# A Rule of Thumb for the Detectability of Gravitational-Wave Bursts

**Patrick J Sutton**

School of Physics and Astronomy, Cardiff University, Cardiff, United Kingdom, CF24 3AA

E-mail: patrick.sutton@astro.cf.ac.uk

**Abstract.** We derive a simple relationship between the energy emitted in gravitational waves for a narrowband source and the distance to which that emission can be detected by a single detector. We consider both linearly polarized and elliptically polarized gravitational waves. We also consider several emission patterns: isotropic emission (unrealistic but simple), and emission patterns appropriate for sources that emit linearly or circularly polarized waves. We ignore cosmological effects.

PACS numbers: 04.80.Nn

## 1. Relating $E_{\mathrm{GW}}$ to $h_{\mathrm{rss}}$

We first relate the total energy emitted in gravitational waves, $E_{\mathrm{GW}}$, to the LIGO-Virgo standard measure for burst amplitude at the detector, $h_{\mathrm{rss}}$.

The flux (energy per unit area per unit time) of a gravitational wave is

$$F_{\mathrm{GW}} = \frac{c^3}{16\pi G}\langle \dot{h}_+^2(t) + \dot{h}_\times^2(t)\rangle \,,$$

where the angle brackets denote an average over several periods. For a burst of duration $\leq T$ we can compute the average by integrating over the duration:

$$F_{\mathrm{GW}} = \frac{c^3}{16\pi G}\frac{1}{T}\int_{-T/2}^{T/2} dt \, \left[\dot{h}_+^2(t) + \dot{h}_\times^2(t)\right] \tag{1}$$

$$= \frac{c^3}{16\pi G}\frac{1}{T}\int_{-T/2}^{T/2} dt \, \Bigg[ \left(\int_{-\infty}^{\infty} df' \exp(i2\pi f't)(i2\pi f')\tilde{h}_+^*(f') \int_{-\infty}^{\infty} df \exp(-i2\pi ft)(-i2\pi f)\tilde{h}_+(f)\right) + (\text{same}, + \to \times)\Bigg] \tag{2}$$

Since $h_{+,\times} \to 0$ outside $-T/2 < t < T/2$, we may extend the time integration to $t \to \pm\infty$. The time integral then evaluates to a delta function, $\delta(f - f')$, giving

$$F_{\mathrm{GW}} = \frac{\pi c^3}{4G}\frac{1}{T}\int_{-\infty}^{\infty} df \, f^2 \left(|\tilde{h}_+(f)|^2 + |\tilde{h}_\times(f)|^2\right) \,. \tag{3}$$

### *1.1. Isotropic emission*

To compute the total energy $E_{\mathrm{GW}}$ emitted, we need to integrate the flux $F_{\mathrm{GW}}$ assuming some emission pattern. Let us first assume isotropic emission. Then

$$E_{\mathrm{GW}} = 4\pi D_{\mathrm{L}}^2 T F_{\mathrm{GW}} \tag{4}$$

$$= \frac{\pi^2 c^3}{G} D_{\mathrm{L}}^2 \int_{-\infty}^{\infty} df\, f^2 \left( |\tilde{h}_+(f)|^2 + |\tilde{h}_\times(f)|^2 \right) . \tag{5}$$

If we assume that the signal is narrowband with central frequency $f_0$, we obtain

$$E_{\mathrm{GW}} = \frac{\pi^2 c^3}{G} D_{\mathrm{L}}^2 f_0^2 h_{\mathrm{rss}}^2 , \tag{6}$$

where the root-sum-square amplitude $h_{\mathrm{rss}}$ is given by ‡

$$h_{\mathrm{rss}}^2 = \int_{-\infty}^{\infty} df \left( |\tilde{h}_+(f)|^2 + |\tilde{h}_\times(f)|^2 \right) . \tag{7}$$

