# Model Waveform and Calibration Accuracy Standards for Gravitational Wave Data Analysis


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## Collaborators:

Duncan Brown (Syracuse), Benjamin Owen (Penn State)

- How accurate must model waveforms and detector calibration be:
  - to prevent a significant rate of missed detections?
  - to prevent a significant accuracy loss for measurements?
  - to avoid unnecessary costs of achieving excess accuracy?

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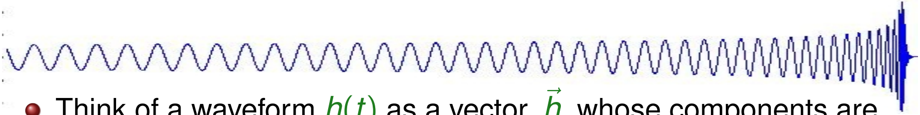

- Think of a waveform $h(t)$ as a vector, $\vec{h}$, whose components are the amplitudes of the waveform at each frequency:

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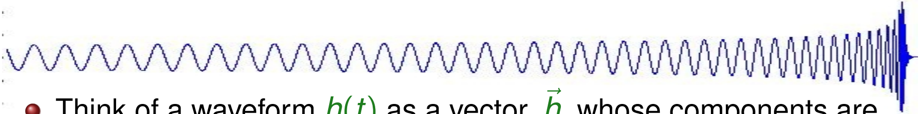

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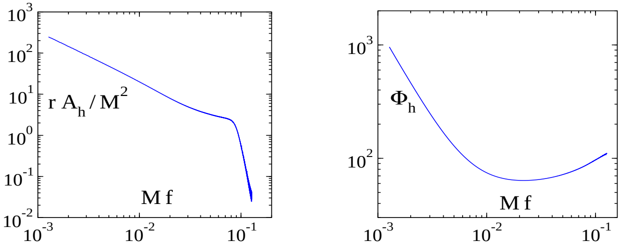


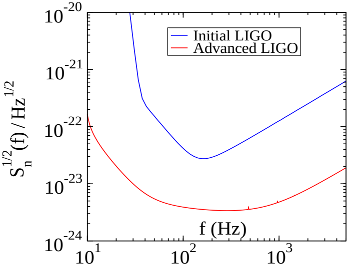
# A Theoretician's View of GW Data Analysis II:

- Let $\vec{h}_e = h_e(f)$ denote the exact waveform for some source, and let $\vec{h}_m = h_m(f)$ denote a model of this waveform.
- Define a waveform inner product that weights components (frequencies) in proportion to the detector's sensitivity:

$$\vec{h}_e \cdot \vec{h}_m = \langle h_e | h_m \rangle = \int_{-\infty}^{\infty} \frac{h_e^*(f) h_m(f) + h_e(f) h_m^*(f)}{S_n(f)} df,$$

where $S_n(f)$ is the power spectral density of the detector noise.

- This inner product is normalized so that $\rho = \sqrt{\langle h_e | h_e \rangle}$ is the optimal signal-to-noise ratio for detecting the waveform $\vec{h}_e$.



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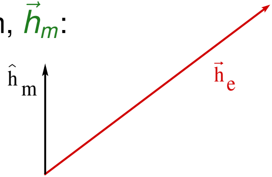

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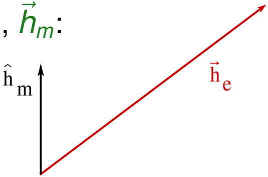


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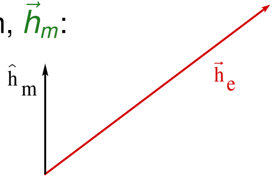


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- A detection is made when $\vec{h}_e$ has a projected signal-to-noise ratio $\rho_m$ that exceeds a pre-determined threshold.
- Measured signal-to-noise ratio, $\rho_m$, is largest when the model waveform $\vec{h}_m$ is proportional to the exact $\vec{h}_e$;
  in this case $\rho_m$ equals the optimal signal-to-noise ratio $\rho$:

$$\rho_m = \frac{\langle h_e | h_e \rangle}{\sqrt{\langle h_e | h_e \rangle}} = \sqrt{\langle h_e | h_e \rangle} = \rho = \sqrt{\int_{-\infty}^{\infty} \frac{2|h_e(f)|^2}{S_n(f)} df}.$$

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- Evaluate standards for the LIGO case.
- Do current LIGO search templates meet the appropriate initial LIGO standards?
- Possible misinterpretations and misapplications of the standards.
- Transform standards into more user-friendly forms.

