LIGO-T020092-00-D

The Effects of Nonlinear Thermoelastic Damping in Highly Stressed Fibres

G. P. Cagnoli
Department of Physics and Astronomy, University of Glasgow, Glasgow, U.K.

P. A. Willems
LIGO Project, California Institute of Technology, Pasadena, CA U.S.A.

(Dated: October 13, 2004)

We present an analysis of thermoelastic damping in materials arising from the temperature dependence of the elastic moduli. We show that this new form of damping can cancel thermoelastic damping arising from thermal expansion for a suitably chosen operating stress. For the case of the suspension fibres used in the gravitational wave detectors, this loss mechanism can be the dominant one.

PACS numbers: 04.80.Nn, 05.40.Jc, 62.40.+i, 46.25.Hf

I. INTRODUCTION

Brownian motion in the suspension wires of the mirrors in interferometric gravitational-wave detectors is one of the fundamental limitations to gravitational wave detector sensitivity [1]. For this reason they are fabricated from materials with very low internal friction, since this friction is related to the displacement noise through application of the fluctuation-dissipation theorem [2]. In order to reduce the displacement noise still further by means of the dilution factor[3], these wires are so thin that the weight of the mirror stresses them to a significant fraction of their breaking strength.

However, in general the diameter can only be made so small, due to the specific strength of the material and in some cases due to thermoelastic damping, which has a frequency dependence which is related to the diameter of the wire. For the fused silica suspension wires currently preferred for GW detectors, thermoelastic damping dominates the intrinsic damping of the material in the frequency bandwidth of the detector at about 200 MPa. The mechanism of linear thermoelastic damping is well understood. When a rod of material is flexed, the material to the inside of the bend is compressed and that to the outside of the bend is expanded. Because of its nonzero thermal expansion coefficient, the material will change temperature with the sign depending on whether it is compressed or expanded. Thus there will be a temperature gradient across the bend. Heat will flow irreversibly across this gradient and dissipate energy.

It can be easily seen how the thermal expansion responsible for thermoelastic damping can also cause thermal fluctuations in the position of a mass suspended from a thin elastic wire. If we consider the wire as having a front half and a back half, it is obvious that a thermal fluctuation in the temperature of the front half of the wire will lead to an expansion or contraction of that part of the wire and therefore a bend in the wire, thus moving the mass. Now consider what happens if this wire is under fixed tension and the Young's modulus is a function of temperature. A thermal fluctuation in one half of the wire will also change the Young's modulus, and thereby the strain of that segment under the tension. This also will bend the wire. It is easy to show that these two effects exactly cancel each other if the tension is such that the static strain is $u_0 = \alpha E_0/E'$, where α is the linear expansion coefficient and E_0 and E' are the Young's modulus and its temperature derivative.

We will show here that the elastic modulus' dependence on temperature also causes a temperature change upon expansion or contraction of the material where the amount of temperature change depends nonlinearly on the amount of strain, and that the sign of the temperature change depends on the sign of the strain. Therefore, the strain can be chosen, taking into account both the thermal expansion coefficient and the temperature dependence of the Young's modulus, such that a small additional strain will result in no net temperature change, and therefore no net thermoelastic damping. Nonlinear thermoelasticity (NTE) has been discussed in the literature [4][5], but the internal damping that results from NTE has— to our knowledge— not been explored prior to this work.

II. DAMPING IN A HOMOGENEOUSLY STRETCHED FIBRE

We begin by treating a wire under simple extension. The free energy for our nonlinear thermoelastic solid is[6]

$$F(T) = F_0(T) + \frac{E(T)}{2(1+\nu(T))}\left[u_{ij}u_{ij} + \frac{\nu(T)}{1-2\nu(T)}(u_{ll})^2\right] - \alpha(T-T_0)\frac{E(T)}{(1-2\nu(T))}u_{ll} \tag{1}$$

where F_0 is the undeformed free energy, u_{ij} is the strain tensor, T is the temperature, α is the linear thermal expansion coefficient, $E(T)$ is Young's modulus, and $\nu(T)$ is Poisson's ratio. T_0 is an arbitrary temperature at which we define the unstressed deformation to be zero. It is understood that repeated indices are summed over. This formula as written explicitly recognizes that E and ν are functions of T. For simple extension of a thin beam in the z direction ($\sigma_{zz} \neq 0, \sigma_{\text{all other}} = 0$, where σ_{ij} is the stress tensor) we have

$$u_{xx} = u_{yy} = -\nu(T)u_{zz} + \alpha(1+\nu(T))\theta, \tag{2}$$

where $\theta \equiv (T - T_0)$. We substitute this into the free energy to get

$$F(T) = F_0(T) + \frac{E(T)}{2}u_{zz}^2 - \alpha\theta E(T)u_{zz} - E(T)\frac{1+\nu(T)}{1-2\nu(T)}\alpha^2\theta^2 \tag{3}$$

and use the thermodynamic relation $\sigma_{ik} = dF/du_{ik}$ to get the stress-strain equation:

$$\sigma_{zz} = E(T)(u_{zz} - \alpha\theta) \tag{4}$$

Note that this equation correctly models the free thermal expansion of a beam, for on setting $\sigma_{zz} = 0$, the result is $u_{zz} = \alpha\theta$. It is also clear upon inspection that this stress-strain equation is correct in the isothermal limit, when the strain changes so slowly that the material is always in thermal equilibrium.

If the strain changes very rapidly, there is not sufficient time for the material to exchange energy with its environment or between its parts, and so the transformation is adiabatic: the entropy remains constant. The entropy is related to the free energy by

$$S(T) = -\left(\partial F/\partial T\right)_{u_{ij}}. \tag{5}$$

In Appendix A, where we expand $E(T) = E_0 + E'\theta$ and $\nu(T) = \nu_0 + \nu'\theta$, we show that if the material is taken from the unstressed state to a large strain u_{max}, then

$$\theta_{\text{ad}} = \frac{T_0}{C_V}\left[-\alpha E_0 u_{\text{max}} + \frac{E' u_{\text{max}}^2}{2}\right], \tag{6}$$

where θ_{ad} is the adiabatic temperature change. In the same appendix we show that if the material is initially at a large strain u_0 to which is added a relatively small strain δu, then

$$\theta_{\text{ad}} = \frac{T_0}{C_V}\left[-\alpha E_0 + E' u_0\right]\delta u \tag{7}$$

Substituting the adiabatic temperature change into the stress-strain equation to find how the stress varies with the strain we get:

$$\sigma_{zz\text{ad}} = E_0 u_{zz} - \alpha E_0 \theta_{\text{ad}} + E'\theta_{\text{ad}} u_{zz} \tag{8}$$

where we have dropped the term proportional to $E'\theta_{\text{ad}}^2$, which is small so long as $\theta_{\text{ad}} \ll E_0/E'$. This is justifiable because the θ_{ad} at the yield strength of most materials, including fused silica, satisfies this condition. We will use $\sigma_{zz\text{iso}}$ to refer to the stress when $\theta = 0$. Note the remarkable fact that Poisson's ratio, ν, does not appear in Eqs. 6, 7, or 8.

We now estimate the strength of the nonlinear thermoelastic damping by considering the energy lost as the material is cycled through a closed loop in stress-strain space. The closed loop consists of 1) an adiabatic extension from the initial state to the extended state, 2) thermalization at fixed strain with a surrounding reservoir, so slow that the material temperature can be considered essentially uniform, and 3) isothermal return to the initial state. This process frees us from the considerations of the timescale of dissipation that would follow from inhomogeneous thermal gradients.

