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Modal Q's, loss angles, and thermal noise in coupled harmonic oscillators

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Abstract

We describe a new procedure for relating the normal mode quality factors (Q's) of a chain of coupled harmonic oscillators to the loss angles (ϕ's) in the individual stages. We also show how to infer the spectrum of the thermal noise in such a chain from the losses in the individual stages.

1 Introduction

There are currently a number of interferometric gravitational-wave detectors under construction world-wide, including VIRGO [1], GEO 600 [2], TAMA [3], and LIGO [4, 5]. To reduce both seismic and thermal noise in these detectors, test masses are supported by low-loss suspensions with low natural resonance frequencies. One common form of such a suspension in the current generation of gravitational-wave detectors is a simple pendulum, realized by cradling each test mass in a single loop of fine steel wire. Advanced gravitational-wave detectors such as the anticipated second generation LIGO II, as well as the current generation GEO 600, are expected to employ compound pendula to further reduce the levels of both seismic and thermal noise [6, 7, 8, 9].

The physics of single-pendulum suspensions has been extensively studied [10, 11, 12]. Pendulum thermal noise for a simple, one-stage suspension is easy to model using the fluctuation-dissipation theorem, which relates the mechanical losses in a system to the spectrum of its thermal noise [13, 14]. For an extended system, however, the fluctuation-dissipation theorem can be difficult to apply simply because of the complexity of the calculation, and it is easy to lose sight of the underlying physics. Simpler methods of calculating the thermal noise, such as normal-mode decomposition, often require a great deal of care and effort to correctly interpret their results, even for compound systems with only two stages [15]. Furthermore the relationship between the losses in the system, the thermal noise, and the quality factors (Q's) of the individual modes is often not clear in a system with many modes.

Thermal noise in two-mode systems has been studied in the time domain by Wang and Uhlenbeck [16] and by Paik [17]. One-dimensional, two-mode systems have also been explored in the frequency domain using normal-mode decomposition [10], and the normal-mode decomposition method has been compared with the fluctuation-dissipation theorem, again in a one-dimensional, two-mode system [15]. Thermal noise in single-cradle suspensions has been explored in three dimensions, including violin modes of the wires, using the fluctuation-dissipation theorem [18] and an electrical transmission line model [19]. In this paper we will explore the relationship between the Q's of the modes and the losses in the individual stages of a one-dimensional chain of harmonic oscillators and show how the off-resonance thermal noise depends on those losses.

2 A general method for calculating thermal noise in coupled oscillators

The fluctuation dissipation theorem predicts the thermal noise in a linear system as [13, 14]

$$x_{th}^2 = \frac{4k_BT}{\omega^2}\mathrm{Re}\left\{Y\right\}.$$

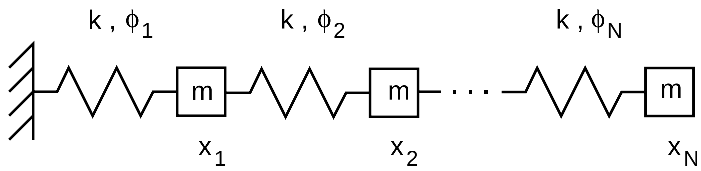

Figure 1: A chain of coupled harmonic oscillators. The first (left-most) mass is attached to a rigid wall, but the last mass is attached only to the previous stage. The analysis described in the paper can be easily modified for different boundary conditions. This system is mathematically equivalent to a compound pendulum in the small-angle approximation.

Where $Y = v/f$ is the mechanical admittance of the system, defined by the velocity of some part of the system (v) in response to a force applied to that part (f). For a sinusoidal driving force, the admittance is just

$$Y = \frac{i\omega x}{f},$$

where x is the displacement of that mass from its static equilibrium point.

