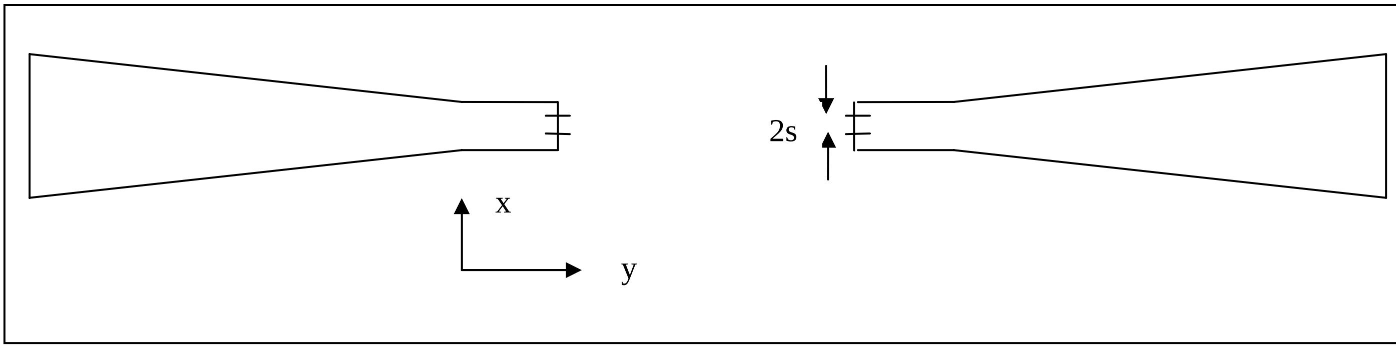


Figure 3.

However to reduce overall footprint of the suspension for Advanced LIGO, the blades are typically crossed, as indicated in Figure 4.

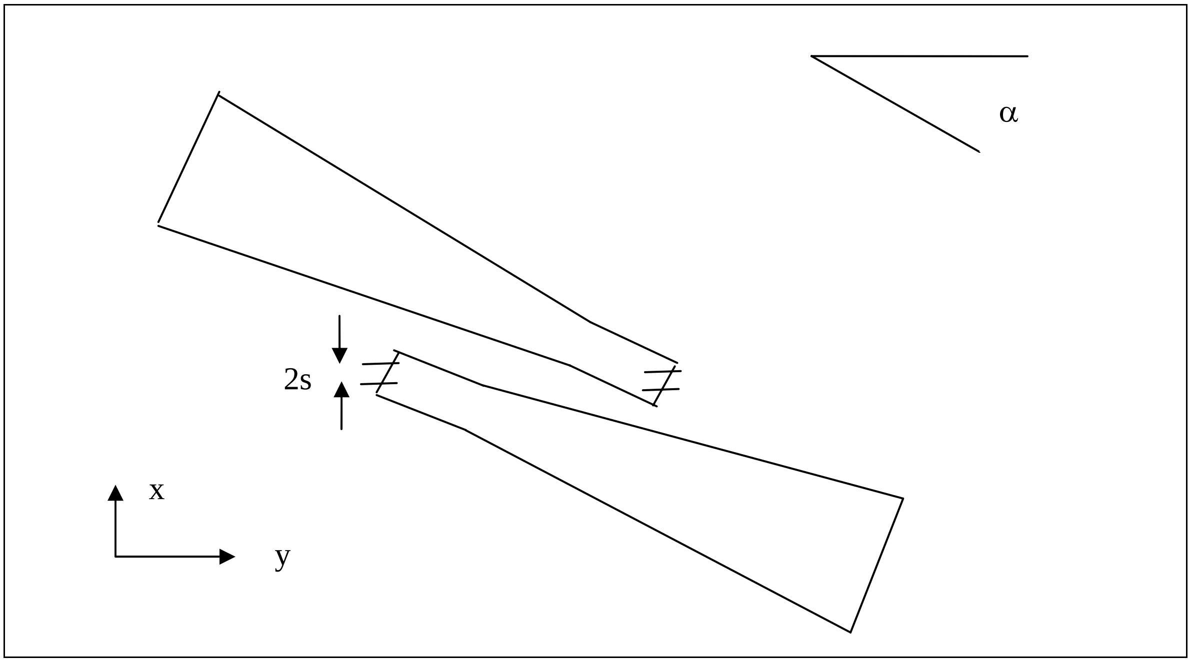


Figure 4.

Let α be the angle through which the blade is rotated from the y-axis. Note that the wires will be attached to the blade using a clamp which ensures that the two wires from each blade will still be lying one directly behind the other in the x direction with the same separation in the x direction as in Fig 3.

Now consider what happens when a blade twists. The resultant torque is applied to the mass below via the two wires attached to the end of the blade. As the tip rotates, one wire is lifted as the other wire is lowered. When the situation is as in Fig 3, the full effect of that rotation is imposed on the mass below via the wires. For example if the blade tip twists by a small angle ϕ in the z-x plane the relative height of the wires changes by $2s\phi$. However when the blades are arranged as in Fig 4, the effect is reduced by $\cos\alpha$. This can be understood by considering the tip of the blade as in Fig 5. The solid dots represent where the wires are attached (via a clamp). If the blade tip twists by a small angle ϕ in the plane defined by z and $x\sin\alpha$ the relative height of the two ends of the wire will change by $2s\cos\alpha \times \phi$. Since this has produced a smaller resultant change for the same angle of twist, (or equivalently the same applied torque), it corresponds to an increase in the effective torsional constant of the blade, by a factor $1/\cos\alpha$.

