

LASER INTERFEROMETER GRAVITATIONAL WAVE OBSERVATORY
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Time Domain Modal Model in End-to-End simulation package
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1 ABSTRACT

In the End to End simulation program for LIGO uses the time domain modal model to simulate the spatial and time evolution of the field in cavities. This note summaries the formulas used in the e2e package. Also included are explicit calculations of fields in a simple cavity to understand how the fields with limited number of spatial modes behave, and to demonstrate model validation methods which are used to validate the e2e code.

2 KEYWORDS

time domain modal model e2e End-to-End simulation

3 TIME DOMAIN MODAL MODEL

3.1. General Formula

In the time domain modal model, a freely propagating field, $E(x,y,z,t)$, in a vacuum is expressed using a set of Hermite-Gaussian functions in the following way.

$$E(x, y, z, t) = \exp(i\omega t) \cdot E(x, y, z) \quad (1)$$

$$E(x, y, z) = \sum \bar{a}_{mn} \cdot Prop(z, \bar{z}) \cdot U_{mn}(x, y, z) \quad (2)$$

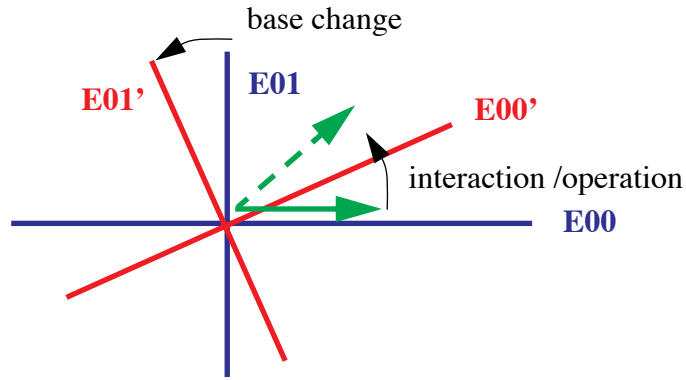
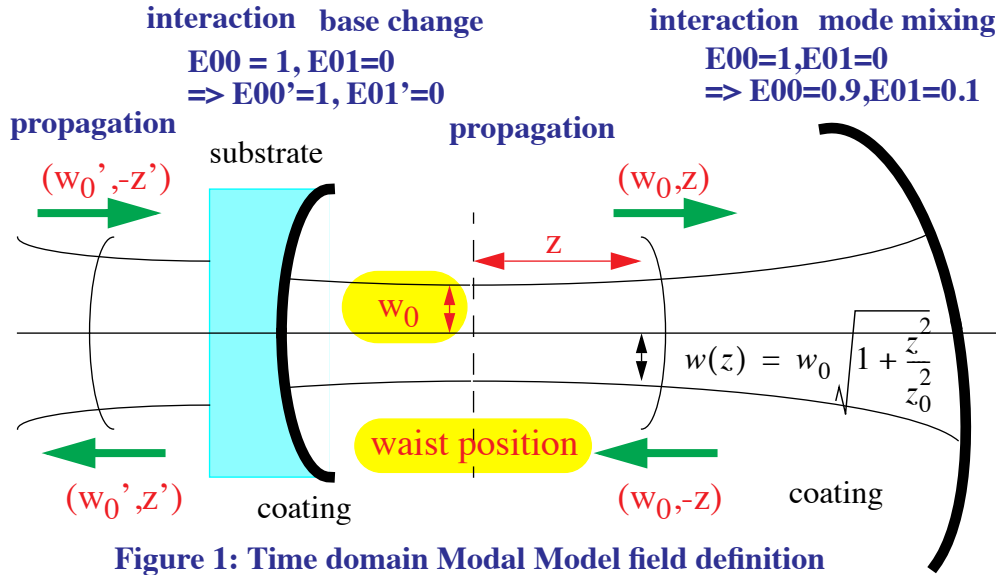
$$Prop(z, \bar{z}) = \exp[-ik \cdot (z - \bar{z})] \cdot \exp[i(m + n + 1)(\eta_{00}(z) - \eta_{00}(\bar{z}))] \quad (3)$$

$$U_{mn}(x, y, z) = u_m(x, z) \cdot u_n(y, z) \quad (4)$$

$$u_m(x, z) = \left(\frac{2}{\pi}\right)^{\frac{1}{4}} \cdot \frac{1}{\sqrt{2^m m! w(z)}} \cdot H_m\left(\frac{\sqrt{2}x}{w(z)}\right) \cdot \exp\left[-x^2\left(\frac{1}{w(z)^2} + \frac{ik}{2R(z)}\right)\right] \quad (5)$$

$$\int dx u_m^*(x, z) \cdot u_n(x, z) = \delta_{mn} \quad (6)$$

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Various field properties are defined in Appendix 1. The base of a Hermite-Gaussian function can be characterized by the waist size, w_0 , and the waist position. In the expression about, the origin of the z coordinate is chosen at the waist position, and the direction of the field propagation is chose to be the positive direction.

Because the Hermite-Gaussian functions, Eq. (5), provides a complete set in the (x,y) space and the field expression, Eq. (1), is a solution of the paraxial approximation of the Maxwell's equation, one can choose any Hermite-Gaussian base, and once chosen, the field is fully described by the coefficients \bar{a}_{mn} and \bar{z} .

3.2. Propagation

An initial condition can be specified by a set of amplitudes, a_{mn} , of a chosen set of Hermite-Gaussian functions at a given time \bar{t} at a given location \bar{z} . Then the field propagation can be calculated by Eq. (3), i.e., when an initial condition is given and the amplitudes of all Hermite-Gaussian modes are calculated, the propagation of fields, or field amplitudes at location z , can be

calculated by multiplying the phase due to the normal wave oscillation, $k(z-\bar{z})$, and the Gouy phase change, $(m+n+1)(\eta_{00}(z)-\eta_{00}(\bar{z}))$.

The simulation of e^{2e} traces the change of coefficients a_{mn} . When a field moves, Eq. (3) is applied to update the coefficients and, when there is any interaction, the coefficient is updated using the matrix operation explained in the next subsection.

3.3. Operation

A operation on a field, like changing the propagation direction or displacing the waist position, can be represented by a mode decomposition matrix, $M_{mn, m'n'}^{Op}$, which converts one set of coefficients a_{mn} to another.

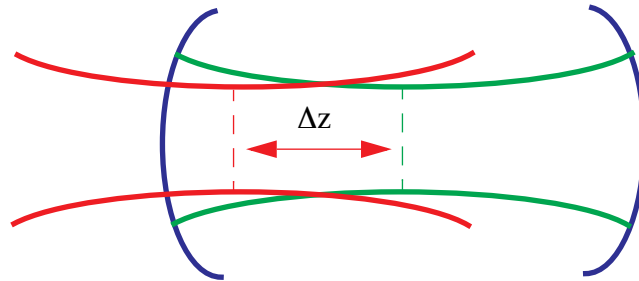
$$\begin{aligned} Op[E(x, y, z)]_{z=\bar{z}} &= \sum_{mn} Op[a_{mn} \cdot Prop(z, \bar{z}) \cdot U_{mn}(x, y, z)]_{z=\bar{z}} \\ &= \sum_{mn} a'_{mn} \cdot U_{mn}(x, y, \bar{z}) \end{aligned} \quad (7)$$

$$\begin{aligned} a'_{mn} &= \int dx dy U_{mn}^*(x, y, \bar{z}) \cdot Op[E(x, y, z)]_{z=\bar{z}} \\ &= \sum_{m'n'} a_{m'n'} \int dx dy U_{m'n'}^*(x, y, \bar{z}) \cdot Op[Prop(z, \bar{z}) \cdot U_{mn}(x, y, z)]_{z=\bar{z}} \\ &= \sum_{m'n'} \bar{a}_{m'n'} \cdot M_{mn, m'n'}^{Op} \end{aligned} \quad (8)$$

In these equations, $[xxx]_{z=\bar{z}}$ means that the value of z is evaluated at \bar{z} after the operation is applied. An example is an operation to rotate the field direction discussed in the following section.

