Simple Calculations for Charge Noise for Advanced LIGO

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1 Abstract

I describe a few simple calculations for 3 ways in which charge on the Advanced LIGO test mass and on the nearby earthquake stops can cause displacement noise for the optic. The first noise mechanism is a force transmitted by a static charge distribution from a moving earthquake stop to a charge distribution on the isolated test mass. The second method is similar and describes how a static charge distribution interacts with bulk polarizability of an uncharged optic. The third mechanism is a variant of Rai's 1995 'Note on Electrostatics in the LIGO suspensions' (T060795). I have modified the calculation to assume that the charge moves in many small steps across the face of the optic, as might be the case if the optic has a leaky layer at the surface.

The result of the fixed charge distribution with a moving frame is that, if both the stop and the optic are charged, the charge density on the stop and the mass should be less than 1.4×10^{-6} C/m^2. This is roughly equivalent to a potential, at the middle of the tip of a lone earthquake stop, of 140 volts, with respect to infinity. For an uncharged optic, there is still considerable interaction with a charged stop, and the allowed charge density on the stop tip increases to approximately 1.6×10^{-6} C/m^2, with a proportional increase in the allowed potential to about 160 V.

2 Assumptions and Problem Definition

2.1 Allowed force on the optic

We define $F_{allowed\_10Hz}$ to be the allowed amplitude spectral density of force on the optic at 10 Hz. This is the total force in the beam direction (not the force at each stop). This is the force which moves the optic by 1/10 of the 10 Hz motion limit of 1e-19 m/rtHz, since we assume that charge noise is a 'technical' noise source. The mass, m is 40 kg. Thus,

$$F_{allowed\_10Hz} = (1/10) * m * (2\pi \cdot 10 \text{ Hz})^2 * 1 \times 10^{-19} = 1.58 \times 10^{-15} \text{ N}/\sqrt{\text{Hz}}. \quad (1)$$

2.2 Earthquake stop geometry

A drawing of the real stops is shown below in figure 1. The tip radius is about 1.77 mm, and the area is about 9.82 square mm. We assume that there are 8 earthquake stops. We model them as 3.14 mm by 3.14 mm squares, so they have the same area as the real stops. The stops are placed 1 mm from the surface of the optic, parallel to the optical surface. Making the tips square makes the numerical integration simple.

We assume that the earthquake stop is an insulator, and that charge on the earthquake stop is uniformly distributed across the surface. We ignore all other geometry of the stop (finite size of dielectric, nearby ground planes, etc). In some of the analytic calculations, we replace the 3.14 mm by 3.14 mm square with a disc of radius 1.77 mm, so that it has the same area.

We assume that the optic is at least 1 cm x 1 cm and parallel to the tip of the stop. In Advanced LIGO, the stops will come up at 45 degrees and contact the chamfer of the optic, so the geometry is of the real system is not the same as what we model. However, the contact surfaces at parallel in the real system, but smaller on the optic than in our model. Since most of the interaction happen right around the tip of the stop, this model is a reasonable first step.

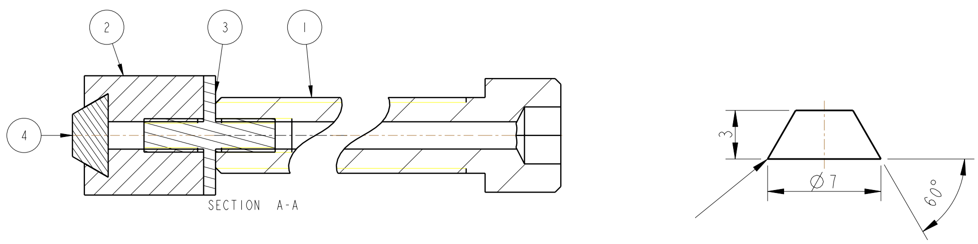

Figure 1: Left: Assembly drawing of the earthquake stop from D060544. (4) is the silica tip, (2) is a florel cylinder 1 cm long and 1 cm in diameter. Right: Dimensions of the silica tip (in mm) from D060548. The tip radius is about 1.77 mm, and the area is about 9.82 square mm.

