Location of the Center of Mass of the Inner Stage, Relative to the Horizontal Actuator-Flexure Plane, and the Table Top

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The Short Answer

If we want the uncontrolled dynamics to be well behaved for the inner stage, and the current set of spring/ mass parameters, the center of mass of the inner stage should be always stay within about 8 cm of the horizontal plane defined by the horizontal actuators and flex points. This being the case, I recommend

1) Put the center of mass of the inner stage (excluding 600 kg payload) anywhere from the actuator-flexure plane to 8 cm below that point.
2) Keep the actuator-flexure plane as close to the table top as possible, so that the mass of the payload is easier to deal with.
3) Make sure it is possible to put some counter-weights well below the actuator-flexure plane, so we can compensate for the payload if need be.

The Model

To consider the importance of the vertical location of the center of mass of the inner stage, relative to the plane of the horizontal actuators and lower flex points, I have made a simple model of the inner stage of the ETF system. The real stage has 3 cantilevers with flexures, the model has 3 idealized pure vertical springs, which mimic the vertical stiffness of the cantilevers, and a single horizontal spring, which mimics the pendulum and bending stiffness of the rod flexures. The three springs are arranged around the table on a 23” circle as shown in figure 1.

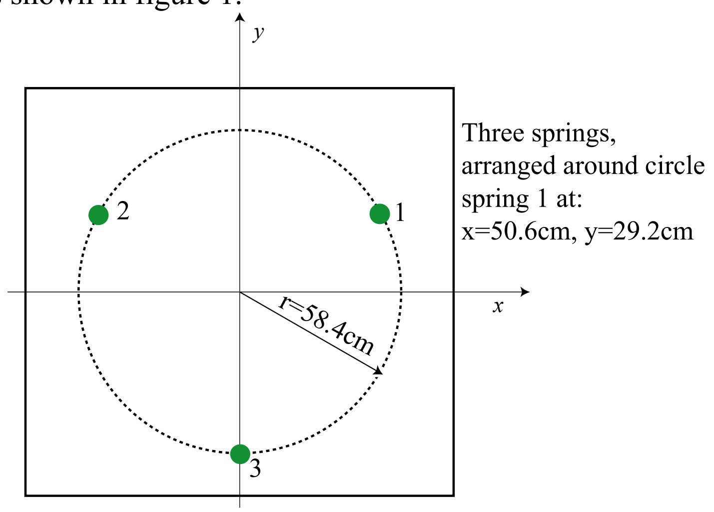

Figure 1: Top view of simplified model of the inner stage

If each of the three cantilever blades has a vertical stiffness of k_v, then the rotational stiffness of the table, K, about the y axis will be determined by the vertical stiffness of springs 1 and 2, and will be $\mathrm{K} = 2 \cdot k_v \cdot (\sqrt{3}/2\; r)^2$. The stiffness about the x axis should be identical. Now we consider a side view of the table, and consider only two degrees of freedom: translation along x and rotation about y, denoted θ.

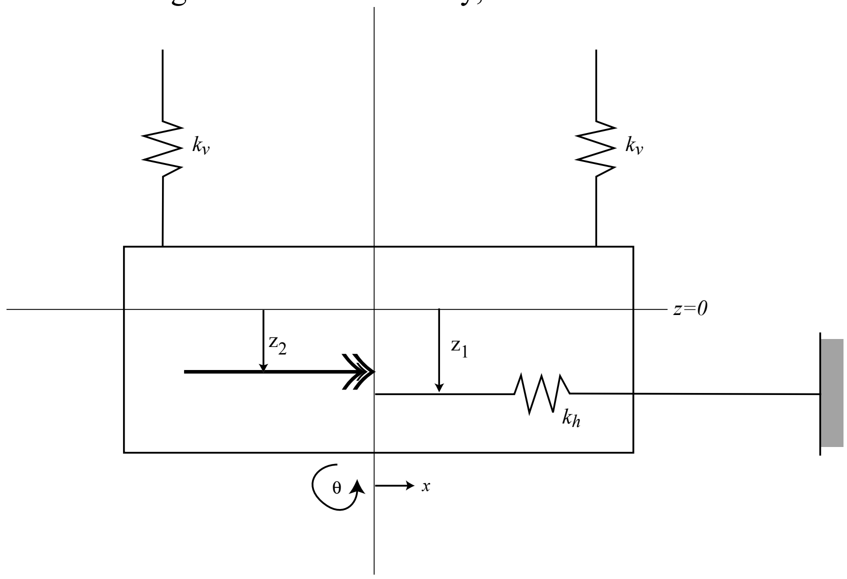

Figure 2. Side view of the simple model, showing the vertical offset of the actuator and the horizontal spring from the center of mass.