One important point to note is that the mode decomposition matrix is calculated at a given location. The matrix relates a set of amplitudes calculated at a position to another set of amplitudes at the same position. Then the propagator defined in Eq. (3) is used to calculate the field at different locations.

One explicit example will be in order. When the waist position of a field is shifted by Δz , the new field can be expressed by using the original field by shifting the z coordinate. A more general discussion is given below, but the following is an explicit calculation up to the first order effect.



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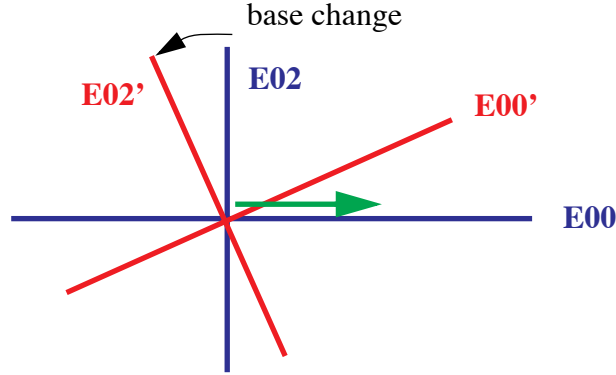
(9)

$$u_0(x, \bar{z} + \Delta z) = \exp[i\Delta\eta] \cdot [a_0 \cdot u_0(x, \bar{z}) + a_2 \cdot u_2(x, \bar{z})]$$

$$\Delta\eta = \eta(\bar{z} + \Delta z) - \eta(\bar{z})$$

$$a_0 = \frac{1}{\sqrt{1 - i\frac{1}{2}\frac{\Delta z}{z_0}}} \quad a_2 = -i\frac{1}{2\sqrt{2}} \exp[i2\eta(\bar{z})] \frac{\Delta z}{z_0} \quad (10)$$

(11)

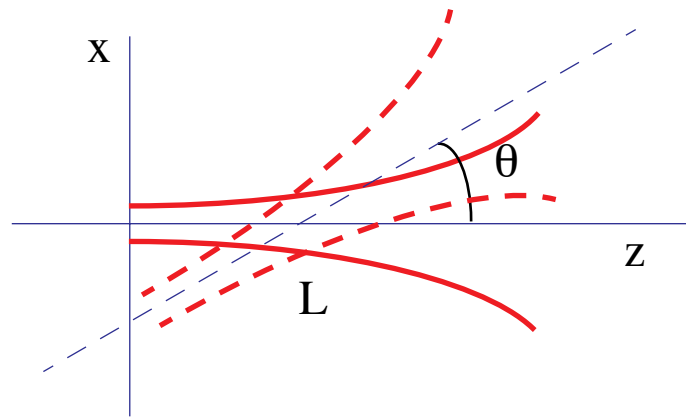


Given these two coefficients, the field at other location z can be calculated using Eq. (3).

4 BASIC OPERATIONS

4.1. Rotation Operator

4.1.1. Rotate around $(0,0,L)$ by θ



$$Op[E(x, y, z)] = E(x \cdot \cos\theta - (z - L) \cdot \sin\theta, y, L + x \cdot \sin\theta + (z - L) \cdot \cos\theta) \quad (12)$$

The size of θ of interest is $\sim 1 \mu$ rad, and the matrix is calculated by expanding in $\theta/\Theta/\cos(\eta)$, where Θ is the divergence angle which is $\sim 10 \mu$ rad in LIGO core optics cavities.

4.1.2. Mode decomposition matrix

$$\begin{aligned}
 M_{mm'}^{Rot} &= \int dx u_m^*(x, L) Rot[u_{m'}(x, L)] \\
 &= i^{-|\Delta m|} \frac{\sqrt{m!m'}}{\bar{m}! \cdot |\Delta m|!} e^{-\frac{\hat{\theta}^2}{2}} \hat{\theta}^{|\Delta m|} g(m, m', \hat{\theta})
 \end{aligned} \tag{13}$$

$$\begin{aligned}
 \Delta m &= m' - m \\
 \bar{m} &= \min(m, m')
 \end{aligned}$$

$$\hat{\theta} = \frac{w(L)}{w_0} \cdot \frac{\theta}{\Theta} = \frac{\theta}{\Theta} / \cos(\eta_{00}) = \sqrt{1 + \left(\frac{L}{z_0}\right)^2} \cdot \frac{\theta}{w_0/z_0} = \frac{\sqrt{z_0^2 + L^2}}{w_0} \cdot \theta \tag{14}$$

$$g(m, m', x) = \bar{m}! \cdot |\Delta m|! \cdot \sum_{r=0}^{\bar{m}} \frac{(-1)^r}{r!(\bar{m}-r)! (|\Delta m|+r)!} x^{2r} \tag{15}$$

$$\begin{aligned}
 \Delta m &= m' - m \\
 \bar{m} &= \min(m, m')
 \end{aligned}$$

When $m \geq m'$,

$$\begin{aligned}
 \frac{1}{\bar{m}! \cdot |\Delta m|!} g(m, m', x) &= \sum_{r=0}^{m'} \frac{(-1)^r}{r!(m'-r)!((m-m')+r)!} x^{2r} \\
 &= (-1)^{m'} \sum_{r=0}^{m'} \frac{(-1)^r}{r!(m'-r)!(m-r)!} x^{2(m'-r)}
 \end{aligned} \tag{16}$$

When keeping only up to the second order in θ , this expression becomes in the following simpler form. (Ref[3])

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$$\begin{aligned}
Op[u_m] \cdot \exp\left(\frac{\hat{\theta}^2}{2}\right) &\sim (1 - m \cdot \hat{\theta}^2)u_m \\
&\quad -i\hat{\theta}(h_{m-1}\sqrt{m} \cdot u_{m-1} + \sqrt{m+1} \cdot u_{m+1}) \\
&\quad -\frac{\hat{\theta}^2}{2}(h_{m-2}\sqrt{m(m-1)} \cdot u_{m-2} + \sqrt{(m+1)(m+2)} \cdot u_{m+2}) \tag{17}
\end{aligned}$$

$$h_i = \begin{cases} 0 & (i < 0) \\ 1 & (i \geq 0) \end{cases}$$

4.1.3. Lower order elements

Table 1: $M_{mm'}^{Rot}$ excluding $\exp\left(-\frac{\hat{\theta}^2}{2}\right)$

$m \setminus m'$	0	1	2	3
0	1	$-i\hat{\theta}$	$-\frac{\hat{\theta}^2}{\sqrt{2}}$	$i\frac{\hat{\theta}^3}{\sqrt{6}}$
1	$-i\hat{\theta}$	$1-\hat{\theta}^2$	$-\sqrt{2}i\hat{\theta}\left(1-\frac{\hat{\theta}^2}{2}\right)$	$-\sqrt{\frac{3}{2}}\hat{\theta}^2\left(1-\frac{\hat{\theta}^2}{3}\right)$
2	$-\frac{\hat{\theta}^2}{\sqrt{2}}$	$-\sqrt{2}i\hat{\theta}\left(1-\frac{\hat{\theta}^2}{2}\right)$	$1-2\hat{\theta}^2+\frac{\hat{\theta}^4}{2}$	$-i\sqrt{3}\hat{\theta}\left(1-\hat{\theta}^2+\frac{\hat{\theta}^4}{6}\right)$
3	$i\frac{\hat{\theta}^3}{\sqrt{6}}$	$-\sqrt{\frac{3}{2}}\hat{\theta}^2\left(1-\frac{\hat{\theta}^2}{3}\right)$	$-i\sqrt{3}\hat{\theta}\left(1-\hat{\theta}^2+\frac{\hat{\theta}^4}{6}\right)$	$1-3\hat{\theta}^2+\frac{3\hat{\theta}^4}{2}-\frac{\hat{\theta}^6}{6}$