The work done in this process is

$$\Delta W = \oint \sigma_{zz} du_{zz} = \int_{u_{\text{min}}}^{u_{\text{max}}} \sigma_{zz\text{ad}} du_{zz} + \int_{u_{\text{max}}}^{u_{\text{min}}} \sigma_{zz\text{iso}} du_{zz} \tag{9}$$

where there is no work done during the thermalization stage since $du_{zz} = 0$. We first consider the case where $u_{\text{min}} = 0$. Here both the adiabatic temperature change Eq. 6 and the stress-strain relation Eq. 8 are nonlinear in u_{zz}. There are two interesting values of the strain. For $u_{zz} = u_1 = \alpha E_0/E'$, the last two terms in Eq. 8 cancel, with the result that

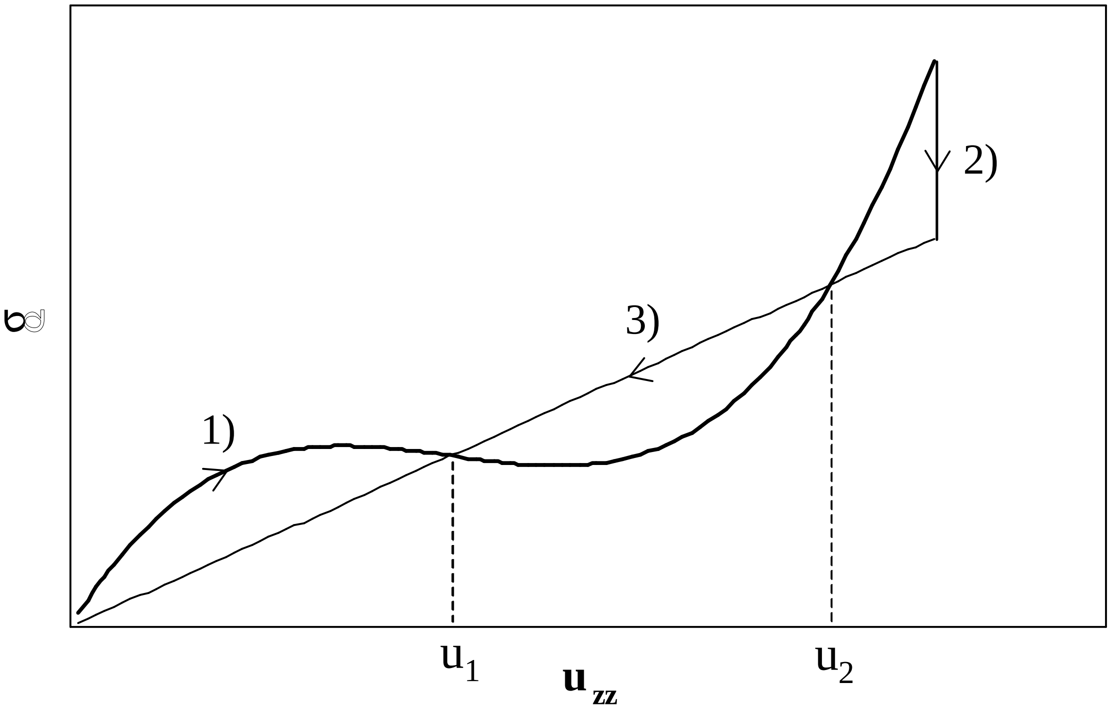

FIG. 1: The closed loop in stress-strain space for large extensions from zero strain. The value of the thermal expansion coefficient has been set to 1 to better show the difference between $\sigma_{\rm ad}$ and $\sigma_{\rm iso}$.

$\sigma_{zz{\rm ad}} = \sigma_{zz{\rm iso}}$, and at $u_{zz} = u_2 = 2\alpha E_0/E'$, the adiabatic temperature change is zero so that again $\sigma_{zz{\rm ad}} = \sigma_{zz{\rm iso}}$. The closed loop in stress-strain space for $u_{\rm max} > u_2$ is shown in Figure 1. For small values of u_{zz}, the stress follows the asymptote $\sigma_{\rm zz} = E_{\rm ad}u_{zz}$, where $E_{\rm ad} = E_0(1+\alpha^2E_0T_0/C_V)$ is the adiabatic Young's modulus from the linear thermoelastic theory (LTE). The dissipation in cycling through the closed loop is equal to the area circled by the curve in the clockwise sense. The integrals in Eq. 9 are easily solved, with the result for the total dissipation being

$$\Delta W = \frac{T_0}{C_V}\left[\frac{\alpha^2E_0^2}{2}u_{\rm max}^2 - \frac{\alpha E_0 E'}{2}u_{\rm max}^3 + \frac{E'^2}{8}u_{\rm max}^4\right]. \tag{10}$$

Inspection shows that the dissipation takes a minimum at $u_{\rm max} = u_2$; in fact the dissipation is exactly zero at this point. Therefore the dissipation is never negative and thus satisfies the laws of thermodynamics. This latter statement is true regardless of the sign of E', α, or $u_{\rm max}$.

We next consider the case where $u_{\rm min} = u_0$ and $u_{\rm max} = u_0 + \delta u_{zz}$. This time the integral Eq. 9 is over the variable $d(\delta u_{zz})$, and Eq. 7 is used for $\theta_{\rm ad}$. In this case both $\sigma_{zz{\rm ad}} = \sigma_{zz{\rm iso}}$ and $\theta_{\rm ad} = 0$ at $u_0 = u_1$. In addition both the adiabatic temperature and stress equations are quasilinear in that for δu very small, $\sigma_{zz{\rm ad}}$ and $\theta_{\rm ad}$ depend on δu_{zz} to first order only. Closed loops in stress-strain space for u_0 less than, equal to, and greater than u_1 are shown in Figure 2. The dissipation is now

$$\Delta W = \frac{1}{2}\frac{T_0}{C_V}(\alpha E_0 - E'u_0)^2(\delta u_{\rm max})^2. \tag{11}$$

The dissipation is clearly always positive, independent of the sign of $\delta u_{\rm max}$, as shown in Figure 2. In both of the above cases, if we set $E' = 0$, we get the familiar thermoelastic damping strength. Therefore, we may generalize to nonlinear thermoelasticity in the quasilinear case by making the substitution $\alpha \rightarrow \alpha' = \alpha - E'u_0/E_0$. In particular, note that if we set the strain $u_0 = E_0\alpha/E'$, there is no net thermoelastic damping for small additional strains. Notice that if u_0 is much larger than this value, then the damping increases as u_0^2.

III. THE CASE OF A BENT WIRE UNDER TENSION

The previous example demonstrated in a simple way how nonlinear thermoelasticity manifests itself in internal damping of a homogeneously strained material where the heat flow is through contact with a thermal reservoir. We