The mechanical admittance is straightforward to calculate for the case of a 1d chain of N harmonic oscillators, such as the one shown in Figure 1. The equations of motion are

$$-m_n\omega^2 x_n = -k_n(1+i\phi_n)(x_n - x_{n-1}) + k_{n+1}(1+i\phi_{n+1})(x_{n+1} - x_n) + f_n,$$

where k_n, m_n, and ϕ_n are spring constants, masses, and loss angles for the individual stages, as shown in Figure 1. The displacements of the masses from their static equilibrium points are given by x_n, and any external force applied to the masses are given by f_n. This can be rearranged to

$$k_n(1+i\phi_n)x_{n-1} + \Big(m_n\omega^2 - k_n(1+i\phi_n) - k_{n+1}(1+i\phi_{n+1})\Big)\, x_n + k_{n+1}(1+i\phi_{n+1})x_{n+1} = -f_n,$$

or in matrix notation,

$$\mathbf{\Omega x} = -\mathbf{f}.$$

This is easily inverted to find the positions of the masses.

$$\mathbf{x} = -\mathbf{\Omega}^{-1}\mathbf{f}$$

If we only apply a force to one mass, then the position of that mass will be determined by a single entry on the diagonal of the matrix $\mathbf{\Omega}^{-1}$, and that entry is given by [20].

$$\begin{aligned} x_n &= -\left[\mathbf{\Omega}^{-1}\right]_{nn} f_n \\ &= -\frac{C_{nn}}{\det(\mathbf{\Omega})} f_n \end{aligned} \tag{1}$$

Finding the position of one mass with a force applied to just that mass then reduces to a calculation of two determinants, one of an $(N-1)\times(N-1)$ matrix (the cofactor C_{nn}), and the other of an $N \times N$

matrix ($\mathbf{\Omega}$). These determinants are fairly easy to compute because of the tri-diagonal nature of $\mathbf{\Omega}$ (See Appendix A).

The admittance at mass n is just $Y = i\omega x_n/f_n$, and a little algebra shows that the thermal noise is

$$x_{th}^2 = -\frac{4k_BT}{\omega}\frac{\text{Re}\left\{iC_{nn}|\mathbf{\Omega}|^*\right\}}{|\text{det}(\mathbf{\Omega})|^2}. \tag{2}$$

3 Loss angles and mode Q's

If we use Equation 2 to predict the thermal noise, we will get an expression for x_{th} in terms of the loss angles in the individual stages, the ϕ_n's. Often it is the quality factors, or Q's, of the normal modes of a system that are measured, not the losses in the individual stages. In this section we will show how the two are related.

One way to measure the Q of a mode is to excite the system with a driving force of constant amplitude and varying frequency. The response of the system will have a maximum at the resonant frequency of each normal mode, and these maxima will show up as peaks in the transfer function of the system. The Q of a mode is conventionally defined as its resonant frequency divided by the width of the peak in the square of its transfer function.

The transfer function of our one dimensional system can be obtained from Equation 1.

$$\frac{x_n}{f_n} = -\frac{C_{nn}}{\text{det}(\mathbf{\Omega})}$$

The maxima in this transfer function will occur very near to the resonant frequencies of an analogous, undamped system. If we assume that the peaks are narrow enough that C_{nn} is slowly varying over the width of the peak, then we may estimate the width of the peak by finding the frequencies where

$$\left(\text{Re}\left\{\text{det}(\mathbf{\Omega})\right\}\right)^2 + \left(\text{Im}\left\{\text{det}(\mathbf{\Omega})\right\}\right)^2 = 2\left(\text{Im}\left\{\text{det}(\mathbf{\Omega})\right\}\right)^2_{\omega=\omega_\alpha}.$$

Here ω_α are the frequencies of the eigenmodes. (We use the greek letter α to designate eigenmodes, and the latin n to number individual stages.) Because the peaks are narrow and the imaginary part is slowly varying, we can write this as

$$\text{Re}\left\{\text{det}(\mathbf{\Omega})\right\} \approx \pm\left(\text{Im}\left\{\text{det}(\mathbf{\Omega})\right\}\right)_{\omega=\omega_\alpha}.$$

We can expand the left side of this expression in a Taylor series in ω to find the widths of the resonances.

$$\text{Re}\left\{\text{det}(\mathbf{\Omega})\right\} \approx \text{Re}\left\{\text{det}(\mathbf{\Omega})\right\}_{\omega=\omega_\alpha} + \frac{\partial}{\partial\omega}\text{Re}\left\{\text{det}(\mathbf{\Omega})\right\}_{\omega=\omega_\alpha}(\omega-\omega_\alpha) + \cdots$$