4.1.4. Sanity check

$$\sum_{m3} M_{m1m3}^{Rot}(-\hat{\theta}) \cdot M_{m3m2}^{Rot}(\hat{\theta}) = \delta_{m1m2} \tag{18}$$

4.1.5. Intuitive meaning of Reflection operator matrix

Simple case reflection coupling from TEM00 mode to TEM01 and TEM10 mode.

$$\begin{aligned}
R_{00 \rightarrow 10}^{Ref-back}(\theta_{yaw}) &= -i2\theta_{yaw} & R_{00 \rightarrow 10}^{Ref-front}(\theta_{yaw}) &= -i2\theta_{yaw} \\
R_{00 \rightarrow 01}^{Ref-back}(\theta_{pitch}) &= -i2\theta_{pitch} & R_{00 \rightarrow 10}^{Ref-front}(\theta_{pitch}) &= i2\theta_{pitch}
\end{aligned} \tag{19}$$

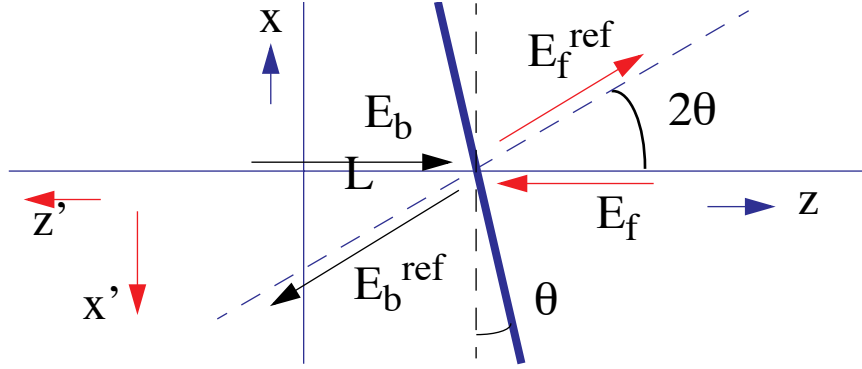


Figure 2: Reflection by θ_y or yaw from front and back

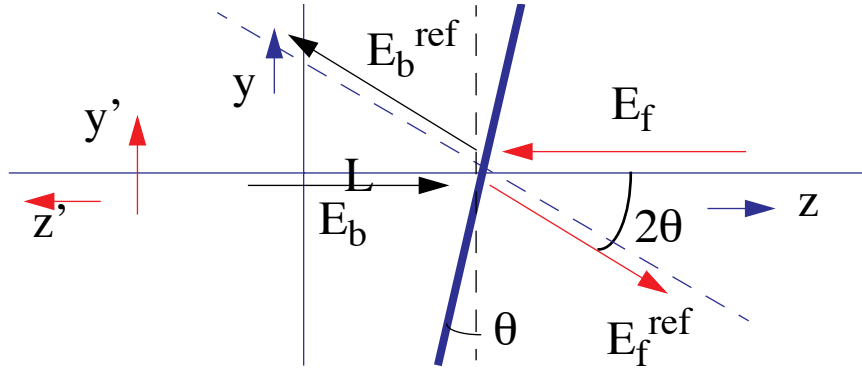


Figure 3: Reflection by θ_x or pitch from front and back

The meaning of the reflection matrix $-i2\theta$ is as follows. In the following calculation, the waist position is assumed to be at the reflection point, and $z \ll z_0$.

$$\begin{aligned}
E_{ref} &= HG_0 - i2\hat{\theta}HG_1\left(\frac{\sqrt{2}x}{w}\right)Exp[i\eta_{00}] \\
&= C \cdot Exp\left[-\frac{x^2}{w^2}\right] \left(1 - i2\frac{\theta}{\Theta} \frac{1}{\sqrt{2}} 2\frac{\sqrt{2}x}{w} Exp[i\eta_{00}]\right) \\
&= C \cdot Exp\left[-\frac{x^2}{w^2}\right] \left(1 + 4\frac{\theta x z}{w^2} - i4\frac{\theta x}{\Theta w}\right)
\end{aligned} \tag{20}$$

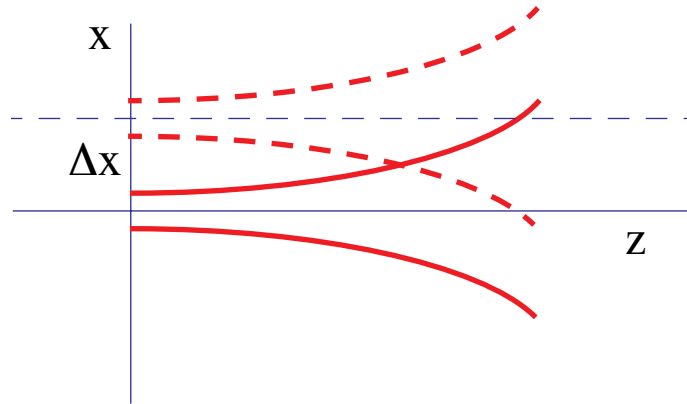
Here, the Gouy phase is approximated by z/z_0 , and only the linear term on θ is kept. The power along the z direction is calculated as follows where only the lowest order of θ is kept.

$$\begin{aligned}
 P_{ref} &= |E_{ref}|^2 \\
 &= |C|^2 \cdot \text{Exp}\left[-\frac{2x^2}{w^2}\right] \left(1 + 8\frac{\theta xz}{w^2}\right) \\
 &= |C|^2 \cdot \text{Exp}\left[-\frac{2x^2}{w^2} + 8\frac{\theta xz}{w^2}\right] \\
 &= |C(z)|^2 \cdot \text{Exp}\left[-\frac{2}{w^2}(x - 2\theta z)^2\right]
 \end{aligned} \tag{21}$$

So, the reflected field has the power maximum along the line $x=2\theta z$, as is naively expected.

4.2. Shift Operator

4.2.1. Shift perpendicular to z axis



$$Op[E(x, y, z)] = E(x - \Delta x, y, z) \tag{22}$$

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4.2.2. Mode decomposition matrix

$$\begin{aligned}
 M_{mm'}^{shift} &= \int dx u_m^*(x, L) Shift[u_{m'}(x, L)] \\
 &= (-1)^{m + \max(m, m')} \frac{\sqrt{m!m'}}{\bar{m}! \cdot |\Delta m|!} \cdot e^{-\frac{\hat{\Delta}^2}{2}} \hat{\Delta}^{|\Delta m|} g(m, m', \hat{\Delta})
 \end{aligned} \tag{23}$$

$$\begin{aligned}
 \Delta m &= m' - m \\
 \bar{m} &= \min(m, m')
 \end{aligned}$$

$$\hat{\Delta} = \frac{\Delta x}{w_0} \tag{24}$$

When keeping only up to the second order in Δ , this expression becomes in the following simpler form. (Ref[3])

$$\begin{aligned}
 Op[u_m] \cdot \exp\left(\frac{\hat{\Delta}^2}{2}\right) &\sim (1 - m \cdot \hat{\Delta}^2) u_m \\
 &\quad - \hat{\Delta} (h_{m-1} \sqrt{m} \cdot u_{m-1} - \sqrt{m+1} \cdot u_{m+1}) \\
 &\quad + \frac{\hat{\Delta}^2}{2} (h_{m-2} \sqrt{m(m-1)} \cdot u_{m-2} + \sqrt{(m+1)(m+2)} \cdot u_{m+2})
 \end{aligned} \tag{25}$$

$$h_i = \begin{cases} 0 & (i < 0) \\ 1 & (i \geq 0) \end{cases}$$

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4.2.3. Lower order elements

Table 2: $M_{mm'}^{Shift}$ excluding $\exp\left(-\frac{\hat{\Delta}^2}{2}\right)$