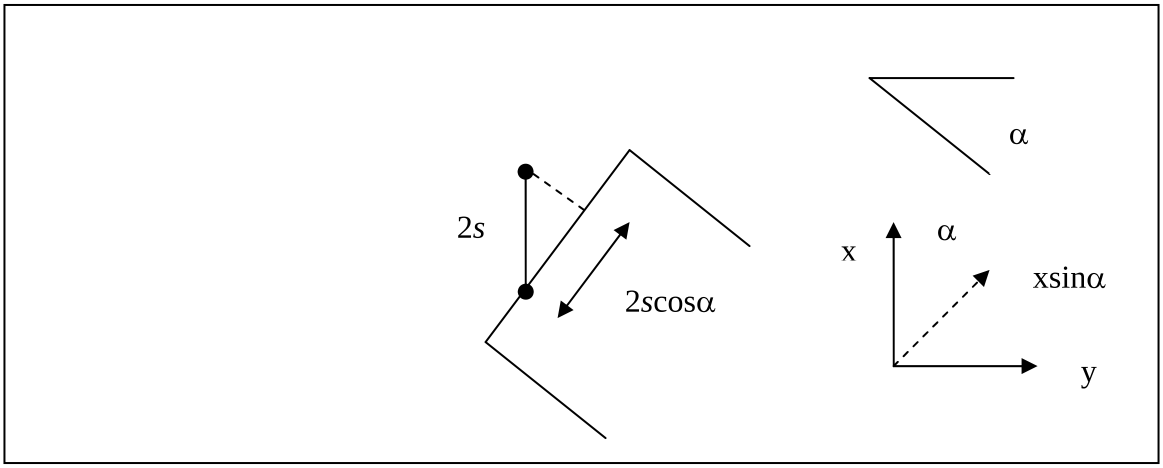


Figure 5

Section 3. Some quantitative analyses.

For maraging steel the relevant physical constants are

E = 1.86e11 Pa, G = 7.1e10 Pa, ν = 0.30.

For the blade used in recent Caltech experiments (old GEO blade) the relevant parameters are B = 82 mm, b= 16 mm, L_1 = 72 mm, L_2 = 370 mm.
Using these values in eqn 12, the integral =11.9, and using eqn 11 with t= 2 mm, the value of K_{blade} = 15.9 Nm.

3.1 Comparison of typical blade torsional constant and the effect due to the wires.

For recent experiment carried out by John Veitch, the values used were k_{wire} = 9.95e4 N/m and s = 0.00215 m. Thus 2 x k_{wire} x s^2 = 0.92 Nm. This is the value of K_{tot} assuming the blade is stiff.

Applying equation 6 gives resultant K_{tot} = 0.87 Nm. The change in K is around 5.4%. Since the pitch frequency is proportional to the square root of K, the resultant change (reduction) in frequency by taking into account the effect of the blade will be around 2.7%. It could be measurable if frequency measurements at this accuracy can be achieved.

I have also considered the blades used in MIT prototype, and the change in K_{tot} taking into account the torsioning of the blades is again at the few percent level.

The above argument assumes the blades are not crossed. Crossing the blades effectively stiffens K_{blade}. If α=20 degrees, the new K_{blade} for the above example is 15.9/cos(20^o) =

16.9 Nm. The resultant K_{tot} is still 0.87 to 2 sig figs. Hence the crossing of the blades produces a negligible change in frequency.

NB The torsional effect on the pitch frequency will only be seen if the "*K*" term dominates in the pitch equation of motion i.e. if we can ignore the terms involving *d*. Recall eqn 1,

$$I_y \ddot{\phi} = \frac{mgd}{l}\left(x_n - x_0\right) + \left(-mgd - \frac{mgd^2}{l} - 4k_{tot}s^2\right)\phi$$

Now replace the final term using K_{tot} (c.f. equations 5 and 6), giving

$$I_y \ddot{\phi} = \frac{mgd}{l}\left(x_n - x_0\right) + \left(-mgd - \frac{mgd^2}{l} - 2K_{tot}\right)\phi \qquad \text{equation 13}$$

When John carried out his measurements recently, the value of *d* was large (approx 2 cm). It can be shown that with this value the restoring torque due to the first term involving *d* in the brackets dominates over the other terms. Thus any change in the effective value of K_{tot} due to including the torsional effect of the blades will not be measurable with such a value of *d*.

Section 4. Conclusions

The torsional motion of the blades when two wires are suspended from one blade has the effect of reducing the overall torsional restoring constant and hence reducing the pitch frequency. However for typical values the frequency is changed by a few percent or less. This is a small effect, and, for example, is not likely to have any impact on damping of modes. Also since it produces a decrease in frequency it improves isolation rather than the contrary.

When the crossing of the blades at an angle is considered, the effect is to stiffen the blades in the pitch direction, raising the effective blade torsional constant by a few percent. Since the effect of the blade is already small, this small change produces a negligible overall effect on the pitch frequency.

It should be noted that the above arguments are to first order, and in particular the torsional effect is treated in a simplified manner. However given the magnitudes of the effects involved, this would appear to give sufficient accuracy for drawing these conclusions.