2.3 Frame motion

We assume that the motion of the frame at 10 Hz is 3e-12 m/rtHz. This number comes from the calculations by Brian Lantz for the critical design review of the BSC. This is substantially larger than the original requirement of 2e-13 m/rtHz set forth in the BSC conceptual design, but is a more likely number, based on the large motion of the top of the BSC piers at 10 Hz.

2.4 Directions

In an attempt to minimize confusion, we define the beam direction to be x so that we can integrate the charge results with the seismic platform motion. Thus, the stop tip is in the $y-z$ plane, as is the surface of the optic. This may actually increase confusion, since most electrostatics problems define the interesting field direction as z, whereas we use x.

2.5 Field at the optic from charge on the tip of the stop

We calculate the electric field of the charge on the tip of the stop with a simple numerical integration in Matlab. The transverse directions on the stop are yy and zz, the transverse directions on the optic are aa and bb. When the stop and the optic are seperated by a distance of x_1, the electric field in the x direction at the point (aa, bb) on the optic surface generated by a point charge of q_p coulombs at point (yy, zz) on the earthquake stop is simply

$$E_x = \frac{q_p}{4\pi\epsilon_0}\frac{1}{r^2}\frac{x_1}{r}, \text{ where } r \equiv \sqrt{(yy-aa)^2 + (zz-bb)^2 + x_1^2} \tag{2}$$

To calculate the total field in the x direction at the point (aa, bb) on the optic, we then sum up the contributions from many small point charges across the surface of the tip of the stop. Currently, I put a total charge of 1 coulomb on the tip, and divided it up into 51 x 51 points spread uniformly across the tip of the stop. This makes the field calculation more generally the field per coulomb of charge on the stop.

The field per coulomb of charge on the stop, for a 1 cm x 1 cm patch of optic centered on the stop, is shown below in figure 2. These calculations generate 2 important numbers, the peak field and the average field. The peak field per coulomb of charge on the tip of the earthquake stop, $E_{x\_peak\_per\_Q1}$, is

$$E_{x\_peak\_per\_Q1} = 2.82 \times 10^{15} \text{ (V/m)/C} \tag{3}$$

The average field across the 1 cm x 1 cm area of the plot, per coulomb of charge on the tip of the earthquake stop, $E_{x\_avg\_per\_Q1}$, is

$$E_{x\_avg\_per\_Q1} = 4.51 \times 10^{14} \text{ (V/m)/C} \tag{4}$$

2.6 Field gradient at the optic from charge on the tip of the stop

We calculate the derivative of the x field in the x direction $\partial E_x/\partial x$ by calculating the E_x field at two planes 1e-5 meters apart, taking the difference, and dividing by the distance between them.

A plot of the field gradient is shown below in figure 3.

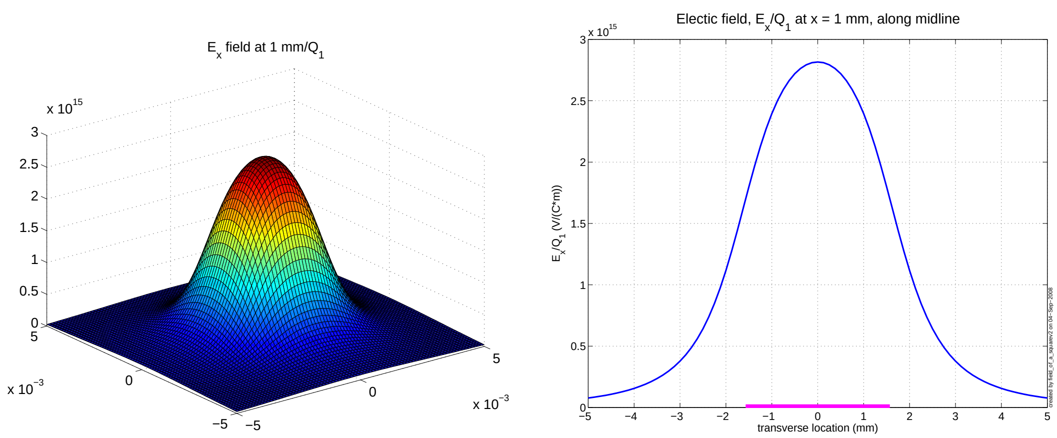

Figure 2: Left: Electric field in the x direction across a 1 cm x 1 cm patch of the surface of the optic, per coulomb of charge, centered above a square stop located 1 mm away. On the right is a slice along the center at aa=0. The magenta line indicates the extent of the source charge.