$m \setminus m'$	0	1	2	3
0	1	$-\hat{\Delta}$	$\frac{\hat{\Delta}^2}{\sqrt{2}}$	$-\frac{\hat{\Delta}^3}{\sqrt{6}}$
1	$\hat{\Delta}$	$1-\hat{\Delta}^2$	$-\sqrt{2}\hat{\Delta}\left(1-\frac{\hat{\Delta}^2}{2}\right)$	$\sqrt{\frac{3}{2}}\hat{\Delta}^2\left(1-\frac{\hat{\Delta}^2}{3}\right)$
2	$\frac{\hat{\Delta}^2}{\sqrt{2}}$	$\sqrt{2}\hat{\Delta}\left(1-\frac{\hat{\Delta}^2}{2}\right)$	$1-2\hat{\Delta}^2+\frac{\hat{\Delta}^4}{2}$	$-\sqrt{3}\hat{\Delta}\left(1-\hat{\Delta}^2+\frac{\hat{\Delta}^4}{6}\right)$
3	$\frac{\hat{\Delta}^3}{\sqrt{6}}$	$\sqrt{\frac{3}{2}}\hat{\Delta}^2\left(1-\frac{\hat{\Delta}^2}{3}\right)$	$\sqrt{3}\hat{\Delta}\left(1-\hat{\Delta}^2+\frac{\hat{\Delta}^4}{6}\right)$	$1-3\hat{\Delta}^2+\frac{3\hat{\Delta}^4}{2}-\frac{\hat{\Delta}^6}{6}$

4.2.4. Sanity check

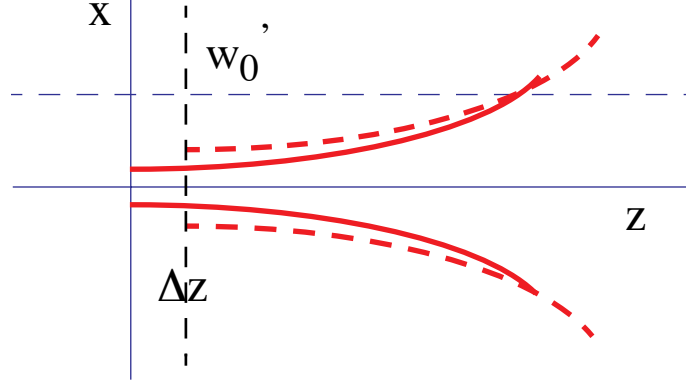
$$\sum_{m3} M_{m1m3}^{shift}(-\hat{\Delta}) \cdot M_{m3m2}^{shift}(\hat{\Delta}) = \delta_{m1m2} \quad (26)$$

4.3. Base Change

4.3.1. Change of base of Hermite Gaussian function

This is an operation to express fields based on one base by by another base. One example usage is as follows. When there is a field which does not mode match with a cavity it is going into. Then this base change operation can be used to express the field using the eigen state of the cavity. Second example is a reflection of a field by a mirror whose curvature does not match with that of the field. This is handled in the next section, but it is a special case of this matrix.

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$$Op[E(x, y, z, w_0)] = E(x, y, z', w_0') \quad (27)$$

4.3.2. Mode decomposition matrix

$$\begin{aligned}
 M_{mm'}^{shift} &= \int dx u_m^*(x, z) Base[u_{m'}(x, z)] \\
 &= \sqrt{\beta \cdot \beta'} \sqrt{\frac{m! \cdot m'}{2^{m+m'}}} \gamma^{-\frac{1}{2}(m+m'+1)} \\
 &\quad \times \sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{r'=0}^{\lfloor \frac{m'}{2} \rfloor} (-1)^{r+r'} \beta^{m-2r} \beta'^{m'-2r'} \gamma^{r+r'} \\
 &\quad \times \frac{(m+m'-2r-2r')!}{r!r'!(m-2r)!(m'-2r')! \left(\frac{m+m'-2r-2r'}{2} \right)!}
 \end{aligned} \quad (28)$$

w and w' are waist sizes of each base in the plane of transformation, β and β' are functions of these values, and the γ is a parameter characterizing the curvature mismatch of two bases. The definitions of these variables are given in Eq. (29). When two bases become the same, γ , β and β' all become 1, or α , $\delta\beta$ and $\delta\beta'$ becomes 0. If a matrix element is expressed by a sum of terms proportional to $\alpha^{n_1} \delta\alpha^{n_2} \delta\beta^{n_3}$, then the order of magnitude (the smallest value of the sum of powers, i.e., $n_1+n_2+n_3$) of the matrix element is $\lfloor m-m'/2 \rfloor$.

$$w = w_0 \sqrt{1 + \frac{z^2}{z_0^2}}, \quad w' = w_0' \sqrt{1 + \frac{z'^2}{z_0'^2}}, \quad R = \frac{z^2 + z_0^2}{z}, \quad R' = \frac{z'^2 + z_0'^2}{z'}$$

$$\gamma = 1 + i\alpha, \quad \alpha = \frac{k}{2} \left(\frac{1}{R'} - \frac{1}{R} \right) \frac{w^2 w'^2}{w^2 + w'^2} \quad (29)$$

$$\beta = \sqrt{\frac{2}{1 + w^2/w'^2}} = 1 + \delta\beta, \quad \beta' = \sqrt{\frac{2}{1 + w'^2/w^2}} = 1 + \delta\beta', \quad \beta^2 + \beta'^2 = 2$$

The lowest order mode coupling can be expressed in the following simple form.

$$Op[u_m] \left/ \left(\sqrt{\beta \cdot \beta'} \gamma^{-m - \frac{1}{2}} \right) \right. \sim (1 + m \cdot (\delta\beta + \delta\beta')) u_m$$

$$- \frac{1}{2} \sqrt{\frac{(m+2)!}{m!}} (i\alpha - 2\delta\beta') \frac{u_{m+2}}{1 + i\alpha}$$

$$- \frac{1}{2} h_{m-2} \sqrt{\frac{m!}{(m-2)!}} (i\alpha - 2\delta\beta) (1 + i\alpha) u_{m-2}$$

$$h_i = \begin{cases} 0 & (i < 0) \\ 1 & (i \geq 0) \end{cases} \quad (30)$$

The explicit energy conservation in the lowest order is shown in the following expression.

$$|Amp(0 \rightarrow 0)|^2 = \frac{\beta\beta'}{\sqrt{1 + \alpha^2}}$$

$$|Amp(0 \rightarrow 2)|^2 = \frac{\beta\beta'}{\sqrt{1 + \alpha^2}} \frac{1}{2} \frac{\alpha^2 + 4\delta\beta'^2}{1 + \alpha^2} \quad (31)$$

$$|Amp(0 \rightarrow 0)|^2 + |Amp(0 \rightarrow 2)|^2 = 1 \text{ up to 2nd order of } \alpha \text{ and } \delta\beta$$

