LASER INTERFEROMETER GRAVITATIONAL WAVE OBSERVATORY

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Technical Note		

Analysis of residual OPD errors after surface polishing of an optic with bulk refractive index inhomogeneities

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Advanced LIGO optics will likely be fabricated from sapphire (Al_2O_3). Sapphire typically has volumetric index of refraction inhomogeneities that are greater than the maximum tolerable amount. The strategy that is being explored in order to mitigate these effects is to surface polish the optic in such a manner that the internal fluctuations in the index of refraction are cancelled to first order by surface figure errors.

Polishing an optic in order to remove first order variations in the optical path difference (OPD) experienced by rays propagating through the optic at different positions will not diminish scattering effects produced by the intrinsic spatial inhomogeneities of the index of refraction of the substrate. It is possible to quantify this effect and to determine the contribution to scatter in terms of the fabrication inhomogeneities, $\delta n_s(x, y)$. Figure 1 shows a schematic of a substrate exhibiting internal index of refraction fluctuations and having a surface figure that was polished in order to cancel the OPD errors for transmission through the bulk. The front plane provides a flat surface against which to refer the fluctuations in OPD, both in vacuum and in substrate. Without loss of generality, the back surface may be assumed to be flat. In the end, the quantities $\overline{l_v} \to 0$ and $\overline{l_s} \to L_0$. Referring to the figure, the following relationships hold,

$$L_0 = (\overline{l_v} + \delta l_v) + (\overline{l_s} + \delta l_s) \tag{1a}$$

$$\delta l_v = -\delta l_s \tag{1b}$$

$$n_0 \equiv 1 \tag{1c}$$

$$n_s = \overline{n_s} + \delta n_s(x, y). \tag{1d}$$

In general, the residual index variations will be functions of $\{x, y, z\}$. To simplify the analysis, the z dependence is averaged, leaving a residual dependence of the variations on $\{x, y\}$ only. The statistical properties of the bulk index are characterized by,

$$\langle \delta n_s(x, y) \rangle_{\mathcal{A}} = \frac{1}{A} \int_{\mathcal{A}} dA(\delta n_s(x, y)) = 0 \tag{2a}$$

$$\begin{aligned} \langle \delta n_s(x, y) \delta n_s(x + \Delta_x, y + \Delta_y) \rangle_{\mathcal{A}} &= \frac{1}{A} \int_{\mathcal{A}} dA \left(\delta n_s(x, y) \delta n_s(x + \Delta_x, y + \Delta_y) \right) \\ &= \sigma_{n_s}^2 R(\Delta), \end{aligned} \tag{2b}$$

with $R(0) = 1$ and $\Delta \equiv \sqrt{\Delta_x^2 + \Delta_y^2}$. By using surface polishing to compensate for internal bulk index variations, the following constraint is imposed at any point {x,y},

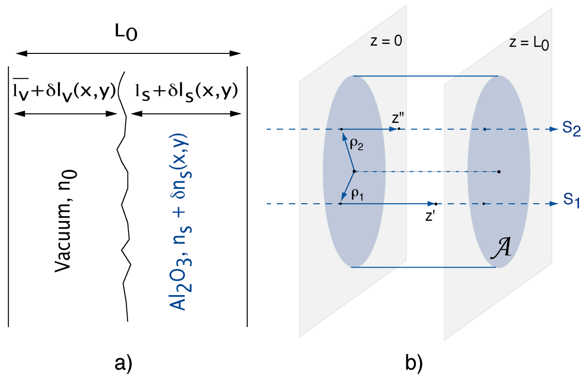

Figure 1: Schematic of an optic with surface and index of refraction fluctuations. (a) shows side view. For Case I, the OPD within the substrate is corrected. For Case II, the OPD between the two reference planes is corrected. (b) shows an isometric projection of a cylindrical optic with two rays traversing the substrate. The optic has an area $\mathcal{A}$ and radius r.

$$\text{OPD} = \text{constant} \tag{3}$$

If the surface polish is used to compensate only for the internal sapphire errors, one has:

Case I:

$$\overline{l_s}\,\overline{n_s} = l_s(x,y)n_s(x,y) = (\overline{l_s} + \delta l_s)(\overline{n_s} + \delta n_s) \tag{4a}$$

$$\delta l_s(x,y) = -\frac{\delta n_s(x,y)\overline{l_s}}{\overline{n_s}} + O[\delta n_s^2] \tag{4b}$$

