**LASER INTERFEROMETER GRAVITATIONAL WAVE OBSERVATORY**

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# Marginally stable cavity simulation : FFT vs Modal Model

Hiro Yamamoto

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**California Institute of Technology**
**LIGO Project – MS 18-34**
**1200 E. California Blvd.**
**Pasadena, CA 91125**
Phone (626) 395-2129
Fax (626) 304-9834
E-mail: info@ligo.caltech.edu

**Massachusetts Institute of Technology**
**LIGO Project – NW17-161**
**175 Albany St**
**Cambridge, MA 02139**
Phone (617) 253-4824
Fax (617) 253-7014
E-mail: info@ligo.mit.edu

**LIGO Hanford Observatory**
**P.O. Box 1970**
**Mail Stop S9-02**
**Richland WA 99352**
Phone 509-372-8106
Fax 509-372-8137

**LIGO Livingston Observatory**
**P.O. Box 940**
**Livingston, LA 70754**
Phone 225-686-3100
Fax 225-686-7189

http://www.ligo.caltech.edu/

# 1 Introduction

When the performance of the marginally stable cavity design of AdvVirgo was studied, it was found that the resonating sideband field calculated using two methods show large difference, one FFT-based program, SIS, and another based on modal model based programs, MIST and Finesse. This note studies the cause of this difference.

An interferometer is designed so that one specific mode resonates in all cavities, which is called the base mode in this note. When the thermal effect induces a lens in ITM, this full mode matching cannot be maintained. For simplicity, the surface deformation is neglected. E.g., on the recycling mirror, the beam size of the eigen mode of the power recycling cavity formed by PRM and ITM is 5.62cm when there is not thermal lens (cold state), but it becomes 1.77cm when the lens with focal length of 140km is induced (hot state). The mode in the arm is not affected by the thermal lens. It is necessary to handle this mixture of modes properly.

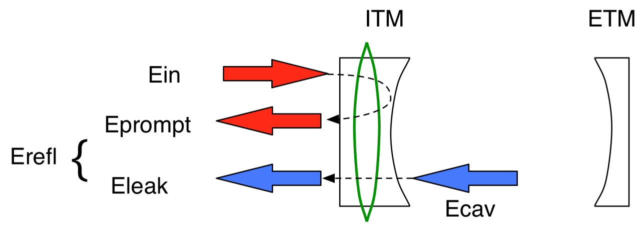


**Figure 1 Reflection by ITM with thermal lens**

The CR field is set to resonate in the arm, and this makes the CR mode resonating in the recycling cavity close to the base mode. This is because the leak field from the arm (Eleak in Fig.1) back to the recycling cavity cancels the leading order lens effect in the promptly reflected field (Eprompt in Fig.1) by ITM with lens. It makes sense to use the input beam set to match with the cold interferometer because the CR mode does not change much in any of the cavities. In this way, it is appropriate to use the cold state base modes in all cavities for CR and for the input beam.

The SB field does not resonate in the arm, and the mode resonating in the recycling cavity is affected by the lens in ITM. This is because there is no canceling mechanism, i.e., Eleak is negligible for SB, which restored the CR mode in the recycling cavity. A better way to handle the situation is that SB in the recycling cavity is handled using the eigen mode determined by the hot recycling cavity formed by ITM with the thermal lens and the recycling mirror, and the mode in the arm is handled by the cold state base mode.

But, if one adapts this choice of SB mode bases, the same mode base cannot be used for CR and SB. When this kind of mode mixing exists in a complex cavity system, one can use higher order mode expansion to calculate fields in a cavity whose eigen mode is different from the base mode. E.g., the resonating field in the recycling cavity with thermal lens in ITM can be expressed using code state mode, not only the base 00 mode, but also higher order mode, like LG10, LG20, etc, or equivalent combination of HGmn modes.

This document explores how good the modal model approach using higher order expansion is for a marginally stable cavity. The main issue is to understand how well the SB resonating in the recycling cavity can be expressed using the base mode, and the leak field from the cavity is small compared to the input beam, a simple FP is used for this study.