The two vertical springs provide a rotational stiffness of K, the horizontal spring is located a distance z_1 below the center of mass, and the horizontal force is applied a distance z_2 below the center of mass. The stage has mass *m* and moment *I*. The stage can translate in *x* and rotate in θ. The displacement of the horizontal spring is then $x + z_1 \cdot \theta$.

The equations of motion, are, therefore:

$m\ddot{x} = -k_h(x + z_1\theta) + F$, and

$I\ddot{\theta} = -\mathrm{K}\theta - k_h(x + z_1\theta) \cdot z_1 + Fz_2$

which is only interesting for two reasons. First, you can shove the mess into Matlab and run some calculations, which you will see in a moment. Second, you can examine the 2x2 matrix which determines all the dynamics, which I call KoM in the Matlab code (short for k over m). We can rewrite the equations of motion as:

$\begin{pmatrix} \ddot{x} \\ \ddot{\theta} \end{pmatrix} = KoM \cdot \begin{pmatrix} x \\ \theta \end{pmatrix} + B \cdot F$, where

$$KoM = -\begin{bmatrix} k_h/m & k_h z_1/m \\ k_h z_1/I & \mathrm{K} + k_h z_1^2/I \end{bmatrix}.$$

If we assume that the horizontal spring is near enough to the center of mass so that the rotational stiffness is dominated by K, i.e. $\mathrm{K} >> k_h z_1^2$, then the eigenvalues, λ, are determined by

$$\lambda^2 - \lambda\left(\frac{k_h}{m} + \frac{\mathrm{K}}{I}\right) + \frac{k_h\,\mathrm{K}}{m\,I} = 0$$

which tells us, unsurprisingly, that the two frequencies are $\sqrt{k_h/m}$ and $\sqrt{\mathrm{K}/I}$, and that the eigenvectors are essentially pure translation and pure rotation. However, as the horizontal spring begins to play a non-negligible role in the rotational stiffness, the modes become coupled. In the current model, $\mathrm{K} = k_h z_1^2$ when $z_1 = 55\ \mathrm{cm}$.

The Characteristic Height

Setting the rotational stiffness of the vertical springs equal to the rotational stiffness about the center of mass of the horizontal springs allows us to define a characteristic height, z_c, as $\mathrm{K} = k_h z_c^2$, or

$$z_c = \sqrt{\mathrm{K}/k_h}\ .$$

Looking at the Matlab results, so long as z_1 is less than about 8 cm, the coupling seems pretty small. At 30 cm, the coupling is large, and the dynamics are different. When the vertical offset reaches the characteristic height of 55 cm, the system gets really bad, as you can see in figures 3 and 4, below.

Hence, for the current system, z_1 should be less than 8 cm. In general, I believe that if we keep the offset to 1/7 of the characteristic height, we should get the same behavior. Hence, we may get some benefit by increasing the characteristic height, which we could do by:

Stiffening the cantilever blades.
Moving the cantilever attach points farther out.
Softening the horizontal spring (lighten the table or lengthen the rods)

We should be careful about lengthening the rods, since that will add move the table top up relative to the lower attach points. This is not a good move, since it moves the science payload farther from the table center, and thus the cg of the science load is harder to deal with.

Results of the Simple Matlab Model

This simple model was coded into Matlab, so one can evaluate the frequency response of the system. The model will be distributed along with this document, and instructions for playing with the model are included at the end of this document. The current parameters are:

```
ETF =
            kv: 85000
             d: 0.5040
            KK: 4.3183e+004
             m: 850
      pend_len: 0.0600
            kh: 1.3883e+005
             I: 212.5000
```

In figures 3 and 4, below, $z_1 = z_2$. First, we look at the tilt response of the inner stage Again, all the inner stage servo loops are off – turning on the vertical loops will stiffen the system by factors of 10 – 100 at the frequency range from 0.1 to 10 Hz.