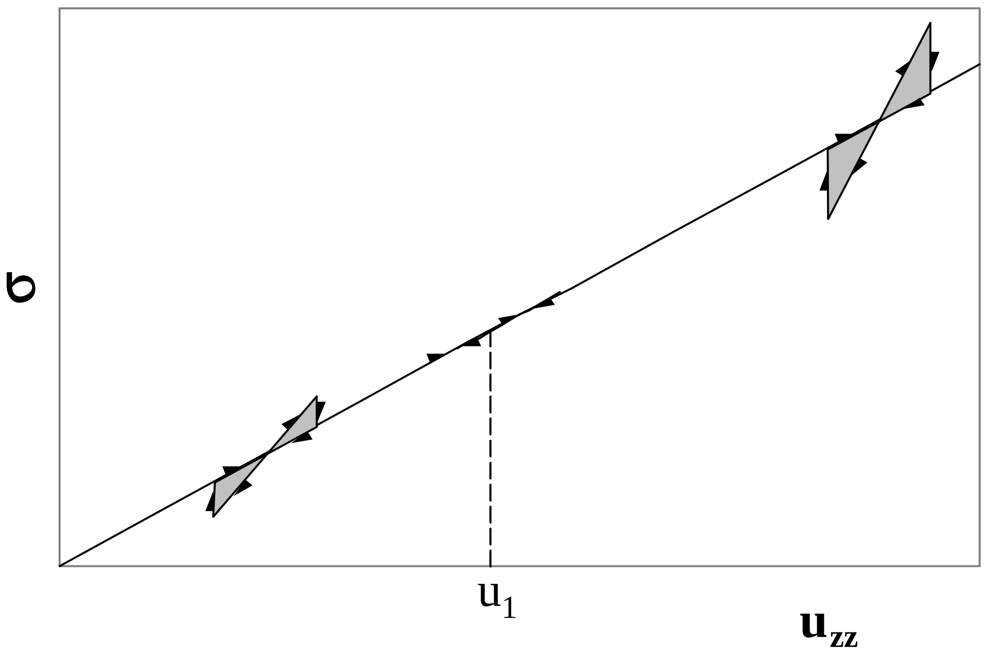

FIG. 2: The closed loops in stress-strain space for small extensions around three different large initial strains. The value of the thermal expansion coefficient has again been set to 1 to better show the difference between $\sigma_{\rm ad}$ and $\sigma_{\rm iso}$.

now consider a more complicated but in practice the more relevant case of nonlinear thermoelastic damping due to the transverse thermal currents that are generated in a bent wire which is under tension. We derive the frequency dependence of the dissipation using two complementary approaches. In both derivations we ignore the presence of other forms of damping not associated with thermoelasticity.

Assume, as is usual in dealing with the bending of thin rods, that any external forces required to cause the bending are negligible compared with internal stresses. As a result we need only consider the σ_{zz} component of stress and the nonzero strain components, which are u_{xx}, u_{yy}, and u_{zz}. We model the wire as a thin beam of length L and rectangular cross-section of width a and thickness b, held under a static tension P and deformed along the x-axis with radius of curvature $R(z)$ that can vary along the beam length. (See Figures 3 and 4.) We make no assumptions about $R(z)$ except that it is large (the bending is moderate). The strain distribution inside the beam is then given by

$$u_{zz} = \frac{P}{E_0} + (x-q)/R = u_0 + (x-q)/R, \tag{12}$$

with u_{xx} and u_{yy} related to u_{zz} by Eq. 2. Here q is the coordinate of the neutral surface of the bending, which contrary to what is usually done cannot be assumed to run through the center of the beam because the temperature gradient as well as other nonlinearities will cause the Young's modulus to vary across the beam. This neutral surface must be treated with some care, because the work required to move it during bending is comparable to the dissipation induced by NTE.

A. The Quasiperiodical Analysis Approach

It is possible to apply the closed-loop analysis of the previous section to this case in a way that explicitly considers the dynamics of the damping. We take $1/R(t) = (1/R_0)e^{i\omega t}$, and allow $\theta(t)$ and $q(t)$ to be quasiperiodic functions of time with period $2\pi/\omega$, where ω is the fundamental frequency of oscillation of the beam. For example, $q(t_1) = q(t_2)$ where $t_2 - t_1 = 2\pi/\omega$. We may then express the closed-loop integral in stress-strain space as the time integral

$$W = \int_{t_1}^{t_2} \sigma_{zz} \frac{du_{zz}}{dt} dt \tag{13}$$

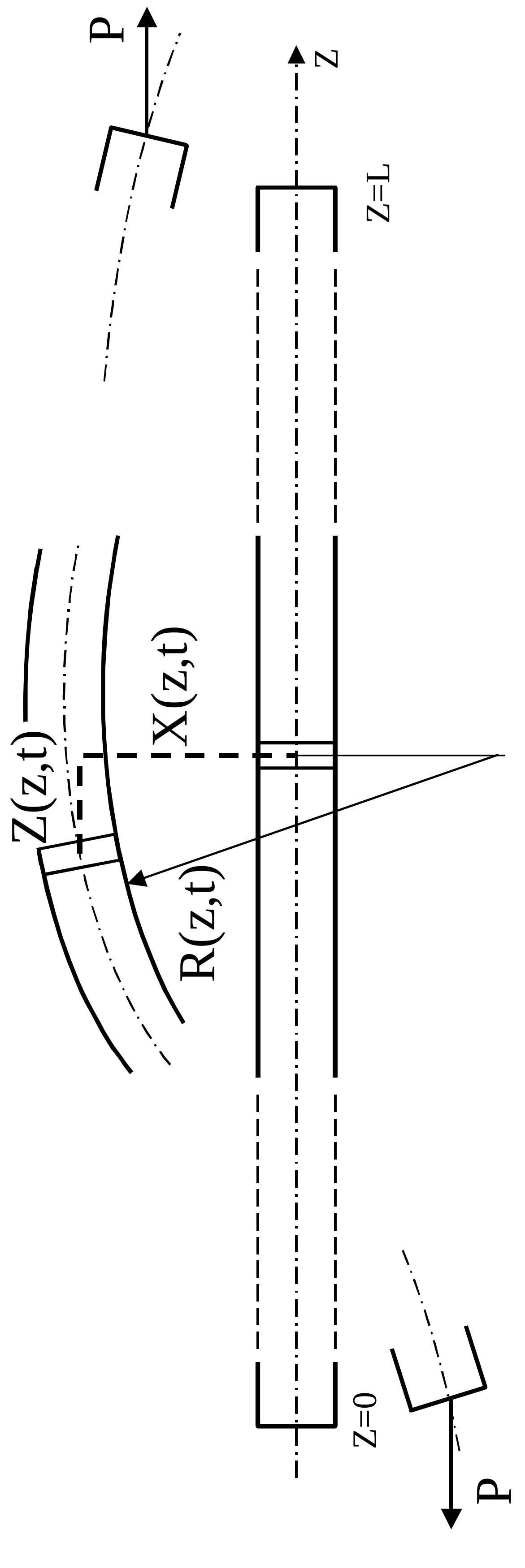

FIG. 3: Model for the bent beam under tension, showing the variables used in the quasiperiodical analysis and dynamic beam equation derivation: diagram showing the variables X, Z, and R.