## 2 Optical system

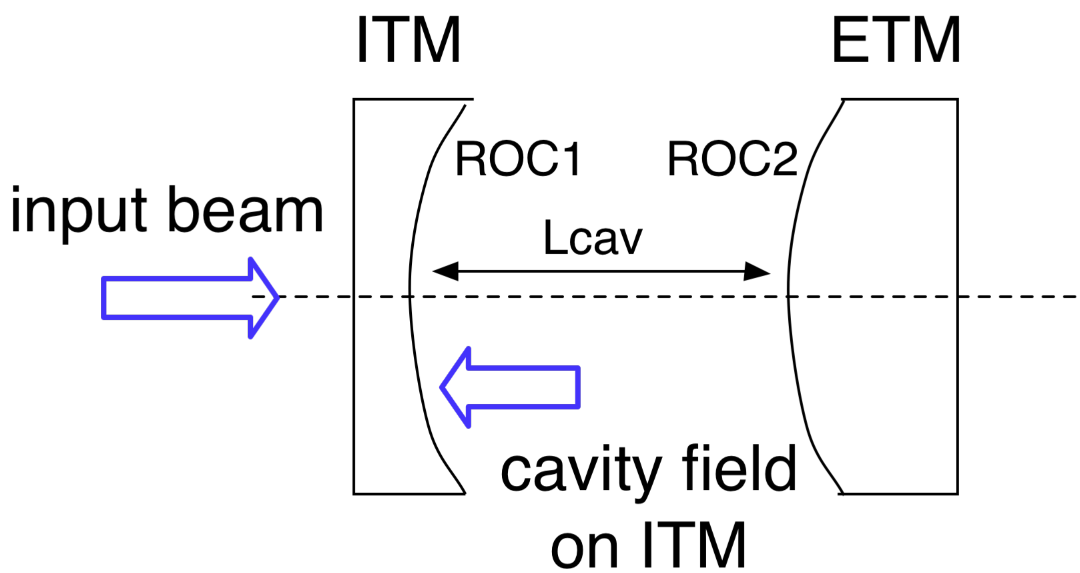


**Figure 2. FP cavity (ITM represents PRM and ETM represents ITM with thermal lens)**

The optical system studied is shown in Fig.2. Various optical parameters are chosen to simulate a marginally stable cavity of AdvVirgo system.

Lcav=11.9522m

ROC1=988.617m

ROC2(cold)=-976.801m

ROC2(hot)=-990.624m (140km thermal lens added)

T(ITM) = 0.04

T(ETM) = 0

The input beam was setup to match to the cold cavity.

The gouy phase, η, equivalent spatial shift, el= λ x η / 2π, and the beam size on ITM of the eigen mode in a cold cavity (ROC1 and ROC2(cold)) and a hot cavity (ROC1 and ROC2(hot)) are summarized in the following table.

| | η (milliradian) | el (nm) | w (cm) |
|---|---|---|---|
| Cold cavity | 1.298 | 0.2199 | 5.618 |
| Hot cavity | 13.05 | 2.2103 | 1,772 |

## 3 Numerical simulation tools – SIS and e2e

For the FFT-based calculation, SIS was used. For the cold case, the SIS locking algorithm was used to find the locked point. This length matched well with el(cold). For the hot cavity case, lengths was changed to scan for the power maximum, and the length matched well with el(hot).

For the modal model based calculation, LIGO time domain simulation code, e2e, was used. The cavity length was changed to find the length where the total or 00 mode cavity power becomes maximum. The null test of the code was done by using input beams which match to the cold cavity and to the hot cavity, and the cavity length and the resonating power distributions, total and 00 mode, were confirmed to match with the analytical calculation. In the following calculations, the input beam and the mode expansion base are fixed to the one for the cold cavity.