# Accuracy Standards for Detection

- The measured signal-to-noise ratio $\rho_m$ for detecting the signal $h_e$ is the projection of $h_e$ onto $\hat{h}_m$:

$$\rho_m = \langle h_e | \hat{h}_m \rangle = \frac{\langle h_e | h_m \rangle}{\langle h_m | h_m \rangle^{1/2}}.$$

- Errors in model waveform, $h_m = h_e + \delta h$, result in reduction of $\rho_m$ compared to the optimal signal-to-noise ratio $\rho$:

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- Evaluate this mismatch $\epsilon$ in terms of the waveform error:

$$\epsilon = \frac{\langle \delta h_\perp | \delta h_\perp \rangle}{2 \langle h_e | h_e \rangle}, \quad \text{where} \quad \delta h_\perp = \delta h - \hat{h}_e \langle \hat{h}_e | \delta h \rangle.$$

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- Consequently model waveform accuracy must satisfy the requirement for detection: $\langle \delta h_\perp | \delta h_\perp \rangle < 2\epsilon_{\text{max}}\rho^2$.

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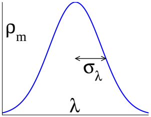


$$\frac{1}{\sigma_\lambda^2} = \left\langle \frac{\partial h}{\partial\lambda} \middle| \frac{\partial h}{\partial\lambda} \right\rangle = \langle \delta h | \delta h \rangle,$$

where the noise weighted inner product is defined by

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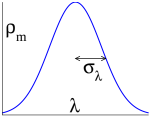


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- Two waveforms are indistinguishable iff the variance $\sigma_\lambda^2$ is larger than the parameter distance between the waveforms: $(\Delta\lambda)^2 = 1 < \sigma_\lambda^2 = 1/\langle\delta h|\delta h\rangle$, that is iff $1 > \langle\delta h|\delta h\rangle$.

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$$h = Rv_e = (R_e + \delta R)\, v_e = h_e + \delta h_R,$$

or equivalently

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- Errors in the measured response function also affect the measured power spectral density of the detector noise, $S_n(f) = e^{2\delta\chi_R(f)} S_e(f)$, with resulting effects on the measured signal-to-noise ratio $\rho_m$.

# Effects of Detector Calibration Error II

- Evaluate the measured signal-to-noise ratio:

$$\rho_m = \frac{\langle h|h_m\rangle}{\sqrt{\langle h_m|h_m\rangle}} = \frac{\langle h_e + \delta h_R|h_e + \delta h_m\rangle}{\sqrt{\langle h_e + \delta h_m|h_e + \delta h_m\rangle}},$$

$$\approx \rho - \frac{1}{2\rho}\langle(\delta h_m - \delta h_R)_\perp|(\delta h_m - \delta h_R)_\perp\rangle,$$

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- Errors in the measured signal-to-noise ratio, $\delta\rho_m$, depend only on the difference between the waveform errors: $\delta h_m - \delta h_R$.
- Waveform accuracy standards are therefore just the ideal detector ($\delta h_R = 0$) standards with $\delta h_m$ replaced by $\delta h_m - \delta h_R$:
$\langle\delta h_m - \delta h_R|\delta h_m - \delta h_R\rangle < 1$ for measurement, and
$\langle\delta h_m - \delta h_R|\delta h_m - \delta h_R\rangle < 2\epsilon_{\max}\rho^2$ for detection.

# Effects of Detector Calibration Error III

- The combined accuracy requirements can be written as

$$\langle \delta h_m - \delta h_R | \delta h_m - \delta h_R \rangle < \begin{cases} 1 & \text{measurement,} \\ 2\epsilon_{\text{max}}\rho^2 & \text{detection.} \end{cases}$$

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which leads to the new accuracy requirements:

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- Choose the relative size of the errors based on cost, or ...? If comparable accuracy standards are adopted, then the calibration standard is $\sqrt{\langle \delta h_R | \delta h_R \rangle} < 1/2$, and the waveform standards are:

$$\sqrt{\langle \delta h_m | \delta h_m \rangle} < \begin{cases} 1/2 & \text{measurement,} \\ \sqrt{2\epsilon_{\max}}\rho - 1/2 & \text{detection.} \end{cases}$$