If, on the other hand, the *in vacuo* propagation from the optic surface to a reference plane (refer to Figure 1a) is also taken into account, then the following constraint is imposed,

$$\begin{aligned}
&\text{Case II:} \\
\overline{l_v}n_0 + \overline{l_s}\overline{n_s} &= l_v(x,y)n_0 + l_s(x,y)n_s(x,y) &(5a)\\
&= (\overline{l_v} - \delta l_s)n_0 + (\overline{l_s} + \delta l_s)(\overline{n_s} + \delta n_s) &(5b)\\
\delta l_s(x,y) &= -\frac{\delta n_s(x,y)\overline{l_s}}{(\overline{n_s} - 1)} + O[\delta n_s^2] &(5c)
\end{aligned}$$

Equations 4b and 5c show the anti-correlation introduced by the polishing process that is needed to cancel OPD errors through the bulk: if the index is higher than the mean, then the optic is thinner than the nominal thickness ($\delta l_s < 0$ where $\delta n_s > 0$). The difference between the two cases is the denominator: $\overline{n_s} \to \overline{n_s} - 1$.

A Mathematica^{TM} program was written to perform the algebraic manipulations necessary to determine the correlated errors. The calculations were performed to 2^{nd} order in the perturbations.

I. CASE I – FIGURING TO MAKE AN OPTIC OF CONSTANT *INTERNAL* OPD

After polishing according to Equation 4b, a flat wavefront impinging on the substrate from the left will exit with the following characteristics,

$$\begin{aligned}
\langle \phi(x,y) - \phi(x',y') \rangle_{\mathcal{A}} &= k\frac{1}{A^2}\int_{\mathcal{A}} dA \int_{\mathcal{A}} dA' \left(\int_0^{L_0} dz \, [\, n(x,y,z) - n(x',y',z)] \right) \\
&= k\frac{1}{A^2}\int_{\mathcal{A}} dA \int_{\mathcal{A}} dA' \frac{\overline{l_s}}{n_s}\{n_0\, [\delta n_s(x,y) - \delta n_s(x',y')] + \\
&\quad (\delta n_s^2(x',y') - \delta n_s^2(x,y))\} &(6a)\\
&= 0 &(6b)\\
\langle (\phi(x,y) - \phi(x',y'))^2 \rangle_{\mathcal{A}} &= k^2 \int_{\mathcal{A}} dA \int_{\mathcal{A}} dA' \left(\int_0^{L_0} dz \, [\, n(x,y,z) - n(x',y',z)] \right)^2 \\
&= \left(\frac{k\overline{l_s}n_0}{n_s} \right)^2 \frac{1}{A^2} \int_{\mathcal{A}} dA \int_{\mathcal{A}} dA' \times \\
&\quad (\delta n_s^2(x',y') - \delta n_s(x,y)\delta n_s(x',y')) &(6c)\\
&= \left(\frac{k\overline{l_s}n_0}{n_s} \right)^2 \sigma_{n_s}^2 \left[1 - \int_0^1 \alpha \; d\alpha K(\alpha) R(\alpha r) \right], &(6d)\\
K(\alpha) &\equiv \frac{16}{\pi}\left[cos^{-1}(\alpha) - \alpha\sqrt{1-\alpha^2} \right]; \; \int_0^1 \alpha \; d\alpha K(\alpha) = 1. &(6e)
\end{aligned}$$

$K(\alpha)$ is the autocorrelation function of a circular aperture and arises from the double integral over the aperture in the second term of Equation 6c.

II. CASE II – FIGURING TO MAKE AN OPTIC OF CONSTANT OPD REFERRED TO PLANES OUTSIDE THE OPTIC

After polishing according to Equation 5c, a flat wavefront impinging on the substrate from the left will exit with the following characteristics,

$$\begin{aligned} \langle \phi(x,y) - \phi(x',y') \rangle_{\mathcal{A}} &= k\frac{1}{A^2}\int_{\mathcal{A}} dA \int_{\mathcal{A}} dA' \left(\int_0^{L_0} dz \, [\, n(x,y,z) - n(x',y',z)] \right) \\ &\equiv 0 \end{aligned} \tag{7}$$

Result 7 follows directly from Equation 5b, which is indepedent of $\{x, y\}$. After much algebra, one similarly obtains (accurate to fourth order),

$$\begin{aligned} \langle (\phi(x,y) - \phi(x',y'))^2 \rangle_{\mathcal{A}} &= k^2 \int_{\mathcal{A}} dA \int_{\mathcal{A}} dA' \left(\int_0^{L_0} dz \, [\, n(x,y,z) - n(x',y',z)] \right)^2 \\ &\equiv 0 \end{aligned} \tag{8}$$

III. CONCLUSION

Polishing an optic to compensate only for the internal OPD variations results in reduced but non-zero residual wavefront errors (Case I). On the other hand, if the surface is polished to correct not only for the internal errors but also to correct for the *in vacuo* propagation errors resulting from the uneven surface, then to very high order it is indeed possible to correct for the internal OPD errors of a substrate (Case II).