The first term in this expansion is zero by definition. That's what defines the normal mode frequencies. (We can ignore the very small shift in these frequencies introduced by the damping.) If the second term in the series does not vanish (*i.e.* there are no degenerate modes [21]), then the width of the resonance peaks is

$$\Delta\omega_\alpha = \left|\frac{2\text{Im}\left\{\text{det}(\mathbf{\Omega})\right\}}{\frac{\partial}{\partial\omega}\text{Re}\left\{\text{det}(\mathbf{\Omega})\right\}}\right|_{\omega=\omega_\alpha},$$

and the Q of the mode is

$$Q_\alpha = \frac{\omega_\alpha}{2} \left| \frac{\frac{\partial}{\partial\omega}\text{Re}\left\{\det(\mathbf{\Omega})\right\}}{\text{Im}\left\{\det(\mathbf{\Omega})\right\}} \right|_{\omega=\omega_\alpha}. \tag{3}$$

Since we are only retaining terms to first order in the various ϕ's, we may invert Equation 3 to get a set of linear equations relating the ϕ's to the Q's.

$$\left|\text{Im}\left\{\det(\mathbf{\Omega})\right\}\right|_{\omega=\omega_\alpha} = \frac{\omega_\alpha}{2Q_\alpha}\frac{\partial}{\partial\omega}\left|\text{Re}\left\{\det(\mathbf{\Omega})\right\}\right|_{\omega=\omega_\alpha}$$

Note that if any one of the losses ϕ_i is large enough to dominate the expression for $\text{Im}\left\{\det(\mathbf{\Omega})\right\}_{\omega_\alpha}$, then that loss will effectively determine the Q of that mode. The measured Q will be determined by that one lossy stage. If this is the case, then we will be unable to deduce the values of the smaller losses from a simple Q measurement.

4 The general behavior of the thermal noise spectrum

It is easy to show that the thermal noise spectrum will have peaks at the natural resonance frequencies of the system. We can also say some very general things about the off-resonance thermal noise, which we present in this section. Proofs of all of these results are straightforward but quite long, and we do not include them here in the interest of brevity.

4.1 Upper and lower bounds on mode frequencies

We can set upper and lower bounds on the eigenfrequencies of the modes. We use the greek letter α to number the eigenmodes of the system, and we define two new quantities $\omega_{\max}$ and $\omega_{\min}$ as the largest and smallest values of $\sqrt{k_p/m_q}$, where p and q may refer to different stages. For $\omega_{\max}$, p and q may not differ by more than 1, but for $\omega_{\min}$, p and q may have any values. Note that these are not the highest and lowest resonant frequencies of the individual stages. Rather, they are the highest and lowest possible resonant frequencies of individual stages that could be assembled by mixing and matching masses and springs, with the restriction on $\omega_{\max}$ that the springs were already attached to either the front or back ends of the masses.

With these definitions, all of the eigenfrequencies must lie in the range

$$\omega_{\min}\sqrt{\frac{2}{N(N+1)}} < \omega_\alpha < 2\omega_{\max}.$$

4.2 Asymptotic behavior

In this section we describe the off-resonance spectrum of the thermal noise in the last stage at high and low frequencies. (Analogous results apply to any arbitrary mass in the chain.)

At high frequencies (well above the highest resonance), the thermal noise falls off as $\sqrt{\phi_N(\omega)}/\omega^{5/2}$, just like a single oscillator, and it is dominated by the last stage, regardless of the relative values of the loss angles in each stage.

$$x_{th}^2(\omega) \approx \frac{4k_BT}{m_N}\frac{\omega_0^2}{\omega^5}\phi_N(\omega)$$

At low frequencies (well below the lowest resonance), the thermal noise also resembles the case of a single oscillator, but here the losses in each stage contribute equally.

$$x_{th}^2(\omega) \approx \frac{4k_BT}{\omega}\sum_{n=1}^{N}\frac{\phi_n(\omega)}{m_n\omega_n^2}$$

Neither of these results are surprising. In the first (high-frequency) case, thermal noise in distant stages does not show up in the last one because of the filtering action of a harmonic oscillator at frequencies well above its resonance. The second (low-frequency) result tells us that, at frequencies where there is no filtering, the thermal noise in all of the stages contributes equally and adds in quadrature.