# Accuracy Standards for LIGO

- It is useful to define the model waveform (logarithmic) amplitude $\delta\chi_m$ and phase $\delta\Phi_m$ errors:
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- The basic accuracy requirements can be written as

$$\frac{\sqrt{\langle\delta h|\delta h\rangle}}{\rho} = \sqrt{\overline{\delta\chi_m}^2 + \overline{\delta\Phi_m}^2} < \begin{cases} 1/(2\rho_{\max}) & \text{measurement,} \\ \sqrt{2\epsilon_{\max}} - 1/(2\rho_{\max}) & \text{detection,} \end{cases}$$

where the signal-weighted average errors are defined as

$$\overline{\delta\chi_m}^2 = \int_{-\infty}^{\infty} \delta\chi_m^2 \frac{2|h_e|^2}{\rho^2 S_n} df, \quad \text{and} \quad \overline{\delta\Phi_m}^2 = \int_{-\infty}^{\infty} \delta\Phi_m^2 \frac{2|h_e|^2}{\rho^2 S_n} df.$$

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- The most restrictive measurement standards are needed for the strongest gravitational wave signals. For Advanced LIGO the maximum signal-to-noise ratio unlikely larger than $\rho_{\max} \approx 100$.

$$\sqrt{\overline{\delta\chi_R}^2 + \overline{\delta\Phi_R}^2} \approx \sqrt{\overline{\delta\chi_m}^2 + \overline{\delta\Phi_m}^2} < \frac{1}{2\rho_{\max}} \approx 0.005.$$

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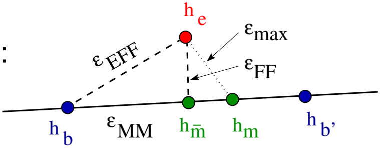


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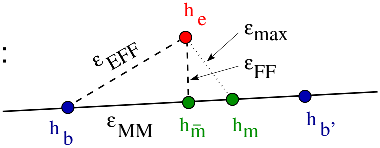


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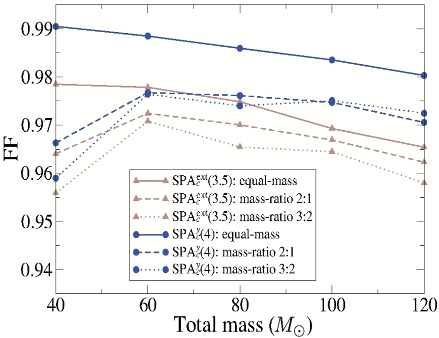
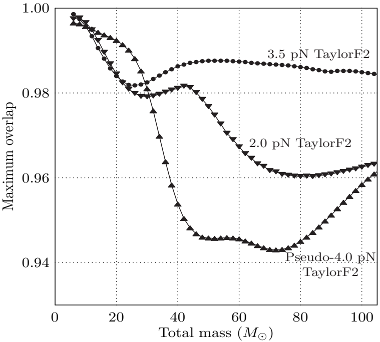
- To ensure this condition, $\epsilon_{\text{max}}$ must be chosen so that $\epsilon_{\text{max}} \leq 0.005$.
- Accuracy requirement for BBH waveforms for detection in LIGO:

$$\sqrt{\overline{\delta\chi_m}^2 + \overline{\delta\Phi_m}^2} < \sqrt{2\epsilon_{\text{max}}} - 1/(2\rho_{\text{max}}) \approx 0.095.$$

# How good are current LIGO templates?

- Studies by Pan, et al. Phys.Rev. D77,024014 (2008), and by Boyle, et al. CQG 26, 114006 (2009) suggest $\epsilon_{FF}$ for current non-spinning LIGO templates may be as large as 0.04.




- The effective range $R_{BBH}$ for BBH detections may therefore be reduced by up to $(1-\epsilon_{FF}-\epsilon_{MM})R_{BBH} \approx 0.93R_{BBH}$, resulting in an event loss rate that may be as large as $1-(1-\epsilon_{FF}-\epsilon_{MM})^3 \approx 0.2$.