### *1.2. Linear motion emission*

Axisymmetric motion will produce linearly polarized emission with pattern

$$h_+(t) = \sin^2(\iota)\, h(t) , \tag{8}$$

$$h_\times(t) = 0 , \tag{9}$$

where $\iota$ is the angle between the symmetry axis and the line-of-sight to the observer, and we have selected a polarization basis aligned with this symmetry axis. The energy emitted in a narrowband signal is then

$$E_{\mathrm{GW}} = \frac{\pi c^3}{4G} D_{\mathrm{L}}^2 \int_{-1}^{1} d(\cos\iota) \int_{0}^{2\pi} d\lambda \int_{-\infty}^{\infty} df\, f^2 \left( \sin^4(\iota)\, |\tilde{h}(f)|^2 \right)$$

$$= \frac{8}{15} \frac{\pi^2 c^3}{G} D_{\mathrm{L}}^2 f_0^2 h_{\mathrm{rss}}^2 , \tag{10}$$

where $\lambda$ is the azimuthal angle in the source frame. This is 8/15 times the result for isotropic emission, (6).

### *1.3. Rotating system emission*

Rotational motion (such as from a circular binary) will produce emission with pattern

$$h_+(t) = \frac{1}{2}(1 + \cos^2(\iota))\, A(t) \cos\Phi(t) , \tag{11}$$

$$h_\times(t) = \cos(\iota)\, A(t) \sin\Phi(t) , \tag{12}$$

where $\iota$ is the angle between the rotation axis and the line-of-sight to the observer, and we have selected a polarization basis aligned with this symmetry axis. We assume $A(t)$ varies slowly enough compared to $\Phi(t)$ that $h_+$ and $h_\times$ are approximately orthogonal.

‡ Strictly speaking, we define $h_{\mathrm{rss}}$ as the root-sum-square amplitude from an *optimally oriented* source. This differs slightly from the standard LIGO-Virgo definition, which includes the inclination factors. In practice, however, all LIGO-Virgo papers to date have only simulated optimally oriented sources.

The energy emitted in a narrowband signal is then

$$E_{\rm GW} = \frac{\pi c^3}{4G} D_{\rm L}^2 \int_{-1}^{1} d(\cos\iota) \int_0^{2\pi} d\lambda \int_{-\infty}^{\infty} df\, f^2 \left( \frac{(1+\cos^2(\iota))^2}{4} + \cos^2(\iota) \right) |\tilde{h}(f)|^2$$
$$= \frac{2}{5} \frac{\pi^2 c^3}{G} D_{\rm L}^2 f_0^2 h_{\rm rss}^2 \,, \qquad (13)$$

where $\tilde{h}(f)$ is the Fourier transform of $A(t)\cos\Phi(t)$. This is 2/5 times the result for isotropic emission, (6).

## 2. Relating $E_{\rm GW}$ to Signal-To-Noise Ratio

The detectability of a generic signal is determined mainly by its expected signal-to-noise ratio $\rho$ for a matched filter. For a narrowband signal, $\rho$ has a simple relationship to the $h_{\rm rss}$ amplitude. We start from

$$\rho^2 = 2 \int_{-\infty}^{\infty} df \, \frac{|F_+ \tilde{h}_+(f) + F_\times \tilde{h}_\times(f)|^2}{S(f)} \,, \qquad (14)$$

where $S(f)$ is the one-sided noise power spectrum, and $F_{+,\times}(\theta,\phi,\psi)$ are the antenna responses to the sky position $(\theta,\phi)$ and polarization $\psi$. We may expand the square in (14) and drop the $\tilde{h}_+\tilde{h}_\times^*$ terms for all signals of interest: for elliptically polarized signals the two waveforms are orthogonal, while for linearly polarized signals $\tilde{h}_\times = 0$. (The waveforms are also orthogonal in the *unpolarized* case, where the two polarizations are independent stochastic timeseries. An example is white-noise bursts.) Assuming a narrowband signal, we find

$$\rho^2 = \Theta^2 \frac{h_{\rm rss}^2}{16 S(f_0)} \,, \qquad (15)$$

where we define the angle factor

$$\Theta^2 \equiv 16 \begin{cases} F_+^2(\theta,\phi,\psi)(\frac{1+\cos^2(\iota)}{2})^2 + F_\times^2(\theta,\phi,\psi)\cos^2(\iota) & \text{elliptical} \\ F_+^2(\theta,\phi,\psi)\, 2\sin^4\iota & \text{linear} \end{cases} \qquad (16)$$

Note that all dependence on the four angles $\theta$, $\phi$, $\psi$, and $\iota$ is contained in $\Theta$ (the factor of 16 is for convenience, and follows the notation used in [1]). Substituting (6), (10), or (13) gives

$$\rho^2 = \Theta^2 \frac{G}{\alpha 16 \pi^2 c^3} \frac{E_{\rm GW}}{S(f_0) D_{\rm L}^2 f_0^2} \,, \qquad (17)$$

where $\alpha = 1$ for isotropic emission, 8/15 for linearly polarized emission, and 2/5 for circularly polarized emission.