Note that in the case of structural damping, where ϕ is constant over a wide range of frequencies [22], our expression appears to predict a divergence in the thermal noise at $\omega = 0$. This divergence does not really happen because ϕ must be an odd function of ω, which means it must go to zero at $\omega = 0$ [23]. This keeps our thermal noise prediction from diverging, but we seldom need to worry about the low-frequency variation of ϕ when calculating thermal noise curves, since ϕ usually doesn't begin to fall off until you get to extremely low frequencies [10].

4.3 Consequences of the asymptotic behavior

We saw at the end of Section 3 that if there is a large mismatch in the losses of the individual stages, we would only be able to measure the largest ϕ_i by measuring the modal Q's. Our asymptotic results show that, in some cases, we can still determine the low-frequency thermal noise spectrum from such a Q measurement, but that we cannot reasonably predict the high-frequency spectrum if the last stage has low loss. Since it is the high-frequency, low-loss case that we are usually interested in for gravitational-wave detector suspensions, a modal Q measurement is of little use for predicting the thermal noise behavior of such a suspension.

5 Some examples

5.1 Thermal noise in a system with three identical stages

As an example we consider here a system with three stages having identical masses and spring constants but different losses. The natural frequency of one of these stages (by itself, not coupled to any others) is $\omega_0 = \sqrt{k/m}$, and we will find it useful to express our measurement frequency with respect to this single-stage natural frequency. Our unitless measurement frequency is defined as

$$\zeta \equiv \frac{\omega}{\omega_0}.$$

The thermal noise of a single stage in isolation is easy to calculate, and we give it here for comparison with the three-mass, coupled case.

$$x_{th}^2 = \frac{4k_B T}{m\omega_0^3}\left(\frac{1}{\zeta}\right)\frac{\phi}{\left(\zeta^2-1\right)^2+\phi^2}.$$

For the three-stage system the thermal noise in the last mass is, according to our procedure

$$\begin{aligned} x_{th}^2 &= \frac{4k_B T}{m\omega_0^3}\left(\frac{1}{\zeta}\right) \\ &\times \frac{\phi_3\zeta^8 - 6\phi_3\zeta^6 + (\phi_2+11\phi_3)\zeta^4 - 2(\phi_2+3\phi_3)\zeta^2 + (\phi_1+\phi_2+\phi_3)}{\left[\zeta^6-5\zeta^4+6\zeta^2-1\right]^2 + \left[(\phi_1+2\phi_2+2\phi_3)\zeta^4 - (3\phi_1+4\phi_2+5\phi_3)\zeta^2 + (\phi_1+\phi_2+\phi_3)\right]^2}, \end{aligned}$$

The loss angles in these expressions may be frequency dependent, as in the case of thermoelastic or viscous damping, but they must all be small ($\phi_n \ll 1$) for this expression to be valid.

Figure 2 shows an example of thermal noise in one and three stage systems. There, the loss angle in the single-stage system is the same as that of the last (highest-Q) element in the three-stage system.

5.2 Q's and loss angles in systems with identical stages

For the three-stage system considered above, the modal Q's will be given by

$$Q_\alpha = \frac{3\zeta_\alpha^6 - 10\zeta_\alpha^4 + 6\zeta_\alpha^2}{(\zeta_\alpha^4 - 3\zeta_\alpha^2+1)\phi_1 + (2\zeta_\alpha^4 - 4\zeta_\alpha^2+1)\phi_2 + (2\zeta_\alpha^4 - 5\zeta_\alpha^2+1)\phi_3}.$$

If ϕ_2,$\phi_3 \ll \phi_1 \ll 1$, then the mode Q's will be dominated by the first stage.

$$Q_\alpha = \frac{1}{\phi_1}\frac{3\zeta_\alpha^6 - 10\zeta_\alpha^4 + 6\zeta_\alpha^2}{\zeta_\alpha^4 - 3\zeta_\alpha^2 + 1},$$

or

$$\begin{aligned} Q(\zeta_{\mathrm{I}} = 0.45) &= \frac{1.92}{\phi_1} \\ Q(\zeta_{\mathrm{II}} = 1.24) &= \frac{2.81}{\phi_1} \\ Q(\zeta_{\mathrm{III}} = 1.80) &= \frac{9.28}{\phi_1}. \end{aligned}$$

An example of this Q suppression by one lossy stage can be seen in Figure 2, where the peaks in the coupled system are much lower than the peak in the spectrum of the isolated stage.