# Verifying Calibration Accuracy

- The standards place limits on the signal- and noise-weighted averages of the frequency-domain amplitude and phase errors of the response function $R = R_e\, e^{\delta\chi_R + i\delta\Phi_R}$:

$$\overline{\delta\chi_R}^2 + \overline{\delta\Phi_R}^2 < 1/(4\rho^2_{\max})$$

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- This can be resolved by enforcing the somewhat stronger sufficient conditions:

$$\begin{aligned}\overline{\delta\chi_R}^2 + \overline{\delta\Phi_R}^2 &= \int_0^\infty \left[(\delta\chi_R)^2 + (\delta\Phi_R)^2\right] \frac{4|h_e|^2}{\rho^2 S_n(f)} df,\\ &\leq max\left[(\delta\chi_R)^2 + (\delta\Phi_R)^2\right] < 1/(4\rho^2_{\rm max}).\end{aligned}$$

# Verifying NR Waveform Accuracy

- The standards also place limits on the signal- and noise-weighted averages of the waveform amplitude and phase errors:

$$\sqrt{\frac{\langle\delta h_m|\delta h_m\rangle}{\rho^2}} = \sqrt{\overline{\delta\chi_m}^2 + \overline{\delta\Phi_m}^2} < \begin{cases} 1/(2\rho_{\rm max}) & \text{measurement,} \\ \sqrt{2\epsilon_{\rm max}} & \text{detection.} \end{cases}$$

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- How can NR waveforms be checked against these standards?
- Express the time-domain waveform in terms of an amplitude $A_e(t)$ and phase $\Phi_e(t)$ of the “exact” waveform,

$$h_e(t) = A_e(t) \cos \Phi_e(t),$$

plus errors,

$$h_m(t) = A_e(t)\left[1 + \delta\mu_\chi g_\chi(t)\right] \cos\left[\Phi_e(t) + \delta\mu_\Phi g_\Phi(t)\right],$$

where $\delta\mu_\chi$ and $\delta\mu_\Phi$ are the maximum amplitude and phase errors so that $|g_\chi(t)| \leq 1$ and $|g_\Phi(t)| \leq 1$.

# Verifying NR Waveform Accuracy II

- Some NR groups have estimated the maximum time-domain waveform errors $\delta\mu_\chi$ and $\delta\mu_\Phi$, and compared them with the standards for $|\overline{\delta\chi_m}|$ and $|\overline{\delta\Phi_m}|$.
- Is this good enough?

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- Consider a model waveform: $h_m(t)$ with errors of the form:

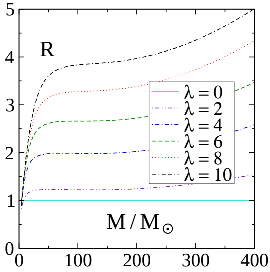
$$h_m(t) = A_e(t)\left[1 + \delta\mu_\chi g_\chi(t)\right]\cos\left[\Phi_e(t) + \delta\mu_\Phi g_\Phi(t)\right],$$

with $g_\chi = g_\Phi = \cos[\lambda\Phi_e(t)]$.

- Compute ratio of frequency- to time-domain error measures,

$$R = \sqrt{\frac{\overline{\delta\chi_m}^2 + \overline{\delta\Phi_m}^2}{\delta\mu_\chi^2 + \delta\mu_\Phi^2}}$$

using the PN+Caltech/Cornell waveform for $A_e$ and $\Phi_e$.



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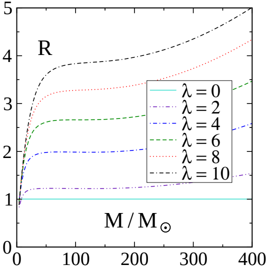


- Bad News! Limiting $\delta\mu_\chi$ and $\delta\mu_\Phi$ to the standards is not sufficient.

# Verifying NR Waveform Accuracy III

- Additional knowledge of the full waveform errors, $\delta\mu_\chi g_\chi(t)$ and $\delta\mu_\Phi g_\Phi(t)$, is needed. Unfortunately the exact time dependencies, $g_\chi(t)$ and $g_\Phi(t)$, will never be known.
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- Probably the most we will ever know will be local-in-time error envelope-functions $G_\chi(t)$ and $G_\Phi(t)$, that satisfy

$$|g_\chi(t)| \leq G_\chi(t) \leq 1, \quad \text{and} \quad |g_\Phi(t)| \leq G_\Phi(t) \leq 1.$$

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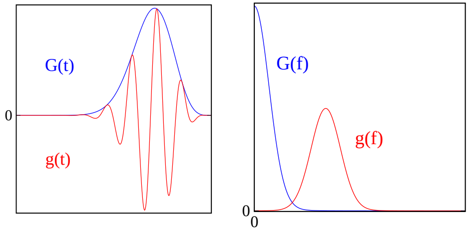
- Do time-domain bounds imply frequency-domain bounds, i.e., does $|g(t)| \le G(t)$ imply $|g(f)| \le G(f)$?
- No!
- It is not possible to verify the accuracy of a waveform using a time-domain error-envelope function.