The peak field gradient per coulomb of charge on the tip of the earthquake stop, $\partial E_{x\_peak\_per\_Q1}/\partial x$, is

$$\frac{\partial E_{x\_peak\_per\_Q1}}{\partial x} = 2.03 \times 10^{18} \text{ (V/m}^2\text{)/C} \tag{5}$$

The average field gradient across the 1 cm x 1 cm area of the plot, per coulomb of charge on the tip of the earthquake stop, $\partial E_{x\_avg\_per\_Q1}/\partial x$, is

$$\frac{\partial E_{x\_avg\_per\_Q1}}{\partial x} = 9.66 \times 10^{16} \text{ (V/m}^2\text{)/C} \tag{6}$$

2.7 Sanity check

One can check the basic functionality of the code by comparing the x field above the center of the stop $(aa, bb) = (0, 0)$ with the analytic expression for the field of a disc of the same charge density and total area. This compares at 2 mm x 2 mm square with a disc of radius 1.13 mm.

Table 1 shows that the numerical method is similar to the analytic methods for a disc of the same area and charge density, for fields and field gradients along the central axis $((aa, bb) = (0, 0))$.

For the electric field in the x direction, at distance x_1 from the center of a disc of radius R and uniform charge density σ_1, the electric field is

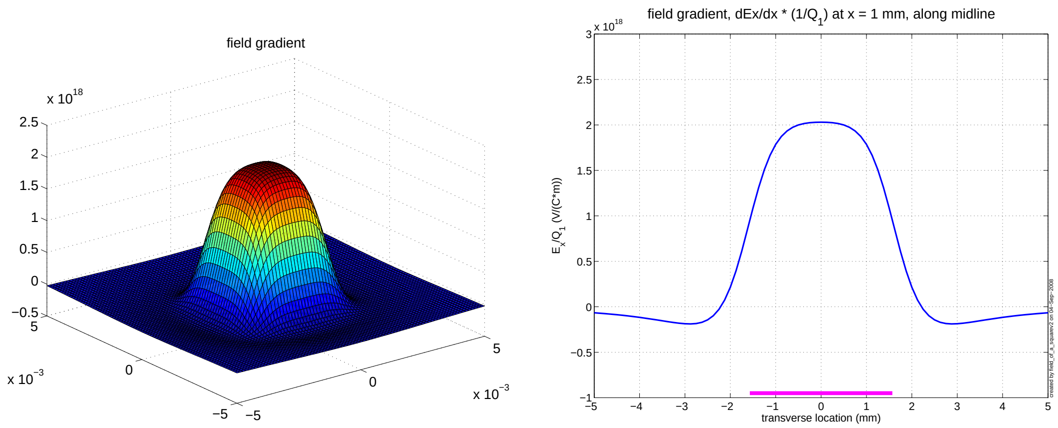

Figure 3: Left: Derivative of the electric field E_x in the x direction ($\partial E_x/\partial x$) across a 1 cm x 1 cm patch of the surface of the optic, per coulomb of charge, centered above a square stop located 1 mm away. On the right is a slice along the center at aa=0. The magenta line indicates the extent of the source charge.

$$E_x = \frac{\sigma_1}{2\epsilon_0}\left(1 - \frac{x_1}{\sqrt{x_1^2 + R^2}}\right) \tag{7}$$

The field derivative in the x direction is

$$\frac{\partial E_x}{\partial x} = -\frac{\sigma_1}{2\epsilon_0}\left(\frac{R^2}{(x_1^2 + R^2)^{3/2}}\right) \tag{8}$$

and the potential above the center of the disc is

$$V = \frac{\sigma_1}{2\epsilon_0}\left(\sqrt{x_1^2 + R^2} - x_1\right) \tag{9}$$

at $x_1 = 0$, the potential simplifies to

$$V = \frac{\sigma_1 R}{2\epsilon_0} \tag{10}$$

We use this expression for the potential of the end of the earthquake stop, even though it is only true at the center of the stop.

seperation	field of square (V/m)	field of disc (V/m)	field gradient of square (V/m^2)	field gradient of disc (V/m^2)
0.5 mm	8.11E+15	8.40E+15	9.08E+18	9.57E+18
1.0 mm	4.62E+15	4.76E+15	4.99E+18	5.25E+18
2.0 mm	1.80E+15	1.83E+15	1.44E+18	1.49E+18
20.0 mm	2.24E+13	2.24E+13	2.23E+15	2.24E+15

Table 1: Compare fields and gradients at $(aa, bb) = (0, 0)$ for numerical and analytic methods, as a simple check on the numerical method. The source charge is 1 coulomb. The answers are in reasonable agreement.