4.3.3. Lowest order elements

Table 3: $M_{mm'}^{Base}$ excluding $\sqrt{\beta \cdot \beta'} \gamma^{-\frac{1}{2}(m+m'+1)}$

$m \backslash m'$	0	1	2	3	4
0	1	0	$\frac{1}{\sqrt{2}}(-\gamma + \beta^2)$	0	$\sqrt{\frac{3}{8}}(\gamma - \beta^2)^2$
1	0	$\beta\beta'$	0	$\sqrt{\frac{3}{2}}\beta\beta'(-\gamma + \beta^2)$	0
2	$\frac{1}{\sqrt{2}}(-\gamma + \beta^2)$	0	$\frac{1}{2}(\gamma^2 - 2\gamma + 3\beta^2\beta'^2)$	0	$-\frac{\sqrt{3}}{4}(\gamma - \beta^2)$ $(\gamma^2 - 2\gamma + 5\beta^2\beta'^2)$
3	0	$\sqrt{\frac{3}{2}}\beta\beta'(-\gamma + \beta^2)$	0	$\frac{1}{2}\beta\beta'$ $(3\gamma^2 - 6\gamma + 5\beta^2\beta'^2)$	0
4	$\sqrt{\frac{3}{8}}(\gamma - \beta^2)^2$	0	$-\frac{\sqrt{3}}{4}(\gamma - \beta^2)$ $(\gamma^2 - 2\gamma + 5\beta^2\beta'^2)$	0	$\frac{3}{8}\gamma^4 - \frac{3}{2}\gamma^3$ $+ \frac{3}{4}\gamma^2(2 + 5\beta^2\beta'^2)$ $-\frac{15}{2}\gamma\beta^2\beta'^2 + \frac{35}{8}\beta^4\beta'^4$

The explicit expressions of low order terms are shown in Table 3. The matrix elements are expressed using γ , β and β' , all of which go to 1 when the two bases are the same. If matrix elements are shown using α , $\delta\beta$ and $\delta\beta'$, expressions become longer.

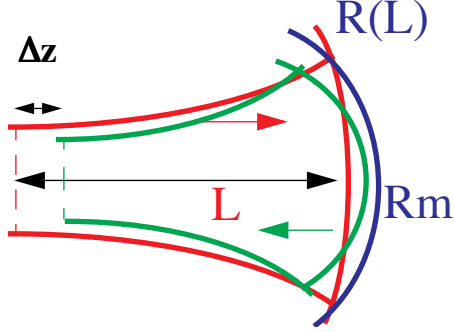
4.3.4. Sanity check

$$\sum_{m3} M_{m1m3}^{Base}(\alpha, w, w') \cdot M_{m3m2}^{Base}(-\alpha, w', w) = \delta_{m1m2} \quad (32)$$

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4.4. Reflection by a mirror with non-matching curvature

4.4.1. Basic



This is a special case of the Hermite-Gaussian base change. The reflection by a mirror is given by Eq. (33) and Eq. (34).

$$z' = \frac{L \left(1 - \frac{2R(L)}{R_m} \right)}{1 + 4 \frac{L}{R_m} \left(\frac{R(L)}{R_m} - 1 \right)} \quad (33)$$

$$z_0' = \frac{z_0}{1 + 4 \frac{L}{R_m} \left(\frac{R(L)}{R_m} - 1 \right)} \quad (34)$$

L is the location of the mirror (distance from the waist of the field), and R_m and $R(L)$ are the curvatures of the mirror and the incoming field on the mirror. If the mirror curvature matches with that of the beam, $z' = -L$ and $z_0' = z_0$, i.e., the beam is just reflected back.

The curvature of the reflected field, R_{ref} , is given by Eq. (35). When the mirror curvature matches with that of the incoming field, $R_{ref} = -R(L)$, i.e., expanding (shrinking) field changes to shrinking (expanding) field.

$$\frac{1}{R(L)} - \frac{1}{R_{ref}} = \frac{2}{R_m} \quad (35)$$

$$\left(\frac{1}{R_{in}} - \frac{1}{R_{out}} = \frac{1}{f} \right)$$

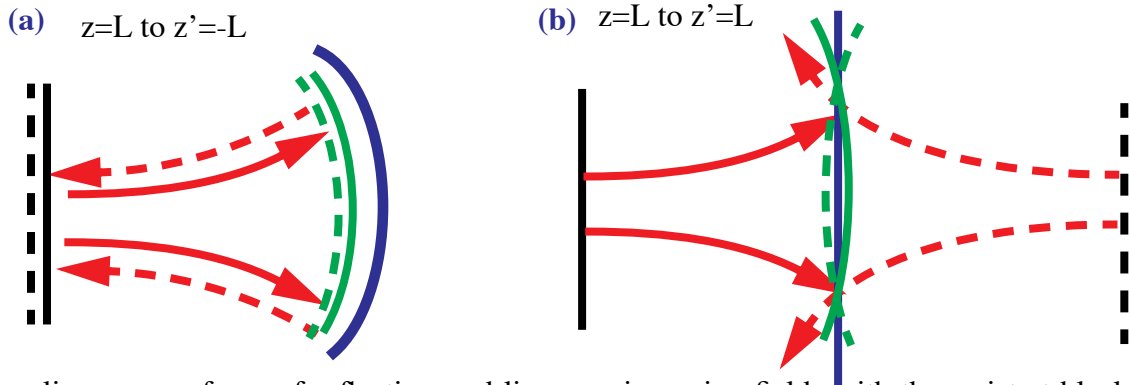
The beam size on the mirror remains the same on reflection.

In the following calculation, the propagation direction of the reflected field is reversed

$$Op[E(x, y, z)] = E(x, y, z'; z_0') \quad (36)$$

There are two possibilities of reflection. One is that the convergence status changes (Figure 4-a) and another is that the status does not change (Figure 4-b). E.g., when a field is reflected by a

mirror with a curvature matching to the field wave front, a converging field becomes diverging, and a diverging field becomes converging. On the other hand, when a field is reflected by an almost flat mirror, a converging field is still converging and a diverging field diverging.



Blue lines are surfaces of reflection, red lines are incoming fields with the waist at black lines and wave fronts shown by green, and red dashed lines are reflected fields with respective waist and wave front shown by dashed lines.

Figure 4: Reflection (a) with convergence condition change and (b) without.

For two special cases, the reflected field can be expressed using the incoming field. One is when the mirror curvature matches with the wave front of the incoming field, i.e., $R_m = R(L)$. With this condition, $z'=-L$ and $z_0' = z_0$. The other case is when the mirror curvature is infinity, i.e., $1/R_m = 0$. With this condition, $z'=L$ and $z_0' = z_0$.

When the mirror surface curvature is different from either of these cases, one needs to use perturbative expansion.

$$\frac{z' + L}{z_0'} = \frac{2L \left(\frac{R(L)}{R_m} - 1 \right) \left(1 - \frac{2L}{R_m} \right)}{z_0} \quad (37)$$

$$\frac{z' - L}{z_0'} = \frac{-2L \left(2L \left(\frac{R(L)}{R_m} - 1 \right) + R(L) \right)}{z_0} \quad (38)$$

Eq. (37) becomes small if R_m is close to $R(L)$, while Eq. (38) becomes small when R_m becomes large. I.e., when the mirror curvature is close to the wave front curvature, the new waist position is located at around $-L$, while for the reflection by an almost flat mirror, the waist is located at around L .

When the new z' has the opposite sign of L , the convergence condition changes. So the choice of the base can be done by calculating the sign of $L \cdot z'$ using Eq. (33). If $z' \cdot L \geq 0$, then Eq. (38) should be used, otherwise, Eq. (37) should be used, with an appropriate Rayleigh range.