# 4 Numerical results

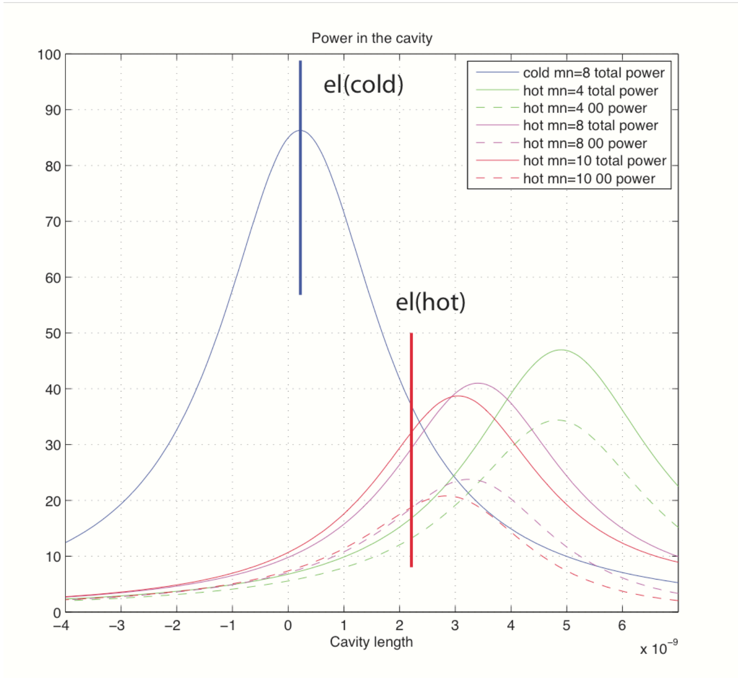


**Figure 3 Cavity power vs cavity length**

Fig.3 shows the power in the cavity as a function of the cavity length calculated using e2e. Two lines, el(cold) and el(hot), are the resonant points for the cold and hot cavity. Solid lines show the total power and dashed lines show the power of the base mode.

For the cold cavity, i.e., the mode matching is good, the maximum length matches well with the base mode resonance point.

For the hot cavity, different number of modes are used to calculate the cavity field. In the figure, mn=X means to use HG(m,n) modes with $m+n \leq X$ to expand the field. If the mode expansion calculation converges well, the result should not depend on the number of modes used. This figure shows that the power maximum moves toward el(hot) as more number of modes are used.

Figure 4 shows the power distribution of the field on ITM, coming from ETM. This is a log of power vs radius$^2$ plot, and a Gaussian shape is shown as a straight line.

Lines named “MM cold/hot” show the total powers calculated using e2e, with the cavity length at the total power maximum. The shape is almost identical when the cavity length at the base power maximum.

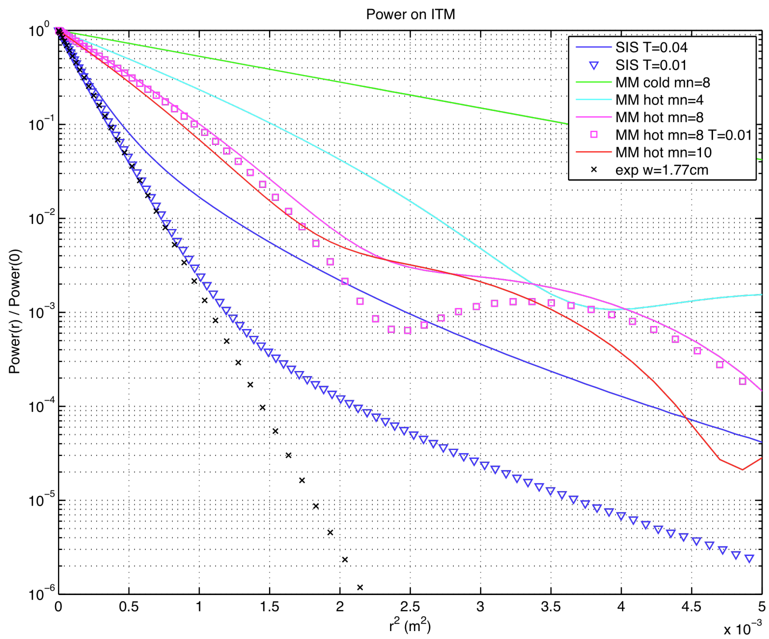


**Figure 4 Power distribution**

The green line is the case in a cold cavity, and the slop is the expected value of 5.618cm. The line shown by black cross' is the Gaussian shape with a slope of 1.772cm.