3 Calculation of the coupling of frame motion to optic motion

3.1 A uniformly charged earthquake stop with a point charge on the optic

If there is a point charge, Q_2 on the optic aligned with the center of the earthquake stop, then the force on that charge will be Q_2 times the peak field from the stop. If the stop is moving at 10 Hz, then the force fluctuation in the beam direction will be the charge times the field gradient times the fluctuating motion of the stop. We assume that the amplitude spectral density of frame motion at 10 Hz is $x_{10Hz\_frame\_ASD} = 3 \times 10^{-12}$ m/rtHz. We assume there are 8 earthquake stops moving independently (although the motions could easily be correlated). The force on the optic becomes

$$F_x = Q_1 * \frac{\partial E_{x\_peak\_per\_Q1}}{\partial x} * x_{10Hz\_frame\_ASD} * \sqrt{n_{stops}} * Q_2 \tag{11}$$

We set this force equal to the maximum allowed force at 10 Hz, and we see that

$$Q1 * Q2 = 1.58 \times 10^{-15}\ \mathrm{N}/\sqrt{\mathrm{Hz}}/\big(\sqrt{8} * 2.03 \times 10^{18}(\mathrm{V/m^2})/\mathrm{C} * 3 \times 10^{-12}\ \mathrm{m}/\sqrt{\mathrm{Hz}}\big). \tag{12}$$

Now we assume that $\mathrm{abs}(Q_1) = \mathrm{abs}(Q_2)$, because we have either transfered charge from one surface to the other by rubbing (i.e. $Q_1 = -Q_2$), or because both are similar materials in similar environments, so if one is charged, it is likely that the other is also charged, then we see that the maximum allowed charge on the end of an earthquake stop, when across from an equal point charge on the optic, is

$$Q_{1\_for\_point} = 9.57 \times 10^{-12}\ \mathrm{C} \tag{13}$$

Q_1 is spread uniformly across the 2 mm x 2 mm tip of the earthquake stop, so the charge density is

$$\sigma_{1\_for\_point} = 0.97 \times 10^{-6} \text{ C/m}^2. \tag{14}$$

The voltage at the center of the tip of the earthquake stop in this case is about 97 V. This is clearly a worst case scenario.

3.2 A uniformly charged earthquake stop with a uniformly charged optic

A more realistic case is when there is a earthquake stop with a uniform charge density on the tip of σ_1 next to an optic which has a uniform charge density σ_2 across a 1 cm x 1 cm area of the optic, A_{optic}, across from the tip of the stop. If the charge density on the optic is uniform, we don't need to do the integral across the optic of the local field times the local charge density, and we can replace that by the product of the average field, the average charge density, and the relevant area on the optic. In this case, the force on the optic from a set of moving stops becomes

$$F_x = Q_1 * \frac{\partial E_{x\_avg\_per\_Q1}}{\partial x} * A_{optic} * x_{10Hz\_frame\_ASD} * \sqrt{n_{stops}} * \sigma_2. \tag{15}$$

Since $Q_1 = \sigma_1 * A_{stop}$, where A_{stop} is the area of the end of the earthquake stop (9.86e-6 m^2), we can rearrange equation 15 to solve for the maximum allowed charge densities.

$$\sigma_1 * \sigma_2 = \frac{F_{allowed\_10Hz}}{\frac{\partial E_{x\_avg\_per\_Q1}}{\partial x} * A_{optic} * A_{stop} * \sqrt{n_{stops}} * x_{10Hz\_frame\_ASD}} \tag{16}$$

if we let $\sigma_1 = \sigma_2$, then the maximum allowed charge density for a distributed charge is 1.40×10^{-6} C/m^2 on both the tip of the stop and the optic.

