Measuring the gravitational wave stochastic background with LIGO

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Stochastic background

origin:

- astrophysical?
- cosmological?

properties:

- isotropic
- gaussian
- completely characterized by $H(f) = \langle h(f)^2 \rangle$

Reference:

- Bruce Allen, "The stochastic gravity-wave background: sources and detection," `gr-qc/9604033v3`.

analysis procedure

Ingredients for stochastic background data analysis:

- Ω_{GW}
- correlation analysis
- overlap reduction function
- optimal filter

$\Omega_{\rm GW}(f)$

$\Omega_{\rm GW}$: energy density in gravitational waves, as a fraction of the critical density, per logarithmic frequency interval.

$$\Omega_{\rm GW}(f) = \frac{f}{\rho_{\rm critical}} \frac{d\rho_{\rm GW}}{df} = \frac{32\pi^3}{3{H_0}^2} f^3 H(f)$$

where $H(f)$ is the strain-squared power spectral density.

Models assume some power-law dependence:

$$\Omega_{\rm GW}(f) = \Omega_\alpha \left(\frac{f}{100 \text{ Hz}} \right)^\alpha$$

We measure $H(f)$ and report Ω_α.

How do we measure it?

Use two detectors and look for correlations. Consider two co-located detectors:

$$s_1(t) = h_1(t) + n_1(t)$$
$$s_2(t) = h_2(t) + n_2(t)$$

where:
h_i is the signal seen in detector i
n_i is the noise seen in detector i
assume $h_1(t) = h_2(t)$ (detectors see same signal)

$$\langle s_1 s_2 \rangle = \langle h^2 \rangle + \langle n_1 h \rangle + \langle n_2 h \rangle + \langle n_1 n_2 \rangle$$

$$\frac{1}{T}\int_0^T s_1(t) s_2(t) dt \longrightarrow \langle h^2 \rangle$$

correlation analysis

We can make an estimator for Ω_α:

$$Y = \int \frac{32\pi^3}{3H_0^2} \left(\frac{100 \text{ Hz}}{f} \right)^\alpha f^3 s_1(f) s_2(f)^* Q(f) df$$

where Q is some weighting function (filter, kernel) we can choose however we want, subject to $\int Q(f)df = 1$.

Note that $\langle Y \rangle = \Omega_\alpha$ because $\langle s_1(f) s_2(f) \rangle = H(f)$.

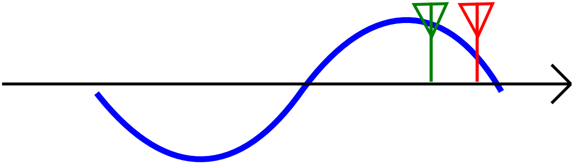
$\gamma(f)$: the overlap function

Due to geometry, the observed correlation is attenuated when detectors are separated or rotated.

- separation in space
- difference in orientation

$\gamma(f)$: the overlap function

separation in space

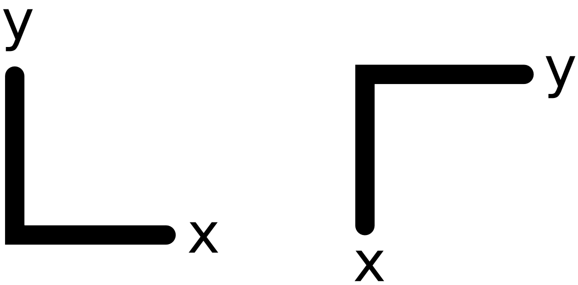


For part of the cycle, the detectors are correlated; for part of the cycle they are anti-correlated.

$$\langle h_1 h_2 \rangle = \langle \sin(\omega t) \sin(\omega t + k\Delta x) \rangle \sim \cos(k\Delta x)$$