The solid blue is the power distribution calculated using SIS with the nominal ITM transmittance 0.04. The slope is close to the hot cavity eigenstate in the central region, but spreads out more in the outer area.

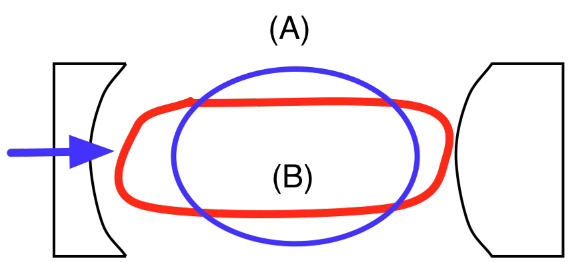


**Figure 5 Stationary field in a cavity**

The stationary field in the cavity is a mixture of modes, one is the mode determined by the cavity parameter, (B) in Fig.5, and the other by the input beam, (A) in Fig.5. The weight of the cavity mode becomes larger as the finesse becomes larger. The line of blue triangles is the power distribution calculated using SIS with the ITM T of 0.01. The resonating shape becomes much closer to the eigen mode of the hot cavity.

Cyan, magenta and red lines are the powers calculated using e2e with different number of modes. As more modes are used, slopes come closer the eigen mode slope. The slope is 2.7cm with m+n=10 and 2.9cm m+n=8 within the 4.5cm central region. The line of magenta square shows the result using ITM T=0.01, high finesse cavity, with m+n=8, and the slope does not change much.

# 5 Analytic calculation using mode expansion using Hermite Gaussian mode

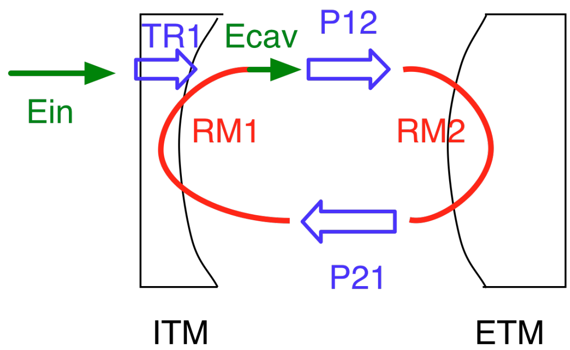


**Figure 6 Fields in a FP cavity**

In this section, the lowest order modal model expansion (HG00, HG02 and HG20) is used to study the cavity field in Fig.6. The mode base is defined by the cold cavity and there is curvature mismatch between the base mode curvature on ETM and the ROC of hot ETM.

The cavity field Ecav in Fig.6 can be expressed using the input field Ein, transmission matrix TR1, two propagators P12 and P21 and two reflection matrixes RM1 and RM2 in the following equation.

$$Ecav = \frac{1}{1 - RM1 \cdot P21 \cdot RM2 \cdot P12} \cdot TR1 \cdot Ein$$

Due to the curvature mismatch, RM2 is not diagonal and is given as follows:

$$\mathrm{RM2} = \begin{pmatrix} \mathrm{H0}[\alpha] & \mathrm{H1}[\alpha] & \mathrm{H1}[\alpha] \\ \mathrm{H1}[\alpha] & \mathrm{H2}[\alpha] & \mathrm{H3}[\alpha] \\ \mathrm{H1}[\alpha] & \mathrm{H3}[\alpha] & \mathrm{H2}[\alpha] \end{pmatrix}$$

$$\mathrm{H0}[\alpha] = \frac{1}{1 - i\,\alpha},\quad \mathrm{H1}[\alpha] = \frac{i\,\frac{\alpha}{\sqrt{2}}}{(1 - i\,\alpha)^2},\quad \mathrm{H2}[\alpha] = \frac{1 - \frac{\alpha^2}{2}}{(1 - i\,\alpha)^3},\quad \mathrm{H3}[\alpha] = \frac{-\frac{\alpha^2}{2}}{(1 - i\,\alpha)^3}$$

In this equation, α is the coupling parameter of different modes and is given by the following expression.

$$\alpha = \frac{kw^2}{2}\left(\frac{1}{ROC(cold)} - \frac{1}{ROC(hot)}\right)$$

For the example in the previous section, $\alpha = 0.13$.