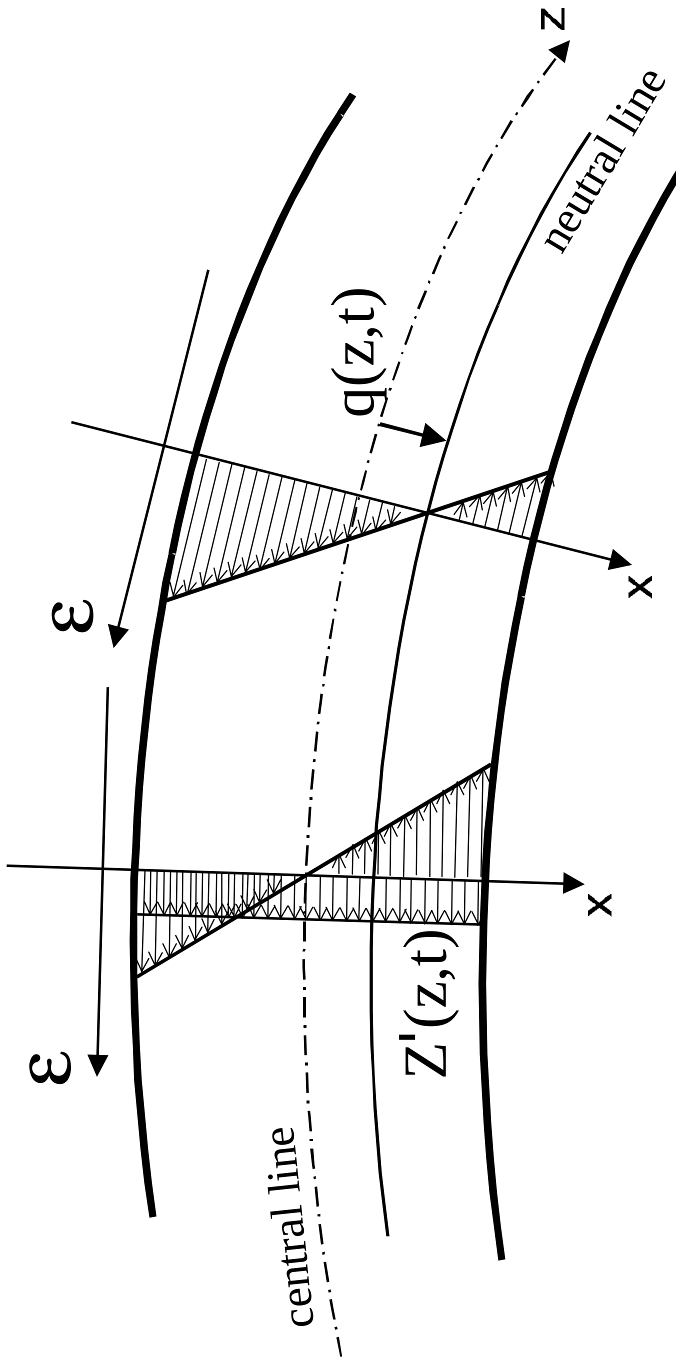

FIG. 4: Model for the bent beam under tension, showing the variables used in the quasiperiodical analysis and dynamic beam equation derivation: diagram showing how a uniform stretching Z' superimposed on an extension which varies linearly across the beam due to the bending deflects the neutral surface from the center to q.

where σ_{zz} obeys Eq. 4 as in the case of homogeneous stretching and where θ now depends upon time and position within the beam through the heat equation

$$\frac{d\theta}{dt} = \chi\nabla^2\theta - \frac{T_0}{C_V}(\alpha E_0 - E'u_0)\frac{du_{zz}}{dt}, \tag{14}$$

where χ is the thermal diffusivity. The final term of Eq. 14 can be seen to be the time derivative of Eq. 7. In using Eq. 4 we make the same approximation used in arriving at Eq. 8, namely that terms of second order in θ are negligible. In Appendix B we show how Eq. 14 and the quasiperiodicity requirement lead to a simple form for the dissipation, provided that we require the neutral surface to obey the parity requirement $q(-R) = -q(R)$. This parity requirement simply asks that the neutral surface moves as far to the right for a rightward bend as it does to the left for a leftward bend, and if it is satisfied the neutral surface makes no contribution to the damping.

The final result is

$$\Delta W = \int W dxdy = \frac{T_0}{C_V}\frac{(\alpha E_0 - E'u_0)^2}{{R_0}^2}2\pi a \mathrm{Im}\left[\frac{1}{k^3}\left(\frac{kb}{2} - \tan\left(\frac{kb}{2}\right)\right)\right] \tag{15}$$

where $k = \sqrt{i\omega/\chi}$ and $1/R_0$ is the amplitude of the bending. The complicated expression involving k is shown in [7] to be the exact solution of the frequency dependence for Zener damping in a beam of rectangular cross section. In addition it is shown that this exact solution for the frequency dependence of the damping is closely approximated by the Zener formula. Using this we may then approximate the NTE damping in the bent beam in the more familiar form $\phi = \Delta W/2\pi W_{\max}$, where $W_{\max} = E_0 ab^3/24R_0^2$ is the energy stored in the bent beam:

$$\phi = \frac{T_0 E_0}{C_V}\left(\alpha - \frac{E'u_0}{E_0}\right)^2 \mathrm{Im}\left[\frac{24}{k^3b^3}\left(\frac{kb}{2} - \tan\left(\frac{kb}{2}\right)\right)\right] \sim \frac{T_0 E_0}{C_V}\left(\alpha - \frac{E'u_0}{E_0}\right)^2 \frac{\omega\tau}{1+(\omega\tau)^2}, \tag{16}$$

where $\tau = b^2/\pi^2\chi$. This shows that NTE has the same frequency dependence as LTE. Note that the parity assumption for q is quite general and applies for many sources of nonlinearity that may move the neutral surface, such as nonlinearity of the isothermal Young's modulus or the nonlinear increase of tension with bending in a clamped beam. It must be remembered that as derived the losses due to NTE may be described by a linear loss function ϕ only in the quasilinear limit.

B. The Dynamic Beam Equation Approach

We may also derive the dynamic beam equation obeyed by the wire under tension by means of a variational approach. In Appendix C we derive a Lagrangian for the bent beam under tension and apply the Euler-Lagrange equations to it. Thereafter we follow Lifshitz & Roukes [7] in deriving the beam equation as a function of several thermal moments of inertia, which are determined by solving the heat equation for the bending beam in the frequency domain. The result is

$$A\rho\ddot{X} - PX'' + IE_0\left(1 + \frac{T_0}{C_V}(\alpha E_0 - E'u_0)^2\left[1 + 24\frac{kb/2 - \tan(kb/2)}{(kb)^3}\right]\right)X^{(iv)} = 0, \tag{17}$$

where $A = ab$ is the beam cross sectional area, ρ is the density, and $I = ab^3/12$ is the moment of inertia of the beam cross section. This equation and Eq. 15 strictly hold only for a beam of rectangular cross section (such as a ribbon), but the equation for a circular beam will be very similar, differing only slightly in the precise frequency dependence of the elastic modulus of the bending. By comparison to the general damped dynamic beam equation

$$A\rho\ddot{X} = \frac{d^2}{dz^2}\left[-IE_0\left(1 + i\phi\right)X'' + PX\right], \tag{18}$$

this formula gives rise to a loss ϕ identical to Eq. 16.

We see, therefore, that the thermoelastic damping in the case of a suspension fibre or ribbon such as are used in gravitational wave detectors is just that of the linear thermoelastic damping with the substitution $\alpha \to \alpha' = \alpha - E'u_0/E_0$. In particular note that for a particular value of tension u_0 (in practice realizable with fused silica because— unlike most materials— its Young's modulus temperature coefficient is positive) the thermoelastic damping can be cancelled exactly. This leaves only other sources of dissipation, which can be at a much lower level. Thus, since thermal noise increases with damping as a result of the fluctuation-dissipation theorem, it is possible to reduce the random noise motion in the detector. Of course, if u_0 is chosen too large, then the thermal noise can become much worse, since