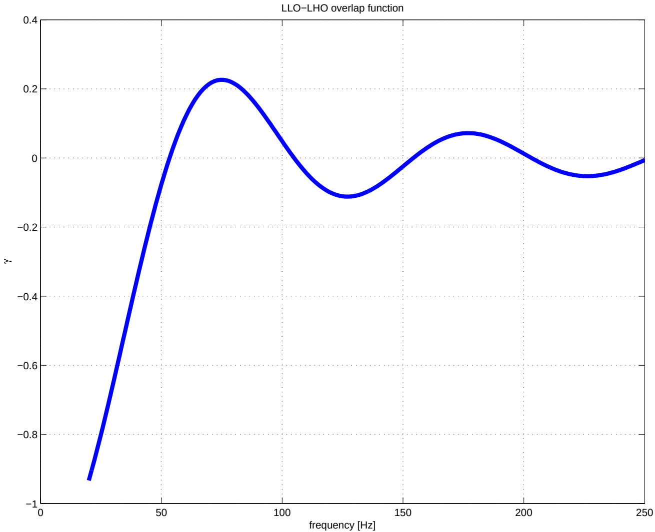
$\gamma(f)$: the overlap function

difference in orientation



co-located detectors, rotated 90 degrees, have $\gamma = -1$.

$\gamma(f)$: the overlap function



Putting $\gamma(f)$ into the estimator

We need to correct for the attenuation quantified by γ:

$$Y = \int \frac{32\pi^3}{3H_0^2} \left(\frac{100 \text{ Hz}}{f}\right)^\alpha \frac{f^3}{\gamma(f)} s_1(f) s_2(f)^* Q(f) df$$

Now we are almost ready to pick $Q(f)$. (Any will do, but some are better...)

optimal estimators

Suppose x_i are independent measurements (random variables) all with mean x but with error bars (standard deviations) σ_i.

We can form a *weighted average*:

$$x = \sum a_i x_i$$

subject to $\sum a_i = 1$ so that $\langle x \rangle = \langle x_i \rangle$.

The weighted average has variance:

$$\sigma^2 = \sum {a_i}^2 {\sigma_i}^2$$

Problem: find $\{a_i\}$ such that σ^2 is minimized.

optimal estimators

In forming a linear combination of measurements

$$\boxed{x = \sum a_i x_i} \text{ subject to } \sum a_i = 1$$

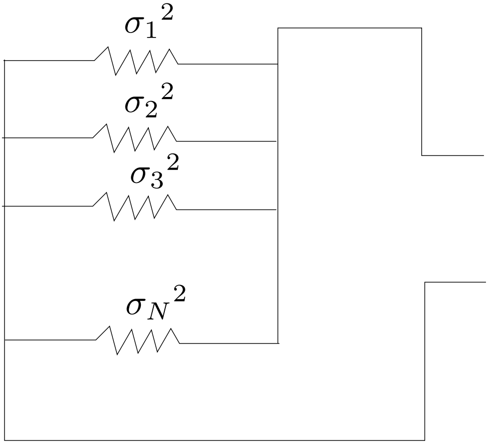
the optimal choice is to weight by inverse variance:

$$\boxed{a_i = N \frac{1}{{\sigma_i}^2}} \text{ with normalization } N = \frac{1}{\sum \frac{1}{{\sigma_i}^2}}$$

in which case:

$$\boxed{\frac{1}{\sigma^2} = \sum \frac{1}{{\sigma_i}^2}}$$

variances add like resistors in parallel!