5.3 Recoil damping

It has been known for some time that the Q suppression described above depends on the relative masses and natural frequencies of the individual stages. In this section we give an example of how

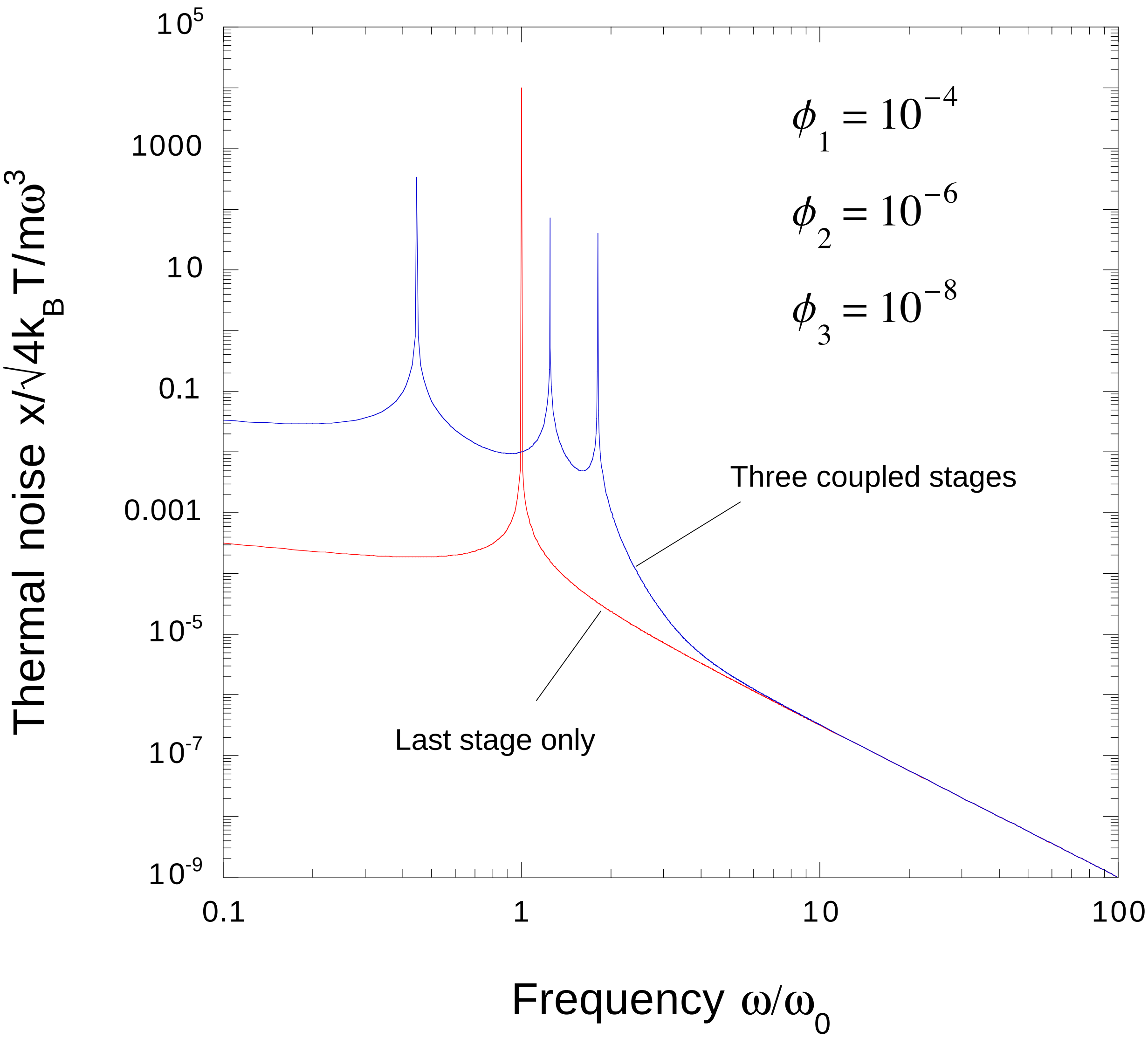

Figure 2: A plot of the thermal noise in a three-stage system with identical masses and spring constants but different loss angles, compared with the thermal noise spectrum of a single stage in isolation. Here all loss angles are independent of frequency, as in the case of structural damping, and the loss in the single stage is identical to the loss in the last stage in the three-mass system. Note the suppression of the Q's of the modes by the lossy first element.

our procedure can be used to model reaction damping in a two-stage system where the masses of the stages are substantially different.