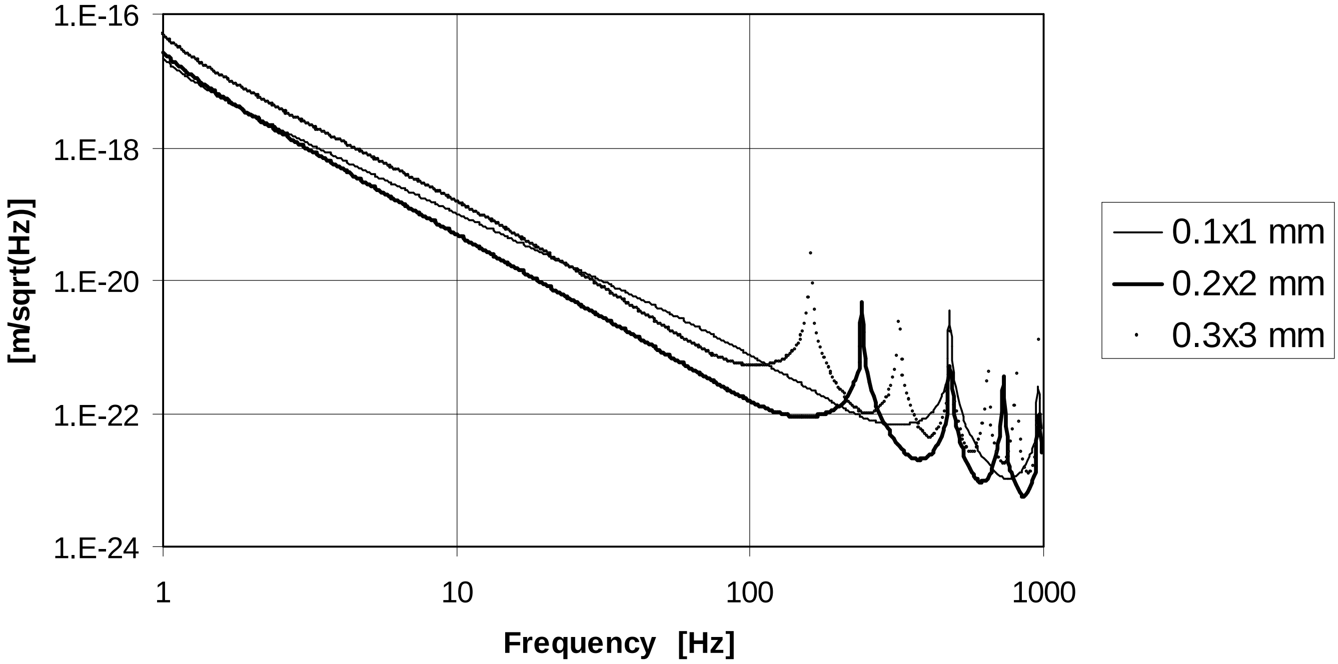

FIG. 5: Calculated thermal noise spectra for fused silica ribbons of three different cross sectional areas in a gravitational wave detector. The peaks at high frequency are due to violin mode resonances of the ribbons.

u_0 appears in the damping formula to the second power. Figure 5 shows the calculated thermal noise for suspension ribbons of fused silica [8] for three different levels of static strain, where the suspension mass was held fixed and the strain varied by changing the ribbons' cross sectional area. The damping in the material is the nonlinear thermoelastic damping plus a frequency-independent structural damping term. It is seen that the minimum thermal noise is obtained when $u_0 = \alpha E_0/E'$, as expected. Figure 6 shows for comparison the thermal noise when E' is set to zero for the strain that gives the lowest noise and for another, much larger, strain. In this case, generally, the greater the static strain the less the thermal noise. We see therefore that consideration of nonlinear thermoelasticity leads to very different conclusions about the optimum fibre thickness than those from consideration of linear thermoelasticity alone [9]. Note that the optimum strain for fused silica ($u_0 = .003$) is much less than the breaking strain $u_0 = .07$ and accordingly is achievable in practice. In the case of ribbons the thickness may be varied independently of the width for best performance. Here nonlinear thermoelastic considerations set the product of thickness and width so that the static strain satisfies $u_0 = \alpha E_0/E'$.

IV. CONCLUSIONS

We have shown in this paper how the existence of a temperature dependence of the elastic modulus leads (through nonlinear thermoelasticity) to dissipation in strained materials. In particular, we have shown that nonlinear thermoelasticity can be the dominant source of thermal noise in fused silica suspension fibres, or instead may cancel the linear thermoelastic damping. It is therefore of importance to interferometric gravitational wave detectors, where suspension thermal noise can limit the ultimate sensitivity at certain frequencies.

Recently the important role of linear thermoelasticity in the thermal noise of the test mass mirrors for gravitational wave interferometers has been pointed out [10]. Thermoelastic noise may in fact be the dominant thermal noise source in the case of sapphire test masses. It is natural to ask whether nonlinear thermoelasticity could be used to reduce this source of noise as well, for example by putting the mirror under a static strain. This seems to the authors to be unlikely, for on inspection of the material properties of sapphire, it is seen that cancellation of the thermal noise occurs at a compression of 5%. Even if sapphire could survive such compression without fracture, we see no way to compress a mirror to such a high level without compromising its optical performance. This conclusion might have to be modified, however, at cryogenic temperatures.

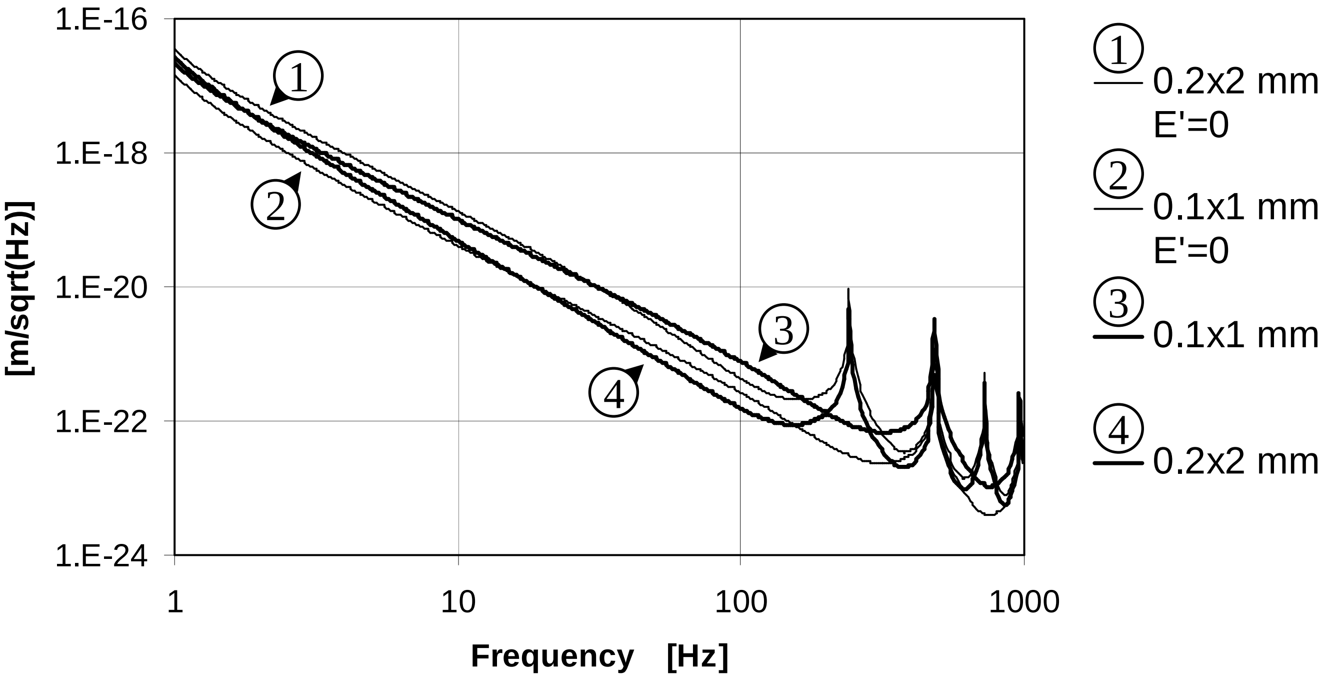

FIG. 6: Calculated thermal noise spectra for fused silica ribbons of two different cross sectional areas in a gravitational detector, with and without NTE.

Acknowledgments

We acknowledge useful discussions with Vladimir Braginsky, Kip Thorne, Stan Whitcomb, and especially Jim Faller and Jim Hough. This work was supported by PPARC and by the U.S. National Science Foundation under awards PHY-9986274, PHY-9801158, PHY-9700601 and PHY-9210038.