With this charge density:

- The total charge on the tip of the stop is 1.4e-11 coulombs, or 8.6e7 electons.
- The potential of the stop is 140 volts.
- The DC force between 1 stop and the optic is 9e-07 N.
- The field at the tip of the stop is about 7.9e+04 V/m.
- The max field at the optic is 3.9e+04 V/m.

Note that in G070204, pg 7, Mitrofanov, et. al saw that a single touch of a fused silica tip of radius ≈3 mm on a fused silica substrate caused a charge transfer of 3e6 electrons, and repeated touches result in a charge transfer of roughly 5e6 electrons, roughly an order of magnitude smaller that the number needed to cause problems, but perhaps close enough to be of some concern.

3.3 Force between a uniformly charged stop and the bulk polarizability of an uncharged optic

The polarizability of the glass means that a charged earthquake stop will still exert forces on an uncharged test mass. This interaction is, for example, what allows the electrostatic drives to work. Estimating the electrostatic forces in this case can be made quite simple if we treat the optic as semi-infinite and use image charges. If the optic were a perfect conductor, then a charge Q_1 at distance d to the left the surface will generate an image charge $Q_{image} = -Q_1$ at distance d to the right of the surface. For a polarizable material, a similar image charge is created at the same location with a smaller magnitude. We define the source charge to be in a material with permittivity $= \epsilon_{r1}\epsilon_0$ and the image charge to be in the bulk material with permittivity $\epsilon_{r2}\epsilon_0$. ϵ_r is relative permittivity, which is 1 for vacuum and about 3.8 for Corning 7940 fused silica. In Lorrain and Corson, 'Electromagnetic Fields and Waves', 2nd ed. eq 4-55, they show that the magnitude of the image charge is

$$Q_{image} = -\frac{\epsilon_{r1}\epsilon_{r2} - 1}{\epsilon_{r1}\epsilon_{r2} + 1} Q_1 \equiv r_{image}\, Q_1 \approx -0.58\, Q_1 \tag{17}$$

where we define r_{image} to be the ratio of the image charge to the source charge. For vacuum and glass, $r_{image} \approx -0.58$.

To calculate the force noise from a moving stop, we use the same formalism as in the previous section, except:

- We use the Matlab code to calculate the field gradient at the image charge location, not the surface of the optic.
- We let $\sigma_2 = r_{image} * \sigma_1$.
- For the motion of the stop, we use $2 * x_{10Hz\_framemotion\_ASD}$, since the relative motion between the source and the image is moving twice as much as the source is moving.
- We only run the averaging of the field gradient only across the 3.14 mm x 3.14 mm size of the image charge, not a 1 cm x 1 cm patch, so we replace the relevant optic area, A_{optic} with $A_{image} = A_{stop} = 9.86 \times 10^{-6}$ m^2.

With this in mind, the force between the stop and the image charge density is

$$F_x = \sigma_1 * A_{stop} * \frac{\partial E_{x\_avg\_per\_Q1,at\_image}}{\partial x} * A_{image} * 2\, x_{10Hz\_frame\_ASD} * \sqrt{n_{stops}} * \sigma_2. \tag{18}$$

Warning: I have not thought carefully about this, so I could easily have missed some subtlety of this problem.

We can rearrange equation 18 to solve for the maximum allowed charge density.

$$\sigma_1^2 = \frac{F_{allowed\_10Hz}}{\frac{\partial E_{x\_avg\_per\_Q1,at\_image}}{\partial x} * A_{stop} * A_{stop} * \sqrt{n_{stops}} * 2\, x_{10Hz\_frame\_ASD} * r_{image}} \tag{19}$$

The calculation for the x field and the x derivative of the x field are shown below in figures 4 and 5. The average field across the area of the image charge is

$$E_{x\_avg\_per\_Q1,at\_image} = 1.14 \times 10^{15} \text{ (V/m)/C} \tag{20}$$

The average field gradient is

$$\frac{\partial E_{x\_avg\_per\_Q1,at\_image}}{\partial x} = 9.66 \times 10^{16} \text{ (V/m}^2\text{)/C} \tag{21}$$