For a two stage system with $m_1 \gg m_2$ and $\omega_1 > \omega_2$, the normal mode frequencies are approximately

$$\begin{aligned} \omega_{\mathrm{I}}^2 &\approx \omega_1^2 \left(1 + \frac{m_2}{m_1}\frac{\omega_2^2}{\omega_1^2 - \omega_2^2}\right) \\ \omega_{\mathrm{II}}^2 &\approx \omega_2^2 \left(1 - \frac{m_2}{m_1}\frac{\omega_2^2}{\omega_1^2 - \omega_2^2}\right). \end{aligned}$$

This situation is a common one. For example, the second stage could represent a low-loss, low-frequency pendulum under study, and the first stage would be the table or building to which the experiment is anchored. In this case, it is the second, low-frequency mode we are interested in, since it corresponds roughly to the natural mode of the second stage in isolation. The Q of this mode i, according to our formula,

$$\frac{1}{Q_{\mathrm{II}}} \approx \phi_2 + \frac{m_2}{m_1}\frac{\omega_1^2\omega_2^2}{(\omega_1^2 - \omega_2^2)^2}\phi_1.$$

Saulson has derived a similar result in the special case of viscous damping using a root-locus method [10], and our result reduces to his if we set our loss angles proportional to frequency.

6 Summary and conclusions

We have developed a simple method for relating the measured Q values of a one-dimensional chain of coupled harmonic oscillators to the losses in the individual stages and hence to the off-resonance thermal noise spectrum.

One of the very general conclusions of our model is that if the losses in the individual stages differ significantly, measuring the modal Q's can allow us to accurately predict the low-frequency thermal noise spectrum, but it is not very revealing of the high-frequency behavior.

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A Calculating determinants of $\mathbf{\Omega}$

The matrix $\mathbf{\Omega}$ is symmetric, and its general form is easy to write down. For a three-mass system,

$$\mathbf{\Omega} = \begin{bmatrix} m_1\omega^2 - k_1 - k_2 - i(k_1\phi_1 + k_2\phi_2) & k_2 + ik_2\phi_2 & 0 \\ k_2 + ik_2\phi_2 & m_2\omega^2 - k_2 - k_3 - i(k_2\phi_2 + k_3\phi_3) & k_3 + ik_3\phi_3 \\ 0 & k_3 + ik_3\phi_3 & m_3\omega^2 - k_3 - i(k_3\phi_3) \end{bmatrix}.$$

Note that all of the diagonal elements *except the last one* have the same form. If $n < N$,

$$\Omega_{nn} = m_n\omega^2 - k_n - k_{n+1} - i\left(k_n\phi_n + k_{n+1}\phi_{n+1}\right),$$

whereas

$$\Omega_{NN} = m_N\omega^2 - k_N - ik_N\phi_N.$$

If we leave the first mass free, not attached to a wall, then the first diagonal element will have a similar form.

The combinatoric method of calculating determinants is particularly well suited to this case [20].

$$\begin{vmatrix} a_1 & b_1 & c_1 & \cdot & \cdot & \cdot \\ a_2 & b_2 & c_2 & \cdot & \cdot & \cdot \\ a_3 & b_3 & c_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_N & b_N & c_N & \cdot & \cdot & \cdot \end{vmatrix} = \sum_{i,j,k\cdots} \varepsilon_{ijk\cdots} a_i b_j c_k \cdots,$$

where $\varepsilon_{ijk\cdots}$ is a multi-argument Levi-Civita symbol, equal to $+1$ when the subscript $ijk\cdots$ is an even permutation of $123\cdots$, -1 when the subscript is an odd permutation of $123\cdots$, and zero if any index is repeated.

This calculation is even easier when the losses are small ($\phi_n \ll 1$), and we only need to retain terms to first order in the loss angles. Note that the assumption that the losses are small does not preclude frequency dependence. Our first-order results would be equally valid for structural damping, where ϕ is constant over a wide range of frequencies [22, 10, 18], thermoelastic damping, viscous damping (both of which give frequency dependence to the loss angle), or any other mechanism that introduces frequency dependence to ϕ.

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