APPENDIX A: DERIVATION OF THE ADIABATIC TEMPERATURE CHANGE

We find the entropy by using the thermodynamic relation $S = -\partial F/\partial T$:

$$S(T) = S_0(T) - \left[\frac{E'}{2(1+\nu)} - \frac{\nu' E}{2(1+\nu)^2}\right] u_{ij}u_{ij} \; - \; \left[\frac{E'\nu + E\nu'}{2(1+\nu)(1-2\nu)} - \frac{E\nu'\nu}{2(1+\nu)^2(1-2\nu)} + \frac{2E\nu'\nu}{2(1+\nu)(1-2\nu)^2}\right](u_{ll})^2 \; + \; \alpha\left[\frac{E+E'\theta}{(1-2\nu)} + \frac{2E\nu'\theta}{(1-2\nu)^2}\right] u_{ll} \tag{A1}$$

where $S_0(T) = -dF_0(T)/dT$.

We consider two types of adiabatic extension. First, an extension of the material from the unstressed state at T_0 to some large extension $u_{\rm max}$ at T. Using Eqs. 2 for simple extension, if we set $S(T) = S(T_0)$ and use the relation $C_V = T(dS/dT)_V$, where C_V is the heat capacity per unit volume, we find after some tedious but straightforward algebra that the temperature change upon adiabatic extension is

$$\theta_{\rm ad} = \frac{T_0}{C_V}\left[-\alpha E_0 u_{\rm max} + \frac{E' u_{\rm max}^2}{2}\right] \tag{A2}$$

In deriving this formula we have dropped as negligibly small all terms proportional to θ in the entropy equation except for that of $S_0(T)$. This approximation is justified for all the materials we consider in this paper and in particular for fused silica. Notice that if $E' \to 0$, then the formula is just that for linear thermoelasticity and is linear in $u_{\rm max}$. However, for $E' \neq 0$ the temperature change is quadratic in $u_{\rm max}$. This is similar to the formula derived by Wong *et al.* [5] to explain observed nonlinearities in the thermal emission of strained wires. If $u_{\rm max} = 2\alpha E_0/E'$, then the total temperature change is zero.

If instead we consider an extension of the material from the strain u_0 at temperature T_0 to $u_0 + \delta u$ at T, where $u_0 \gg \delta u$, the temperature change becomes

$$\theta_{\rm ad} = \frac{T_0}{C_V}\left[-\alpha E_0 + E' u_0\right]\delta u \tag{A3}$$

In this case the temperature change is zero at $u_{\max} = \alpha E_0/E'$, and in addition it is linearized with respect to δu.

APPENDIX B: DERIVATION OF THE QUASIPERIODICAL ANALYSIS

If we substitute the stress equation Eq. 4 into the formula for the work done in an integration cycle Eq. 13, remembering to ignore the small second-order terms in θ, we may integrate by parts to get

$$W = (\alpha E_0 - E' u_0)\int_{t_1}^{t_2} \delta u(t)\frac{d\theta(t)}{dt}dt \tag{B1}$$

where we have used $u_{zz} = u_0 + \delta u$ as before, with $\delta u = (x-q)/R$ as given in Eq. 12. Because the bending is small we may assume the additional strain δu due to the bending is small compared to the strain u_0 due to tension. This justifies the use of the quasilinear heat equation Eq. 14. If we substitute Eq. 14 into Eq. B1 and again integrate by parts the result is

$$W = (\alpha E_0 - E' u_0)\int_{t_1}^{t_2} \delta u(t)\chi\nabla^2\theta(t)dt. \tag{B2}$$

We must now solve the heat equation. We restrict $1/R$ to vary sinusoidally ($1/R = 1/R_0 \mathrm{Re}[e^{i\omega t}]$), and make the expansions

$$\theta = \mathrm{Re}\left[\sum_{n=1}^{\infty}\theta_n(x)e^{ni\omega t}\right], \qquad q = \mathrm{Re}\left[\sum_{n=1}^{\infty} q_n e^{ni\omega t}\right] \tag{B3}$$

The parity requirement $q(-R) = -q(R)$ on q then implies that all the even terms in the expansion Eq. B3 for q are zero. Substituting Eq. 12 and Eq. B3 into the heat equation Eq. 14 and taking care to multiply only real quantities yields

$$\begin{aligned} i\omega\sum_{n=1}^{\infty} n\theta_n e^{ni\omega t} &= \chi\sum_{n=1}^{\infty}\nabla^2\theta_n e^{ni\omega t} \\ &+ i\frac{\Gamma\omega}{R_0}\left(xe^{i\omega t} - \sum_{n \text{ odd}}^{\infty}\left[\frac{n+1}{2}q_n e^{(n+1)i\omega t} + \frac{n-1}{2}q_n e^{(n-1)i\omega t}\right]\right) \end{aligned} \tag{B4}$$

where $\Gamma = (\alpha E_0 - E' u_0)T_0/C_V$. Assuming that there is no heat flow out the surface of the beam and that heat flow along the beam is much less than heat flow across the beam, the solution to Eq. B4 is [7]

$$\chi\nabla^2\theta_1 = \frac{\chi\Gamma k}{R_0}\frac{\sin(kx)}{\cos(kb/2)}, \qquad \chi\nabla^2\theta_{n\neq 1} = 0 \tag{B5}$$

where $k = \sqrt{i\omega/\chi}$. We may then substitute this and the expression for δu into Eq. B2, again taking care to multiply only real quantities; the result is:

$$W = \frac{T_0}{C_V}\frac{(\alpha E_0 - E' u_0)^2}{{R_0}^2}\int_{t_1}^{t_2}\mathrm{Re}\left[e^{i\omega t}\right]\left(x - \mathrm{Re}\left[\sum_{n \text{ odd}}^{\infty} q_n e^{ni\omega t}\right]\right)\mathrm{Re}\left[\chi k\frac{\sin(kx)}{\cos(kb/2)}e^{i\omega t}\right]dt \tag{B6}$$

Examination of the terms involving q_n shows that they involve products of $e^{i\omega t}$, $e^{i\omega t}$, and $e^{ni\omega t}$. Since only the q_n with odd n are nonzero, these products are all sinusoids in t and therefore integrate to zero. This means that the work required to move the neutral surface is reversible and does not contribute to the dissipation. Integrating the work over the beam cross section yields

$$\int W dxdy = \frac{T_0}{C_V}\frac{(\alpha E_0 - E' u_0)^2}{{R_0}^2}2\pi a \mathrm{Im}\left[\frac{1}{k^3}\left(\frac{kb}{2} - \tan\left(\frac{kb}{2}\right)\right)\right], \tag{B7}$$

which is the dissipation due to thermoelastic damping.

We have devoted considerable effort to the question of the neutral surface and its influence on the loss. We now justify our concern. Consider the bending of a thin rod at high frequency, when there is is very little heat flow. The temperature change is then approximately adiabatic.