In the following section, $R(L)$ means the curvature of the incoming field at L , and $R_{out}(L)$ is the curvature of the reflected field of the mode base. In short, $R_{out}(L) = -R(L)$ for case a) and $R_{out}(L) = R(L)$ for case b).

4.4.2. Reflection by a tilted mirror

The derivation here is based on the ABCD matrix for a reflection by a mirror. The ABCD matrix of the reflection of a field with an arbitrary incident angle is given by

$$\begin{bmatrix} 1 & 0 \\ -2/(R_m \cdot \cos(\theta_{inc})) & 1 \end{bmatrix} \quad (39)$$

so the following formula can be used for an arbitrary incident field by using Eq. (40) in place of R_m .

$$R_m \cdot \cos(\theta_{inc}) \quad (40)$$

4.4.3. Effect of a lens

When a field goes through a lens, the base, waist position and size, is affected. The ABCD matrix for a lens with a focal length f ($f > 0$ for converging lens) is

$$\begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \quad (41)$$

which is the same as that for the reflection by a mirror, Eq. (39), when R_m is replaced by $2f$. From this relation, the matrix for the lens is given by the following formula by replacing R_m by $2f$. The argument for the choice of outgoing beam divergence applies in the same way.

4.4.4. Mode decomposition matrix of the reflection by curvature mismatched mirror

$$\begin{aligned} M_{mm'}^{Mis} &= \int dx u_m^*(x, L) Ref[u_{m'}(x, L)] \\ &= \frac{1}{\sqrt{\pi 2^{m+m'} m! m'!}} \int d\xi H_m(\xi) H_{m'}(\xi) Exp[-\xi^2 (1 + i\alpha)] \\ \alpha &= \frac{kw(L)^2}{4} \left(\frac{1}{R_{ref}} - \frac{1}{R_{out}} \right) = \frac{L}{2z_0} R(L) \left(\frac{1}{R_{ref}} - \frac{1}{R_{out}} \right) \end{aligned} \quad (42)$$

For the case a), i.e., reflection by a almost curvature matched mirror, α is given by the following expression by eliminating R_{ref} using Eq. (35) and $R_{out} = -R(L)$,

$$\alpha = \frac{L}{z_0} \left(1 - \frac{R(L)}{R_m} \right) \quad (43)$$

while for the case b), i.e., reflection by an almost flat mirror, α is given as follows using Eq. (35) and $R_{out} = R_m$:

$$\alpha = -\frac{L R(L)}{z_0 R_m} \quad (44)$$

Eq. (42) is the Vinet's formula, except the sign of "i" due to the different convention of $\exp(-ikz)$. As is derived in Appendix 3, the curvature mismatch matrix can be written in the following way. The matrix is symmetric in m and m' , i.e., $M_{mm'} = M_{m'm}$, and the following expression is the one for $m \geq m'$.

$$M_{mm'}^{Mis} = \frac{1 + (-1)^{m+m'}}{2} \frac{(i\alpha)^{n_{min}}}{(1+i\alpha)^{\frac{m+m'+1}{2}}} \times \sum_{n=0}^{n_{max}} cm(m, m', n) \cdot \alpha^{2n} \quad (45)$$

$$n_{min} = \frac{m-m'}{2}, n_{max} = \frac{\left[\frac{m}{2}\right] + \left[\frac{m'}{2}\right] - n_{min}}{2} \quad [m \geq m']$$

and the explicit form of the function $cm(m, m', n)$ is given by the following formula.

$$cm(m, m', n) = (-1)^n \sqrt{\frac{m!m'!}{2^{m+m'}(n_{min}+2n)!}} \times \sum_{r=r_{min}}^{r_{max}} (-1)^r \frac{(m+m'-2r)!r!}{\left(\frac{m+m'}{2}-r\right)!(r-r_{min})!} \sum_{s=0}^{s_{max}} \frac{1}{(m-2u)!(m'-2v)!u!v!} \quad (46)$$

$$r_{min} = \frac{m-m'}{2} + 2n, \quad r_{max} = \left[\frac{m}{2}\right] + \left[\frac{m'}{2}\right] \quad [m \geq m']$$

where s_{max} , u and v are functions of m, m', r and s given as follows.

$$\begin{aligned} r_1 &= \left[\frac{m'}{2}\right], \quad r_2 = \left[\frac{m}{2}\right] \quad [m \geq m'] \\ 0 \leq r \leq r_1 &\Rightarrow s_{max} = r, \quad u = s, \quad v = r - s \\ r_1 < r \leq r_2 &\Rightarrow s_{max} = r_1, \quad u = r + s - r_1, \quad v = r_1 - s \\ r_2 < r \leq r_{max} &\Rightarrow s_{max} = r_{max} - r, \quad u = r + s - r_1, \quad v = r_1 - s \end{aligned} \quad (47)$$

The $Z_{mm'}(\alpha)$ by Vinet is related to our formulation as follows:

$$Z_{mm'}(-\alpha) = \sqrt{2^{m+m'}} m!m'!(i\alpha)^{n_{min}} \sum_{n=0}^{n_{max}} cm(m, m', n) \cdot \alpha^{2n} \quad (48)$$

4.4.5. Lower order elements

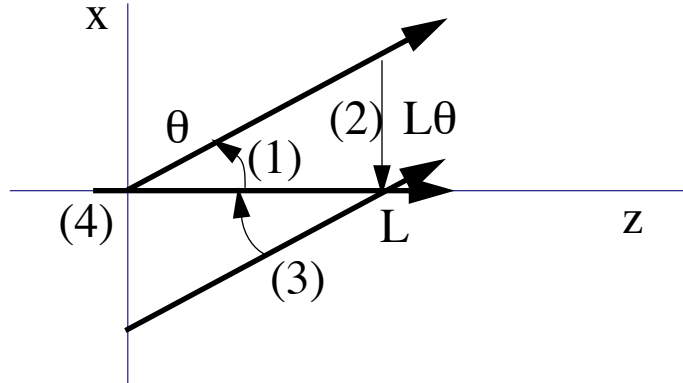
Table 4: $M_{mm'}^{Mis}$ excluding $(1 + i\alpha)^{\frac{m+m'+1}{2}}$

m\m'	0	1	2	3	4	5
0	1	0	$-i\frac{\alpha}{\sqrt{2}}$	0	$-\sqrt{\frac{3}{8}}\alpha^2$	0
1	0	1	0	$-i\sqrt{\frac{3}{2}}\alpha$	0	$-\sqrt{\frac{15}{8}}\alpha^2$
2	$-i\frac{\alpha}{\sqrt{2}}$	0	$1 - \frac{\alpha^2}{2}$	0	$-i\sqrt{3}\alpha\left(1 - \frac{\alpha^2}{4}\right)$	0
3	0	$-i\sqrt{\frac{3}{2}}\alpha$	0	$1 - \frac{3}{2}\alpha^2$	0	$-i\sqrt{5}\alpha\left(1 - \frac{3}{4}\alpha^2\right)$
4	$-\sqrt{\frac{3}{8}}\alpha^2$	0	$-i\sqrt{3}\alpha\left(1 - \frac{\alpha^2}{4}\right)$	0	$1 - 3\alpha^2 + \frac{3}{8}\alpha^4$	0
5	0	$-\sqrt{\frac{15}{8}}\alpha^2$	0	$-i\sqrt{5}\alpha\left(1 - \frac{3}{4}\alpha^2\right)$	0	$1 - 5\alpha^2 + \frac{15}{8}\alpha^4$

This is special case of Table 3. With the following substitution, they match each other : $\gamma = 1 - i\alpha$, $\beta = 1$, $\beta' = 1$. This this case, the beam spot size does not change on reflection, so $w=w'$, and $\beta = 1$ and $\beta' = 1$.