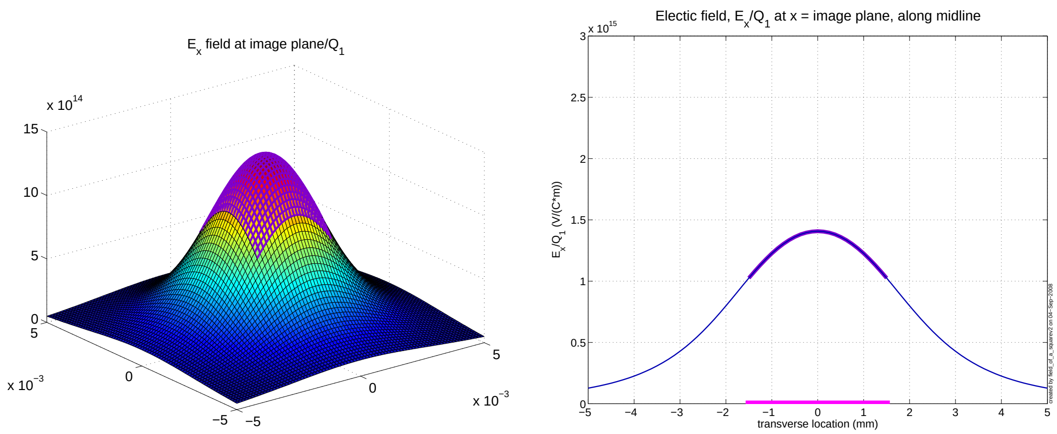

Figure 4: Left: Electric field in the x direction at the image charge location, across a 1 cm x 1 cm patch, per coulomb of charge, centered above a square stop located 2 mm away, on the right is a slice along the center at aa=0. The extent of the image charge is highlighted.

So if the end of the stop is uniformly charged and the optic is not charged, the allowed charge on the stop is allowed to be 1.59×10^{-6} C/m^2, which is only slightly larger than the case for a charged optic.

With this charge density:

- The total charge on the tip of the stop is 1.57e-11 coulombs.
- The potential of the stop is 159 volts.
- The DC force between 1 stop and the optic is 1.63e-07 N.
- The field at the tip of the stop is about 8.98e+04 V/m.

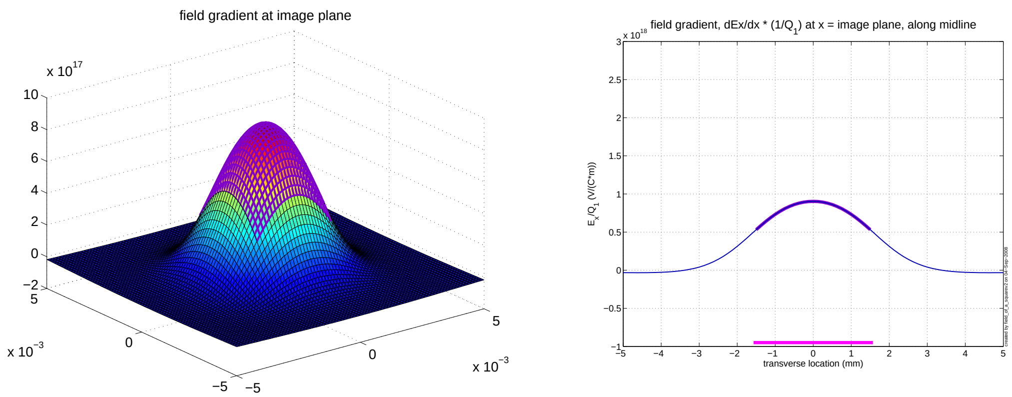

Figure 5: Left: Derivative of the electric field E_x in the x direction ($\partial E_x/\partial x$) across a 1 cm x 1 cm patch of the surface of the optic, per coulomb of charge, centered above a square stop located 2 mm away, on the right is a slice along the center at aa=0.

3.4 Charge hopping noise

One concern about conductive coatings for the optic is that they might make the "charge hopping noise" described by Rai much worse by lowering the time constant. For conductors, this noise should be much smaller than Rai's estimate for insulators, both because the static force can be kept much smaller, and because the charges move relatively smoothly across the surface.

However, using the field calculations from the previous sections, one can make some rough estimates about the worst case charge hopping noise.