Because the strain varies linearly across the bent rod, so will the temperature in the adiabatic limit, as can easily be seen in Eq. B4 by ignoring the heat flow term $\chi\nabla^2\theta$. The Young's modulus will therefore also vary linearly across the bent rod. It is easy to show that for small bending $1/R_0$, if the Young's modulus varies as $E(x) = E_0 + E_1 x$, then the neutral surface will move to $q = E_1 b^2/12E_0$ in order that the strain averaged over the beam cross section remain equal to the tension P. In the adiabatic limit,

$$E_1 = \frac{dE}{dx} = \frac{dE}{dT}\frac{dT}{dx} = E'\frac{dT}{dx} = E'\frac{T_0}{C_V}(\alpha E_0 - E' u_0)\frac{(x-q)}{R_0}, \tag{B8}$$

and therefore

$$q = \frac{E' b^2}{12 E_0 R_0}\frac{T_0}{C_V}(\alpha E_0 - E' u_0) \tag{B9}$$

The energy per unit length required to bend a beam under tension is given by

$$W = \int_{-a/2}^{a/2} dy \int_{-b/2}^{b/2} dx \int_{u_{\min}}^{u_{\max}} \sigma_{zz} du_{zz} \approx \int_{-a/2}^{a/2} dy \int_{-b/2}^{b/2} dx \frac{E_0}{2}(u_{\max}^2 - u_{\min}^2) \tag{B10}$$

where $u_{\min} = u_0$ is the unbent strain and $u_{\max} = u_0 + (x-q)/R_0$ is the bent strain. The work required to move the neutral surface may be estimated by comparing this energy to what it would be if $q = 0$:

$$W_{\text{neutral surface}} = \int_{-a/2}^{a/2} dy \int_{-b/2}^{b/2} dx \frac{E_0}{2}\left[-\frac{2u_0 q}{R_0} + \frac{-2xq + q^2}{R_0^2}\right] \tag{B11}$$

Plugging in Eq. B9 for q and dropping terms of higher order than $(1/R_0)^2$ as negligibly small yields

$$W_{\text{neutral surface}} = -\frac{ab^3}{12R_0^2}\frac{T_0}{C_V} u_0 E'(u_0 E' - \alpha E_0) \tag{B12}$$

Dividing by the total energy required to bend the beam $W_{\max} = E_0 ab^3/24R_0^2$ gives the fractional energy stored in the neutral surface motion as

$$\frac{W_{\text{neutral surface}}}{W_{\max}} = 2\frac{T_0 E_0}{C_V}\frac{u_0 E'}{E_0}(\alpha - \frac{u_0 E'}{E_0}). \tag{B13}$$

Comparing this formula with the loss fraction in Eq. 16 shows that the energy stored in the neutral surface motion caused by NTE is comparable to the dissipated energy and cannot be ignored as small, thus justifying our more careful analysis.

APPENDIX C: DERIVATION OF THE DYNAMIC BEAM EQUATION

We begin with the strain, where, rather than using Eq. 12 to model the bending, we note that the motion of the neutral surface can equally well be modelled by an additional strain homogeneous over the beam cross section:

$$u_{zz} \approx Z'(z) + \frac{1}{2}\left(X'(z)\right)^2 - xX''(z). \tag{C1}$$

(See Fig. 3.) Here we have made the identity $X''(z) = 1/R(z)$ and use $Z'(z) + (X'(z))^2/2$ to describe the part of the stretching that is homogeneous over the cross section. The term $(X'(z))^2/2$ arises because we have chosen a coordinate system fixed in the beam equilibrium position and a tilt of the beam relative to the coordinate z-axis must lead to stretching if it is not accompanied by a contraction $Z' < 0$. We will see that this term leads to the restoring force due to tension. This approximation is good for small bending angles.

We draw a distinction between the variables x and z and X and Z. The lowercase symbols x and z are independent variables that represent points within the beam and therefore move with the fiber, as shown in Fig. 3. The uppercase symbols X and Z are dependent variables that represent the displacements of points within the beam along the x

and z directions of the undeflected beam. For small deflections the dynamics may equivalently use fixed x, z or x, z moving with the beam. For integrating over the beam cross section it is more convenient to set x to move with the midpoint of the beam.

We can then use this form for the extension of the beam to get the elastic potential energy by substituting into the free energy equation Eq. 3 and integrating over x and y. Because we will vary the Lagrangian only with respect to X and Z, we drop all terms depending only on the temperature θ:

$$\begin{aligned}\mathcal{V}_e &= \frac{E_0 I + E' I_\theta^{(2)}}{2}(X'')^2 + \alpha(E_0 I_\theta^{(1)} + E' I_{\theta\theta}^{(1)})X'' + \frac{E_0 A + E' I_\theta}{8}(X')^4 - \frac{\alpha}{2}(E_0 I_\theta + E' I_{\theta\theta})(X')^2 \\ &+ \frac{E_0 A + E' I_\theta}{2}(Z')^2 - \alpha(E_0 I_\theta + E' I_{\theta\theta})Z' - E' I_\theta^{(1)} Z' X'' + \frac{E_0 A + E' I_\theta}{2} Z'(X')^2 - \frac{E' I_\theta^{(1)}}{2} X''(X')^2,\end{aligned} \tag{C2}$$

where A is the beam cross section, $I = \int\int x^2 dxdy$ is the bending moment of inertia, and the thermal moments are defined by:

$$I_\theta = \int\int \theta dxdy \tag{C3}$$

$$I_{\theta\theta} = \int\int \theta^2 dxdy \tag{C4}$$

$$I_\theta^{(1)} = \int\int x\theta dxdy \tag{C5}$$

$$I_\theta^{(2)} = \int\int x^2\theta dxdy \tag{C6}$$

$$I_{\theta\theta}^{(1)} = \int\int x\theta^2 dxdy \tag{C7}$$

There is in addition to the elastic energy required to stretch the wire the work V_P done against the forces at the end of the fiber which maintain the tension, which is

$$V_P = P[Z(L) - Z(0)]. \tag{C8}$$

The kinetic energy of a section of the beam is (ignoring the small rotational kinetic energy)

$$\mathcal{K} = \frac{1}{2} A\rho(\dot{X}^2 + \dot{Z}^2) \tag{C9}$$

where ρ is the density of the beam material.

Forming the Lagrangian $L = \int_0^L \mathcal{L} dz - V_P = \int_0^L (\mathcal{K} - \mathcal{V}_e)dz - V_P$ and the action $S = \int dtL$ and setting $\delta S = 0$ we obtain

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{X}} + \frac{d}{dz}\frac{\partial L}{\partial X'} - \frac{d^2}{dz^2}\frac{\partial L}{\partial X''} = 0, \tag{C10}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{Z}} + \frac{d}{dz}\frac{\partial L}{\partial Z'} = 0, \tag{C11}$$

where we adopt the boundary conditions

$$\frac{\partial L}{\partial X''}\Big|_{z=0,L} = 0, \tag{C12}$$

$$\left(\frac{\partial L}{\partial X'} - \frac{d}{dz}\frac{\partial L}{\partial X''}\right)\Big|_{z=0,L} = 0, \tag{C13}$$

$$\frac{\partial L}{\partial Z'}\Big|_{z=0,L} = P. \tag{C14}$$

While these boundary conditions are fairly arbitrary, they reduce the nonlinearity of the beam equation to only that caused by the nonlinear thermoelasticity.

Inserting the explicit form of the Lagrangian yields the equations

$$A\rho\ddot{X} + \frac{d^2}{dz^2}\left(\left[E_0 I + E' I_\theta^{(2)}\right] X'' + \alpha\left[E_0 I_\theta^{(1)} + E' I_{\theta\theta}^{(1)}\right]\right) - \frac{d}{dz}(E_0 A Z' X') = 0, \tag{C15}$$

$$A\rho\ddot{Z} - \frac{d}{dz}\left([E_0 I + E' I_\theta]Z' - \alpha\left[E_0 I_\theta + E' I_{\theta\theta}\right] - E' I_\theta^{(1)} X'' + \frac{E_0 A + E' I_\theta}{2}(X')^2\right) = 0, \tag{C16}$$