$$\frac{1}{\sigma^2} = \sum \frac{1}{\sigma_i{}^2}$$

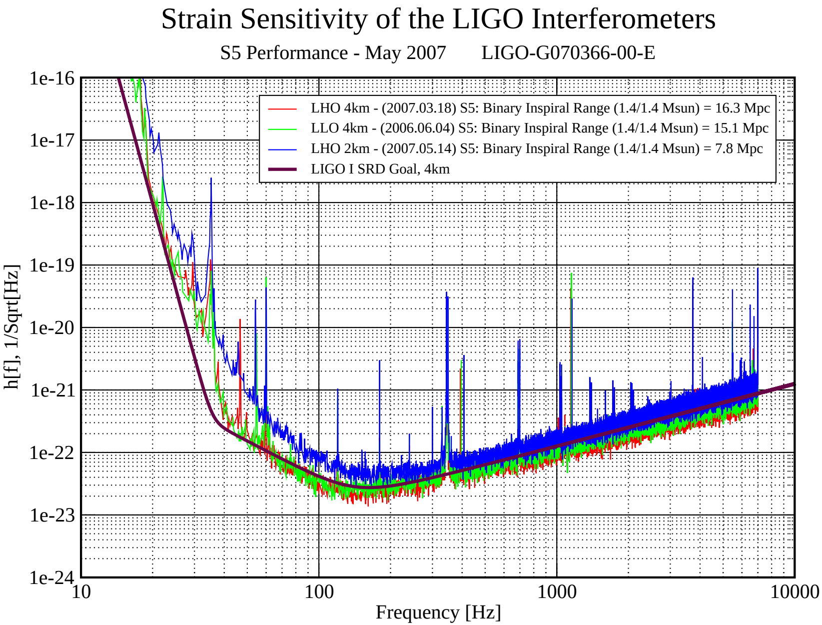
How do we know the variance of our estimates?

$$\langle s_1 s_2 \rangle = \langle h^2 \rangle + \cancel{\langle n_1 h \rangle} + \cancel{\langle n_2 h \rangle} + \cancel{\langle n_1 n_2 \rangle}$$

$$\mathrm{var}\{s_1 s_2\} = \cancel{\mathrm{var}\{h^2\}} + \cancel{\mathrm{var}\{n_1 h\}} + \cancel{\mathrm{var}\{n_2 h\}} + \mathrm{var}\{n_1 n_2\}$$

- By stationarity assumption, $\mathrm{var}\{h^2\} = 0$
- In low signal-to-noise regime, $\mathrm{var}\{n_1 h\} << \mathrm{var}\{n_1 n_2\}$.
- In low signal-to-noise regime, the variance of the correlation is given by the power spectral density of the detector output – which we measure all the time!

Detector PSDs - noise variance



Putting it all together

$$Y = \int Y(f)df \text{ and } {\sigma_Y}^2 = \int \sigma_Y(f)^2 df$$

$$Y(f) = \frac{32\pi^3}{3H_0^2}\left(\frac{100\text{ Hz}}{f}\right)^\alpha \frac{f^3}{\gamma(f)} s_1(f)s_2(f)^* Q(f)$$

$$\sigma^2(f) = \left[\frac{32\pi^3}{3H_0^2}\left(\frac{100\text{ Hz}}{f}\right)^\alpha \frac{f^3}{\gamma(f)}\right]^2 P_1(f)P_2(f)$$

$$Q(f) \sim \frac{1}{\sigma(f)^2}$$

$$\boxed{Y = \frac{3H_0^2}{32\pi^3}\int\left(\frac{f}{100\text{ Hz}}\right)^\alpha \frac{\gamma(f)}{f^3 P1(f)P_2(f)} s_1(f)s_2(f)^* df}$$

the *Nature* paper

- The LIGO Scientific Collaboration and the Virgo Collaboration, "An upper limit on the stochastic gravitational-wave background of cosmological origin," *Nature* **490**, 990-994 (2009).

$$\boxed{\Omega_0^{\mathrm{G}W} < 6.9 \times 10^{-6} \text{ at 95\% confidence}}$$

the big-bang nucleosynthesis (BBN) limit

The BBN limit is:

$$\int_0^{\infty} \Omega_{\mathrm{GW}}(f) d(\log f) < 1.5 \times 10^{-5}$$

Our limit is:

$$\int_{41.5\ \mathrm{Hz}}^{169.25\ \mathrm{Hz}} \Omega_{\mathrm{GW}}(f) d(\log f) < 9.7 \times 10^{-6}$$

Did we beat it?