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4.5. Another sanity check



- (1) Rotate at (0,0) by θ , with waist at (0,0)
- (2) Shift by $-L\theta$ toward $-x$ direction, with waist at $(-L\theta, 0)$
- (3) Rotate at (0,L) by $-\theta$, with waist at $(0, -L\theta^2/2)$
- (4) Move the waist position by $L\theta^2/2$ toward $+z$ direction to move waist to (0,0)

$$M^{Base}\left(\Delta z = \frac{L\theta^2}{2}\right)M^{Rot}(L, -\theta)M^{Shift}(-L\theta)M^{Rot}(0, \theta) = I \quad (49)$$

5 NULL TEST

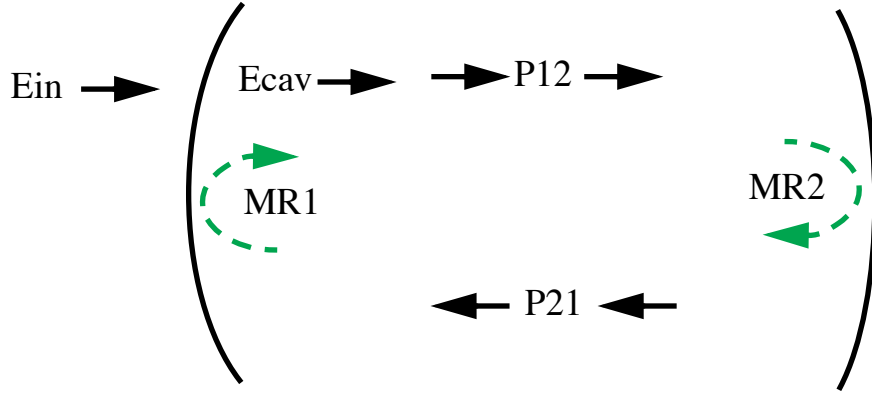
Because the same phenomena can be calculated using different set of Hermite-Gaussians, there are several cases which can be used to validate calculating by comparing results using different set of Hermite-Gaussians. In this section, two cases are studied. One is a field in a flat-curved FP cavity with the end mirror tilted, and the other is a field whose waist position is slightly shifted from that of an resonant mode of a FP cavity.

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5.1. Tilt

5.2. Waist shift

How many modes are necessary?



$$E_{cav} = \frac{t_1}{1 - MR1 \cdot P21 \cdot MR2 \cdot P12} E_{in} \quad (50)$$

$$E_{00}^{cav} = \frac{t_1 \cdot E_{00}^{in}}{(1 - r_1 \cdot r_2 \cdot \exp(i \cdot \varphi)) \cdot (1 + c_1) + c_2} \quad (51)$$

With $n+m \leq 2$, $c_1 \sim \text{real}$, $O(\alpha^2)$, $c_2 = \text{real}$, $O(\alpha^4)$

With $n+m \leq 4$, $c_1 \sim \text{real}$, $O(\alpha^2)$, $c_2 = \text{real}$, $O(\alpha^5)$

$$\alpha = \frac{k \cdot w(z)^2}{2 \cdot R(z)} \left(1 - \frac{R(z)}{R_m} \right) = \frac{\Delta z}{z_0} \cos(2\eta) \quad (52)$$

$$E_{00}^{cav} = \frac{t_1 \cdot E_{00}^{in}}{(1 - r_1 \cdot r_2)} \cdot \frac{1}{(1 + c_1) + \frac{c_2}{1 - r_1 \cdot r_2}} \quad (53)$$

For a high finesse cavity, like locked state LIGO arm, $F \sim 10k$,
4th order mode is necessary.

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6 PHYSICS QUANTITIES

Table 5: Physics parameters of interest

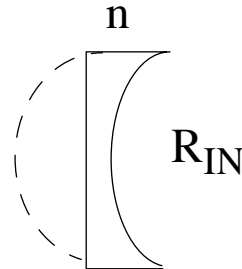
<i>name</i>	<i>value</i>	<i>comment</i>
LIGO ITM curvature	14.18 km	
LIGO ETM curvature	7.4 km	
LIGO Recycling curv	14.9 km	
LIGO ITM Trans / Loss	3% / 50 ppm	
LIGO ETM Trans / Loss	< 20ppm / 50 ppm	
LIGO Recycling T/L	3% / 50 ppm	

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APPENDIX 1 SUMMARY OF FIELD PROPERTIES

Table 6: field parameters

<i>name</i>	<i>expression</i>	<i>in COC (4k/2k)</i>	<i>in PSL/IOO</i>
waist size	w_0	3.51 / 3.13 cm	
Rayleigh length	$z_0 = \frac{\pi \cdot w_0^2}{\lambda} = \frac{k \cdot w_0^2}{2} = \frac{w_0}{\Theta}$ $= \sqrt{\frac{L(R_1 - L)(R_2 - L)(R_1 + R_2 - L)}{(R_1 + R_2 - 2L)^2}}$	3.63 / 2.89 km	
Distance to waist	$z_1 = \frac{L(R_2 - L)}{R_1 + R_2 - 2L}, z_2 = \frac{L(R_1 - L)}{R_1 + R_2 - 2L}$	1000, 3000 m / 614, 1386 m	
spot size	$w = w_0 \sqrt{1 + \left(\frac{z}{z_0}\right)^2}$	3.64, 4.55cm / 3.20, 3.47cm size @ITM,ETM	
curvature of phase front	$R(z) = \frac{z^2 + z_0^2}{z}$		
divergence angle	$\Theta = \frac{\lambda}{\pi \cdot w_0} = \frac{w_0}{z_0}$	9.7 / 10.8 μ r	
Gouy phase	$\eta(z) = \text{atan}\left(\frac{z}{z_0}\right) = \text{angle}\left(\frac{z_0 + iz}{ z_0 + iz }\right)$	-15.4, 39.6 deg -12.0, 25.6 deg η @ITM/ETM	
wave number	$k = \frac{2\pi}{\lambda} = \frac{2}{\Theta \cdot w_0} = \frac{2z_0}{w_0^2}$	5.91e6~YAG 0.513 ~24.5MHz 0.618 ~29.5MHz	

$$R_{\text{OUT}} = R_{\text{IN}} / n$$


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APPENDIX 2 HERMITE GAUSSIAN FUNCTIONS

$H_m(x)$ is the Hermite polynomial of order m . The following relations are used repeatedly in the calculations which follow:

$$\int_{-\infty}^{\infty} U_m^\dagger(x, z) U_n(x, z) dx = \delta_{mn} \quad (54)$$

$$2xH_m(x) = H_{m+1}(x) + 2mH_{m-1}(x) \quad (55)$$

$$\frac{d}{dx}H_m(x) = 2mH_{m-1}(x) \quad (56)$$

$$\int_{-\infty}^{\infty} U_m^\dagger(x, 0) \frac{H_i(\sqrt{2}x/w_0)}{H_k(\sqrt{2}x/w_0)} U_k(x, 0) dx = \sqrt{\frac{2^i i!}{2^k k!}} \delta_{mi} \quad (57)$$

Eq. (54) is the orthonormality condition; eqns. Eq. (55) and Eq. (56) are recursion relations to be used to derive Hermite polynomials of any order, beginning with $H_0(x) = 1$.