For this problem, we assume that you start with a charged stop and a point charge on the optic surface at the center of the stop (as in subsection 3.1), with an initial force between the optic and the stop of $F_{initial}$. Then we assume that a leaky layer on the surface of the optic allows the charge on the optic to relax to a uniform distribution (as in subsection 3.2), which has a much smaller force, F_{final}, for the same total charge. If this happens in S discrete steps, then the average force per step F_s is just

$$F_s = \frac{F_{initial} - F_{final}}{S} \tag{22}$$

If the relaxation takes place in a characteristic relaxation time, τ, then the average number of steps per second is $\dot{S} = S/\tau$, and the average amplitude spectral density of steps should be $\sqrt{2\dot{S}}$. Thus, the average force spectral density, between the optic and the stop as the charge relaxes, $\tilde{F}_{relaxing}$, should be

$$\tilde{F}_{relaxing} = F_s * \sqrt{2\dot{S}} = \frac{F_{initial} - F_{final}}{S}\sqrt{\frac{2S}{\tau}} \tag{23}$$

or

$$\tilde{F}_{relaxing} = (F_{initial} - F_{final})\sqrt{\frac{2}{S\tau}} \tag{24}$$

To put some numbers to these quantities, we recall from equation 11 that for a charge Q_1 on the stop and point charge Q_2 on the optic, the static forces between the stop and the mass for a point charge on the mass is

$$F_{initial} = Q_1 * E_{x\_peak\_per\_Q1} * Q_2. \tag{25}$$

From equation 15, we recall that the static force between the stop and a charge density σ_2 on a small 1 cm x 1 cm patch of the optic is

$$F_{final} = Q_1 * E_{x\_avg\_per\_Q1} * A_{optic} * \sigma_2 = Q_1 * E_{x\_avg\_per\_Q1} * Q_2 \tag{26}$$

so the difference between the initial force and the final force is

$$F_{initial} - F_{final} = Q_1 * Q_2 * (E_{x\_peak\_per\_Q1} - E_{x\_avg\_per\_Q1}) \equiv Q_1 * Q_2 * E_{\Delta/Q} \tag{27}$$

for our geometry,

$$E_{\Delta/Q} = 2.82 \times 10^{15} - 4.51 \times 10^{14} = 2.37 \times 10^{15} \text{ (V/m)/C} \tag{28}$$

To estimate the number steps, we say that there are $N = Q_2/e$ electrons (e is the electronic charge) which each move, on average, 2.5 mm, which is the distance from the center to half way to the of the 1 cm x 1 cm area we are examining. We assume that they hop along in distances of about 1 SiO_2 unit. I don't know what this length is, but we will guess at something like 3 angstroms. Thus, the number of discrete steps taken by the charge distribution is roughly

$$S = \frac{Q_2}{e}\frac{2.5 \times 10^{-3}}{3 \times 10^{-10}} \equiv Q_2 * S_{perQ} = Q_2 * 5.2 \times 10^{25} \tag{29}$$

We put these expressions back into equation 24 and see that

$$\tilde{F}_{relaxing} = Q_1 * Q_2 * E_{\Delta/Q}\sqrt{\frac{2}{Q_2 * S_{perQ} * \tau}}. \tag{30}$$

We put in the numerical values and let $Q_1 = Q_2$ to get

$$\tilde{F}_{relaxing} = {Q_1}^{3/2} * \tau^{-1/2} * E_{\Delta/Q}\sqrt{\frac{2}{S_{perQ}}} = {Q_1}^{3/2} * \tau^{-1/2} * 460 \tag{31}$$

We can now solve this for allowable values of τ.

$$\tau > {F_{allowed\_10Hz}}^{-2} * {Q_1}^2 * Q_2 * {E_{\Delta/Q}}^2 * \frac{2}{S_{perQ}} \tag{32}$$

If we assume that Q_1 is the maximum allowed charge for case where we have a moving stop and a spread charge on the optic, ie $Q_1 = 1.4$e-11 coulombs, and we set the force noise to be equal to the allowed force noise at 10 Hz, then $\tau > 225$ seconds. This is a worst case scenario for a fully charged stop, and the sudden appearance of an equal amount of charge across from the stop. If this were to happen, then the force impulse from the arrival of the charge would certainly have a very large affect on the interferometer, even if the gradual dispursement of that charge did not impact the noise floor, so it is not clear exactly what to do with this relationship. I believe that the conclusion to draw is that we should make the stop conductive, so that Q_1 stays small, then any ionic layer we can actually make on the optic will be fine.