The TEM00 and TEM02 mode amplitudes are given as follows. This is a formula for a cavity where curvature mismatches exist on ITM and ETM. $\alpha_1$ is the mismatch on ITM and $\alpha_2$ is the mismatch on ETM. Amplitudes are calculated keeping up to the second order of αs.

$$\mathbf{E00} = \frac{t_1 E_{00}{}^{\text{in}}}{1 - R_{12}(1 - \delta_R)}$$

$$R_{12} = r_1 r_2 \,\mathrm{Exp}[I\,\phi],$$
$$\phi = -2\,k\,L + 2\,\eta + \chi_1 + \chi_2 + \chi_{12},$$
$$\eta = \mathrm{atan}(z_2 / z_0) - \mathrm{atan}(z_1 / z_0)$$
$$\chi_j = \mathrm{Atan}[\alpha_j]$$
$$\chi_{12} = \frac{1}{2}\left(\mathrm{Cos}[2\,\eta]\,\alpha_1^2 + 2\,\alpha_1\,\alpha_2 + \mathrm{Cos}[2\,\eta]\,\alpha_2^2\right) / \mathrm{Sin}[2\,\eta]$$

$$\delta_R = (1 - R_{12})\,\Delta$$
$$\Delta = \frac{1}{4}\left(\alpha_1^2 + 2\,\mathrm{Cos}[2\,\eta]\,\alpha_1\,\alpha_2 + \alpha_2^2\right) / \mathrm{Sin}[2\,\eta]^2$$

$$\mathbf{E02} = \frac{\mathrm{E00}\,R_{12}\,\mathrm{m2}}{1 - (\mathrm{m3} + \mathrm{m4})\,R_{12}}$$

$$\mathrm{m2} = \frac{i\left(\alpha_1 + e^{2\,i\,\eta}\,\alpha_2\right)}{\sqrt{2}}, \quad \mathrm{m3+m4} = e^{4\,i\,\eta}$$

The 02 mode amplitude is suppressed by α / (1-$R_{12}$ exp(4 i η)) compared to the 00 mode amplitude. For a stable cavity with large η, this is a small number when α is small. The higher order mode is excited when the main 00 mode field is reflected by the surface with curvature mismatch. But, for a unstable cavity with small η, the excited mode can be amplified by the same gain as the base mode, and the higher order mode is no more small compared to the base mode even when the couple α is small.

The power of these two modes can be expressed as follows.

$$\mathrm{Power00} = \left(\frac{t_1 E_{00}{}^{\text{in}}}{1 - R_{12}}\right)^2 (1 - 2\,\Delta)$$

$$\mathrm{Power02} = \mathrm{Power00} * \frac{1}{2}\,\Delta$$

The higher order correction is not of the order of $\alpha^2$, but is $(\alpha / \sin(2\eta))^2$. This value is larger than 1 for the sample FP cavity used in the previous section. This simply means that the lowest order calculation is not enough and more higher order modes need to be included.

Naively speaking, when amplitudes are calculated using modes with orders up to n+m=8, it is a calculation with corrections up to $O(\alpha^8)$, and seems to be very small for α=0.13, or for even larger values of α. But this simple example shows that the calculation should be done very carefully, because the degenerate cavity amplifies higher order modes in the same way as the base mode.

## 6 Calculation using discrete Hankel transform

The cavity field was calculated using the matrix simulation based up the discrete Hankel transform. (Vinet VPB 3.2; Vinet-Hello J. Mod. Phys. 40, 1981-1993 (1993).

The cavity length which gives the maximal power, the total power and 00 mode power, the power distribution and the front mirror transmittance dependence agreed well with the SIS calculation.