## 3. Effective Range

We can now combine the results for $E_{\rm GW}$ and $\rho$ to compute the typical distance to which a source is detectable. We will follow the approach used in Section V of [1].

Consider a homogenous isotropic distribution of sources with rate density $\dot{N}$. A signal from a given source will be detectable if the received signal-to-noise is above some threshold value $\rho_{\rm det}$. The mean rate of detections will then be

$$\dot{N}_{\rm det} = 4\pi\dot{N}\int_0^\infty dr r^2 P(\rho^2 > \rho_{\rm det}^2)\,. \tag{18}$$

Here $P(\rho^2 > \rho_{\rm det}^2)$ is the probability that the signal-to-noise of a source at given distance $r$ with random $\theta$, $\phi$, $\psi$, and $\iota$ will be above threshold. Using (17), we may write this probability as

$$P(\rho^2 > \rho_{\rm det}^2) = P(\Theta^2 > \frac{r^2}{r_0^2})\,, \tag{19}$$

where we have defined the fiducial distance

$$r_0^2 = \frac{G}{\alpha 16\pi^2 c^3}\frac{E_{\rm GW}}{S(f_0)f_0^2\rho_{\rm det}^2}\,. \tag{20}$$

Our detection rate is thus

$$\dot{N}_{\rm det} = \frac{4}{3}\pi r_0^3\dot{N}\left[3\int_0^\infty dx\, x^2 P(\Theta^2 > x^2)\right]\,. \tag{21}$$

The integral is easily evaluated numerically:

$$\int_0^\infty dx\, x^2 P(\Theta^2 > x^2) = \begin{cases} 1.838 \pm 0.002 & \text{elliptical} \\ 3.436 \pm 0.005 & \text{linear} \end{cases} \tag{22}$$

Following [1], we define the effective detection range $D_{\rm L}^{\rm eff}$ as the radius enclosing a spherical volume $V$ such that the rate of detections is $\dot{N}V$:

$$D_{\rm L}^{\rm eff} = r_0\left[3\int_0^\infty dx\, x^2 P(\Theta^2 > x^2)\right]^{1/3} \tag{23}$$

$$= \beta\left(\frac{G}{\pi^2 c^3}\frac{E_{\rm GW}}{S(f_0)f_0^2\rho_{\rm det}^2}\right)^{1/2}\,. \tag{24}$$

where

$$\beta \equiv (16\alpha)^{-1/2}\left[3\int_0^\infty dx\, x^2 P(\Theta^2 > x^2)\right]^{1/3} = \begin{cases} 0.698 & \text{elliptical} \\ 0.745 & \text{linear} \end{cases}\,. \tag{25}$$

We note that for both linear and elliptical polarization, $\beta$ is equal to $1/\sqrt{2}$ to within a few percent. A convenient approximation is thus

$$D_{\rm L}^{\rm eff} \simeq \left(\frac{G}{2\pi^2 c^3}\frac{E_{\rm GW}}{S(f_0)f_0^2\rho_{\rm det}^2}\right)^{1/2}\,. \tag{26}$$

**Acknowlegements**

The author would like to thank Eric Chassande-Mottin for motivating this investigation, and for his careful reading of and helpful suggestions on a previous draft. This work was supported in part by STFC grant PP/F001096/1. This draft has been assigned LIGO document number LIGO-P1000041-v1.

## References

[1] Lee Samuel Finn and David F. Chernoff. Observing binary inspiral in gravitational radiation: One interferometer. *Phys. Rev. D*, 47(6):2198–2219, Mar 1993.