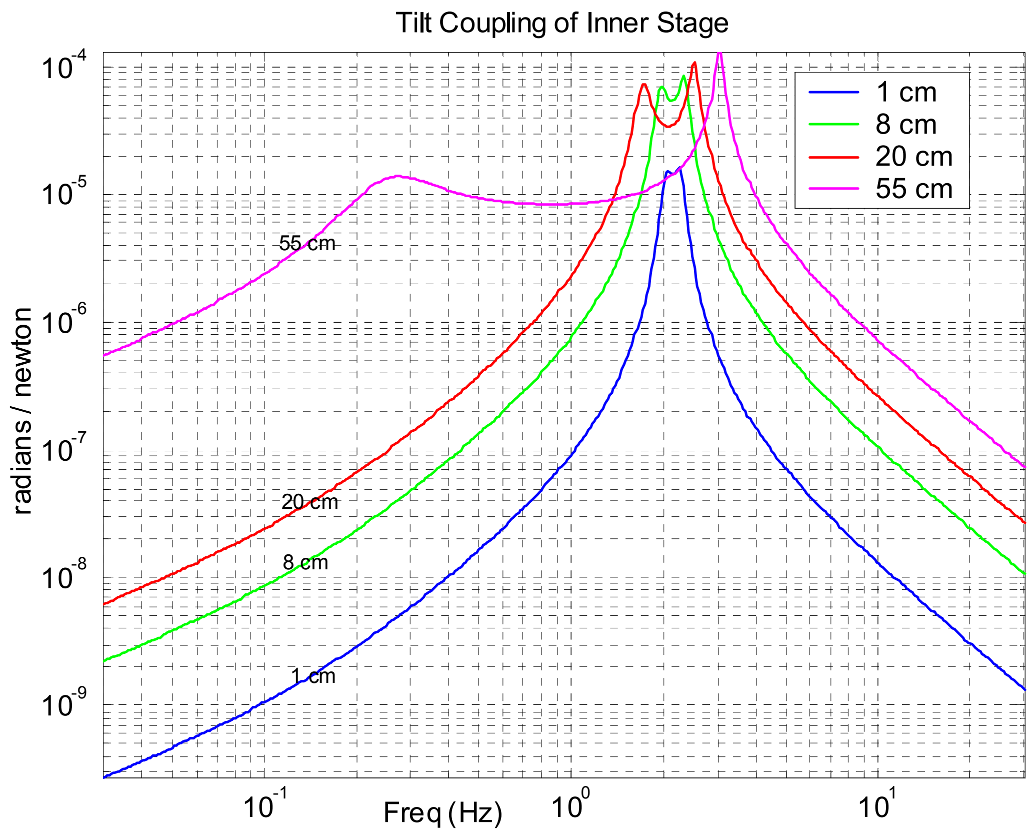

Figure 3. Tilt response of the inner stage. The horizontal spring and the actuator are aligned to one another, but are offset from the center of mass by various amounts.

In figure 3, we see the tilt response of the system at an 8 cm offset is acceptable, but at 20cm, the modes have obviously become coupled, and at 55 cm, the system looks pretty bad. We can also look at what happens to the horizontal seismometers as we increase the offset. Even though there is no offset between the horizontal actuator and the horizontal restoring force, the tilt motion is clearly evident in the horizontal sensors as the offset approached the 55 cm characteristic length.