We now insert the formula for the strain Eq. C1 into the heat equation Eq. 14 to determine the thermal moments. We expand $Z' = u_0 + Z'_d$, where u_0 and Z'_d are the static and dynamic components, respectively. The solution in the frequency domain is [7]

$$\theta(x,z) = \Gamma\left[f(x)X''(z) + Z'_d(z)\right] \tag{C17}$$

where $k = \sqrt{i\omega/\chi}$ and $\Gamma = (T_0/C_V)(\alpha E_0 - E' u_0)$ as before, and

$$f(x) = x - \frac{\sin(kx)}{k\cos(kb/2)}. \tag{C18}$$

In solving this equation we assume that only thermal currents transverse to the beam axis are important, i.e. that the radius of bending along the beam does not change too rapidly. With this solution for θ the thermal moments take the values

$$I_\theta = -\Gamma A\left(Z'_d + (X')^2/2\right) \tag{C19}$$

$$I_{\theta\theta} = \Gamma^2\left[A\left(Z'_d + (X')^2/2\right)^2 + h(X'')^2\right] \tag{C20}$$

$$I_\theta^{(1)} = \Gamma g X'' \tag{C21}$$

$$I_\theta^{(2)} = -\Gamma I\left(Z'_d + (X')^2/2\right) \tag{C22}$$

$$I_{\theta\theta}^{(1)} = -2\Gamma^2 g X''\left(Z'_d + (X')^2/2\right). \tag{C23}$$

Here $g = \int x f(x) dx$ and $h = \int f^2(x) dx$. Making these substitutions into Eqs. C15, C16 yields the desired equations

$$A\rho\ddot{X} - PX'' + E_0 I\left[1 + (\alpha - \frac{E'}{E_0}u_0)\Gamma\frac{g}{I}\right] X^{(iv)} = 0, \tag{C24}$$

$$A\rho\ddot{Z}_d - E_0 A\left[1 + (\alpha - \frac{E'}{E_0}u_0)\Gamma\right] Z''_d = \frac{d}{dz}\mathcal{O}(X'^2, X''^2, \ldots), \tag{C25}$$

where in the first equation all nonlinear terms are neglected as small, and in the second equation they are collected to one side as a driving term. We expand g and Γ in Eq. C24 to get the desired form

$$A\rho\ddot{X} - PX'' + IE_0\left(1 + \frac{T_0}{C_V}(\alpha E_0 - E'u_0)^2\left[1 + 24\frac{kb/2 - \tan(kb/2)}{(kb)^3}\right]\right) X^{(iv)} = 0. \tag{C26}$$

If we compare Eq. C26 to the standard form of the beam equation $A\rho\ddot{X} - PX'' + EIX^{(iv)} = 0$ and expand $E = E_0(1 + i\phi)$, we can extract the loss angle ϕ to be

$$\phi = \frac{T_0 E_0}{C_V}\left(\alpha - \frac{E' u_0}{E_0}\right)^2 \mathrm{Im}\left[\frac{24}{k^3 b^3}\left(\frac{kb}{2} - \tan\left(\frac{kb}{2}\right)\right)\right] \tag{C27}$$

which is exactly the same form derived in the quasiperiodical analysis. By inspection of Eq. C25 for Z_d, we see that the stretching of the fiber is driven parametrically by the bending of the fiber, and has a homogeneous solution that is nondissipative, which indicates that the dynamics of the neutral surface are nondissipative, as was also shown via the quasiperiodical analysis. Closer inspection shows the restoring force for Z_d to be given by the adiabatic Young's modulus, regardless of frequency. This is a consequence of our having ignored thermal currents along the beam axis in calculating the thermal moments. This is a good approximation for our primary application of this theory to gravitational wave detectors, in which the suspension thermal noise is significant at audio frequencies, because longitudinal thermal currents become significant only well above 1 MHz [11].

The beam equation derived here uses admittedly unusual boundary conditions. More realistic boundary conditions for a beam under tension, such as clamping of both ends of the beam, impose constraints on X and Z that couple their dynamics far more strongly than does NTE; for example, if the ends of the beam are fixed, then geometric considerations alone require it to stretch as it bends. This coupling is also nonlinear, and relates the net stretching of the whole beam to the net bending of the whole beam. The situation in a gravitational wave detector, where one or both ends of the suspension wire are rigidly fixed to heavy suspended masses, will be approximately the same as the clamped-clamped case for frequencies above the pendulum and vertical bounce frequencies. The boundary conditions we have chosen free us from consideration of such global constraints, leaving only the coupling due to NTE, while showing the independence of the dissipation from the homogeneous part of the stretching. In the quasiperiodical analysis we showed that in general the motion of the neutral surface (which is equivalent to a homogeneous stretching) does not affect the dissipation, regardless of its cause, provided it satisfies the parity requirement.

APPENDIX D: MATERIAL PARAMETERS

We take C_V to be the heat capacity per unit volume, rather than per unit mass as is more common. The derivative of the Young's modulus with respect to temperature for fused silica has various measured values in the published literature, which may reflect differences in the samples studied or differences in the frequency of measurement, but they are clustered around 15 MPa/K at room temperature.[12]

	E_0 (GPa)	α (1/K)	E' (MPa/K)	$u_0 = \alpha E_0/E'$
fused silica	75	5.5×10^{-7}	15	.003
sapphire	400	5×10^{-6}	-40	-.05

[1] A. Abramovici *et al.,* Science 256 (1992), 325; H. Luck, Class. Quant. Grav. 14 (1997), 1471; R. Passaquieti, Nuc. Phys. B-Proc. Supp. 85 (2000), 241.
[2] H. B. Callen and T. A. Welton, Phys. Rev. 83 (1951), 34.
[3] G. I. González and P. R. Saulson, J. Acoust. Soc. Amer. **96**, 207 (1994).
[4] O. W. Dillon, Jr., J. Mech. Phys. Solids **10**, 123 (1962); G. Herrmann, ZAMP **15**, 253 (1964).
[5] A. K. Wong, R. Jones, and J. G. Sparrow, J. Phys. Chem. Solids **48**, 749 (1987); A. K. Wong, J. G. Sparrow, and S. A. Dunn, J. Phys. Chem. Solids **49**, 395 (1988).
[6] L. D. Landau and E. M. Lifschitz, *Theory of Elasticity*, 3rd ed., (Butterworth-Heinemann, Oxford, 1986).
[7] R. Lifshitz and M. L. Roukes, Phys. Rev. B **61**, 5600 (2000).
[8] For details on how such spectra are calculated, see A. M. Gretarsson *et al.*, Phys. Lett. A **270**, 108 (2000), or G. González and P. Saulson, J. Acoust. Soc. Am., 96 (1994) 207.
[9] In making this statement we have ignored considerations of additional damping in the surface of the suspension fibre, which also leads to a optimum static strain much less than the breaking strength. See, for example, A. M. Gretarsson and G. M. Harry, Rev. Sci. Instrum. **70**, 4081 (1999); A. M. Gretarsson *et al.*, Phys. Lett. A **270**, 108 (2000). The stress in the fibre may also be set at a value not optimized for NTE in order to move the violin and vertical bounce frequencies outside of the gravitational wave detector bandwidth, at the cost of an increase in thermal noise.
[10] V. B. Braginsky, M. L. Gorodetsky, and S. P. Vyatchanin, Phys. Lett. A **264**, 1 (1999).
[11] This ignores the sharp gradients in bending that can occur when the ends of the beam are clamped. Clamping the end of the beam forces the bending to go rapidly to zero there and can result in longitudinal heat flow into the clamp. See Y. L. Huang, PhD. thesis, Syracuse University, 1996.
[12] J. W. Marx and J. M. Sivertsen, J. Appl. Phys. **24**, 81 (1952); H. J. McSkimin, J. Appl. Phys. **24** 988 (1953); S. Spinner and G. W. Cleek, J. Appl. Phys. **31**, 1407 (1960); J. A. Bucaro amd H. D. Dardy, J. Appl. Phys. **45**, 5324 (1974); H. Kobatashi, T. Kosugi, and Y. Kogure, Physica B **219&220**, 276 (1996).