In two dimensions the Hermite-Gaussian modes are given by

$$U_{mn} = U_m(x, z)U_n(y, z)e^{-ikz} \quad (58)$$

The explicit forma of a few low order Hermite polynomials are:

$$H_0(x) = 1; H_1(x) = 2x; H_2(x) = -2 + 4x^2; H_3(x) = -12x + 8x^3; \quad (59)$$

A few examples of the distribution of the Hermite Gaussians are shown below.

The following formula is used in the modeling of the photo detector with simple shapes.

$$\begin{aligned} & \int_0^{\infty} H_n(x)H_m(x)\exp(-x^2) \\ &= \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{s=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^{r+s} \cdot \frac{n!m!2^{n+m-1}}{(2r)!!(2s)!!(n-2r)!(m-2s)!} \cdot \\ & \left(\begin{array}{l} \left(\frac{n+m-1}{2} - r - s\right)! \cdot 2^{-r-s} \text{ for odd } n+m \\ \frac{(n+m-2r-2s-1)!!}{\sqrt{2}^{n+m}} \sqrt{\pi} \text{ for even } n+m \end{array} \right. \end{aligned} \quad (60)$$

$$\text{Table 7: } HH[m, n] = \frac{1}{\sqrt{\pi}2^{m+n}m!n!} \int_0^{\infty} H_m(x)H_n(x)\exp(-x^2)$$

m \ n	0	1	2	3	4	5
0	$\frac{1}{2}$	$\frac{1}{\sqrt{2\pi}}$	0	$-\frac{1}{2\sqrt{3\pi}}$	0	$\frac{1}{4}\sqrt{\frac{3}{5\pi}}$
1	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{\pi}}$	0	$-\frac{1}{4\sqrt{3\pi}}$	0
2	0	$\frac{1}{2\sqrt{\pi}}$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{\frac{3}{2\pi}}$	0	$-\frac{1}{4}\sqrt{\frac{5}{6\pi}}$
3	$-\frac{1}{2\sqrt{3\pi}}$	0	$\frac{1}{2}\sqrt{\frac{3}{2\pi}}$	$\frac{1}{2}$	$\frac{3}{4\sqrt{2\pi}}$	0
4	0	$-\frac{1}{4\sqrt{3\pi}}$	0	$\frac{3}{4\sqrt{2\pi}}$	$\frac{1}{2}$	$\frac{3}{8}\sqrt{\frac{5}{2\pi}}$
5	$\frac{1}{4}\sqrt{\frac{3}{5\pi}}$	0	$-\frac{1}{4}\sqrt{\frac{5}{6\pi}}$	0	$\frac{3}{8}\sqrt{\frac{5}{2\pi}}$	$\frac{1}{2}$

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)\exp(-x^2) = \begin{cases} \sqrt{\pi}2^n n! & \text{for } n=m \\ 0 & \text{for } n \neq m \end{cases} \quad (61)$$

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$$\begin{aligned}
H_n(x) &= \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^r (2r-1)!! {}_n C_{2r} 2^{n-r} x^{n-2r} \\
&= (-1)^m \sum_{r=0}^m (-1)^r \frac{(2m)!}{(m-r)!(2r)!} (2x)^{2r} \quad (n=2m) \\
&= 2x(-1)^m \sum_{r=0}^m (-1)^r \frac{(2m+1)!}{(m-r)!(2r+1)!} (2x)^{2r} \quad (n=2m+1) \\
H_n(x+y) &= \sum_{r=0}^n H_{n-r}(x) 2^r {}_n C_r y^r \quad (62)
\end{aligned}$$

APPENDIX 3 CURVATURE MISMATCH MATRIX

to be filled

APPENDIX 4 STEADY STATE MULTIMODE FIELDS IN A FABRY-PEROIT CAVITY

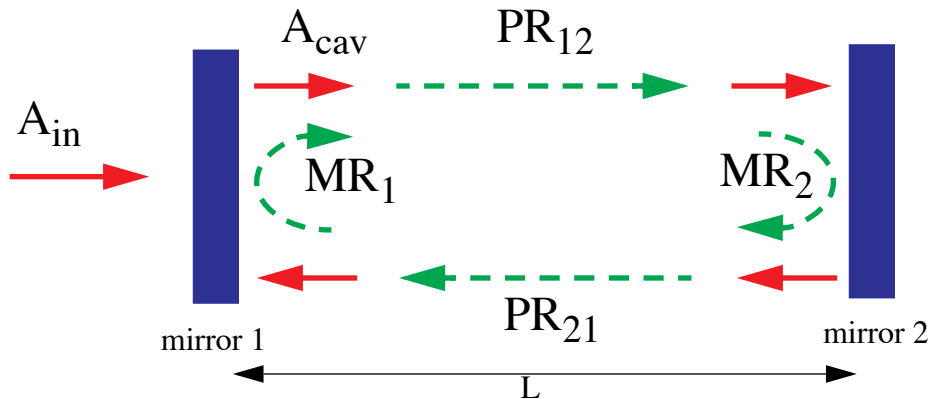


Figure 5: FP cavity schematics

Steady state fields in a Fabry-Pérot cavity can be calculated by solving the following consistency equation Eq. (63).

$$A_{cav} = t_1 \cdot A_{in} + r_1 \cdot r_2 \cdot MR_1 \cdot PR_{21} \cdot MR_2 \cdot PR_{12} \cdot A_{cav} \quad (63)$$

In this equation, A_{in} and A_{cav} are arrays of coefficients, a_{mn} in Eq. (2), of incoming and intra cavity fields.

$$A_x = \begin{bmatrix} a_{m_1 n_1} \\ a_{m_2 n_2} \\ \dots \end{bmatrix} \quad (64)$$

t_j and r_j are amplitude transmittance and reflectance of mirror j . PR_{ij} is a propagator from mirror i to mirror j , corresponding to Eq. (4), and MR_j is a mode decomposition matrix discussed in Section 4.

The propagator can be explicitly written as follows:

$$PR_{ij} = Exp[-ikL + i\Delta\eta_{ij}] \begin{bmatrix} Exp[i(m_1 + n_1)\Delta\eta_{ij}] & 0 & 0 \\ 0 & Exp[i(m_2 + n_2)\Delta\eta_{ij}] & 0 \\ 0 & 0 & \dots \end{bmatrix} \quad (65)$$

where L is the distance between the two mirrors and $\Delta\eta$ is the difference of the Gouy phases,

$$\Delta\eta_{ij} = \eta(z_j) - \eta(z_i) \quad (66)$$

When the field is propagating back from mirror 2 to mirror 1, the z axis direction is reversed, and the numerical value of $\Delta\eta_{21}$ is the same as $\Delta\eta_{12}$. Hereafter, $\Delta\eta$ is used to denote the change of gouy phase from mirror 1 to mirror 2, $\Delta\eta_{12}$.

The solution of this consistency equation is

$$A_{cav} = \frac{t_1 \cdot A_{in}}{1 - r_1 \cdot r_2 \cdot MR_1 \cdot PR_{21} \cdot MR_2 \cdot PR_{12}} \quad (67)$$

When the Hermite-Gaussian base is the eigenstate of the FP cavity, there is no coupling among modes on reflection, i.e., MR_i is diagonal. When the FP cavity is slightly off from this idealistic case, mode mixing occurs when the field is reflected by mirrors, i.e., MR_i is not diagonal.

A mode in the cavity, $A_{cav,mn}$, induced by a specific mode of the incoming field, $A_{in,0}$, can be rewritten in the following form. (The mode of the incoming field can be any. Here suffix 0 is used to clarify that $A_{in,0}$ refers to an amplitude of a certain mode.)

$$A_{cav,mn} = \frac{t_1 \cdot A_{in,0}}{1 - R_{12}(1 - \delta)} \quad (68)$$

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