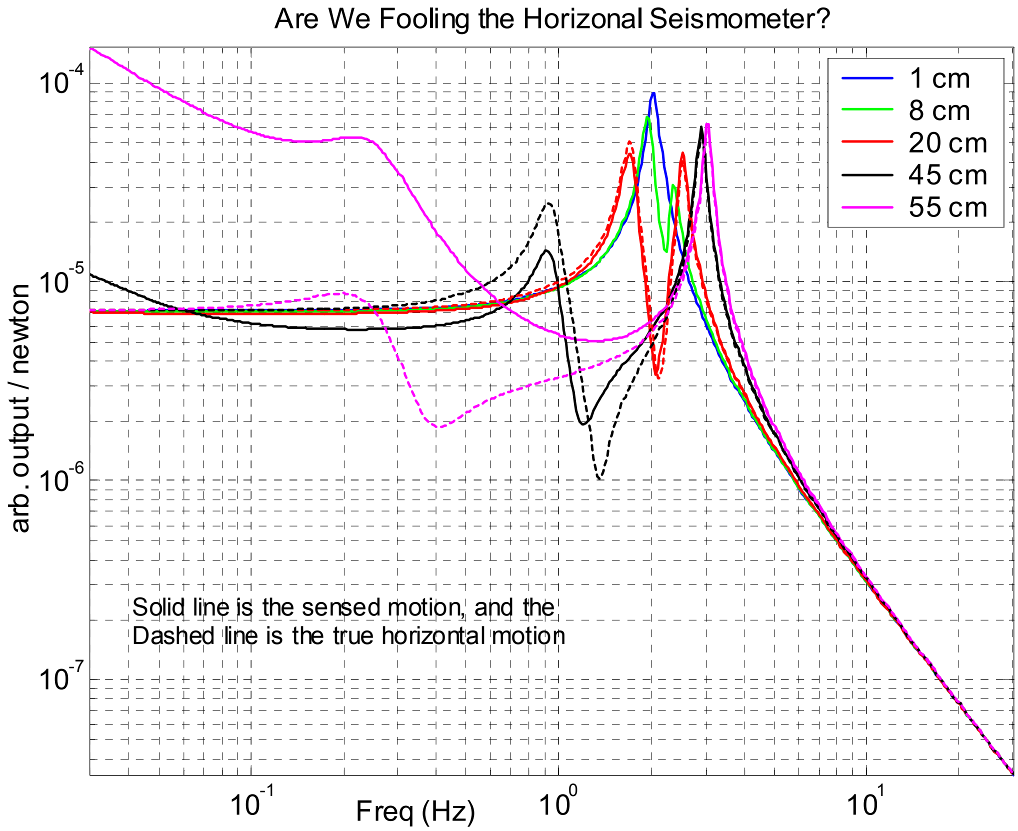

Figure 4. Horizontal motion as sensed by the horizontal seismometer.

Figure 4 shows the output of an idealized seismometer, the plot shows

$$output = x + \frac{g}{\omega^2}\theta$$

The solid lines in figure 4 are the output as defined in the expression above. The dashed lines are the true horizontal motion, which are only plotted for offsets of 20 cm or more. At 20 cm, the tilt makes a negligible contribution to the seismometer output. With a 45 cm offset, the tilt begins to noticeably impact the seismometer output. When the offset equals the characteristic height, we see that the tilt makes a serious impact on the horizontal seismometer output. I don't have a nice explanation for the low frequency slopes of these curves yet. How much does this impact the servos? The dynamics of the ideal case and the 8cm offset are very similar. The dynamics at a 20 cm offset are noticeably different, but are in no way destabilizing. Currently, the blend frequency of the position sensor to inertial sensor on the inner stage happens at 0.3 Hz. Without the vertical servos, the blending will still be stable with a 20 cm offset, may be stable with a 45 cm offset, and will be hopeless with a 55 cm offset.

Suspended Mass of the Multiple Pendulums

So far, we have ignored the suspended mass of the payload. This could be as much as 300 kg, and it could be suspended by as much as 940 mm above the table top. At frequencies below the pendulum modes (about ½ Hz), this inverted pendulum has a rotational stiffness, K_p, of

$$K_p = -m_p \cdot g \cdot h$$

where m_p is the suspended mass of the pendulum (about 300 kg) and h is the height above the center of mass (about 1 meter). Thus, $K_p \approx -3{,}000$ N-m. Below the natural frequencies of the pendulum, its mass also acts to stiffen the horizontal pendulum mode of the table by $k_{hp} = m_p\, g / l \approx 49{,}000$ N/m . This moves the characteristic length down to

$$z_c = \sqrt{\frac{K + K_p}{k_h + k_{hp}}} \approx \sqrt{\frac{4.3e4 - 3e3}{1.39e5 + 4.9e4}} = 46\,\text{cm}$$

This is an almost 20% change, and we will probably need to do some compensation with ballast masses to make up the difference.

Conclusion

If we can keep the center of mass of the inner stage table within about 8 cm of the plane defined by the horizontal actuators and flexure points, the dynamics of the translation and tip modes will not be strongly coupled by this offset. An ideal case would have the cg of the table without any of the 600kg payload be about 8 cm below the actuation plane, and have the table top be as close as possible to the actuation plane. This way we can put lots of mass on top of the table without needing to add ballast mass to a keel.

P.S. Running the Matlab Program

To run the program to generate the plots you see in this document you need three files:

ETF_params.m	A function which fills a structure with current values of the ETF design.
ETF_sys.m	A function which uses the data structure created above to create a dynamic 2-DOF model of the system. Input is force. Output 1 is x, output 2 is theta.
ETF_cg.m	This script calls the other functions with various parameters and plots up the results. This is the file to look at and to run.