# Alternate Waveform Accuracy Requirements

- This seems like a disaster: error envelope functions are probably the most we will ever know about waveform errors, yet they do not provide useful estimates of the relevant error norms.
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- A local-in-time error envelope $G(t)$ does provide a bound on the $L^2$ norm of the frequency-domain waveform error:

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- A waveform accuracy requirement based on $L^2$ norms, rather than the usual noise-weighted norm, could therefore be implemented using local-in-time error bounds

# $L^2$ Norm Accuracy Standard

- We can derive an accuracy requirement based on $L^2$ norms:

$$\langle \delta h_m | \delta h_m \rangle = 2 \int_{-\infty}^{\infty} \frac{|\delta h_m|^2}{S_n(f)} df \leq \frac{2||\delta h_m(f)||^2}{\min S_n(f)},$$

where $||\delta h_m(f)||^2 = \int_{-\infty}^{\infty} |\delta h_m|^2 df$ is the $L^2$ norm of $\delta h_m(f)$.

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- We can therefore convert the basic accuracy requirements (on measurement in this case) into the following sufficient condition:

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- This accuracy requirement demands the waveform $h_m$ and its error-envelope estimate $\delta h_m$ to have the proper scale.
- NR simulations only determine the scale invariant $r\, h_m/M$ and $r \delta h_m/M$, so what value of the scale $r$ should be used?

# $L^2$ Norm Accuracy Standards II

- A scale invariant accuracy standard can be constructed by introducing the obvious $L^2$ norm waveform scale:

$$\frac{||\delta h(f)||}{||h_m(f)||} = \frac{||\delta h(t)||}{||h_m(t)||} < \frac{\sqrt{\min\ S_n}}{2\sqrt{2}||h_m||}.$$

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- Unfortunately, the right side of this new condition depends on $||h_m||$, which must still be scaled properly.
- Introduce the scale invariant quantity $C$, defined as

$$C^2 = \frac{\rho^2}{2||h_m(f)||^2/\min S_n(f)} \leq 1,$$

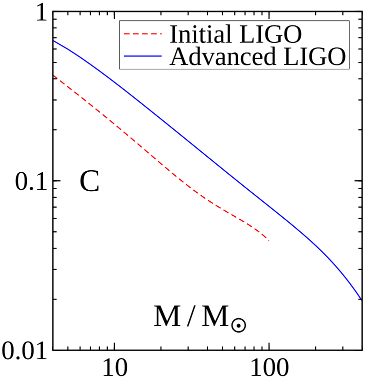
and use it to re-write the accuracy standards,

$$\frac{||\delta h(f)||}{||h_m(f)||} = \frac{||\delta h(t)||}{||h_m(t)||} < \frac{C}{2\rho},$$

in a way that depends on the waveform scale only through the standard signal-to-noise ratio $\rho$.

# Sufficient Conditions for LIGO

- The signal-to-noise quantity $C^2 = \rho^2 \min S_n / 2||h_m||^2 \leq 1$ has been evaluated for equal-mass non-spinning BBH waveforms using LIGO noise.



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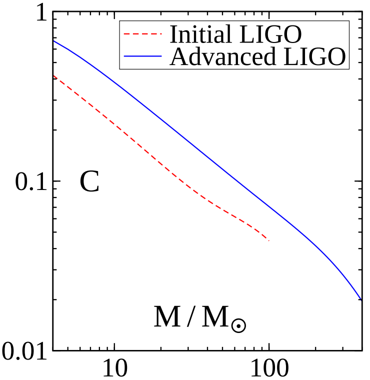
- Sufficient accuracy requirements for BBH waveforms for Advanced LIGO are therefore:




$$\frac{||\delta h_m(t)||}{||h_m(t)||} \lesssim \begin{cases} C/2\rho \approx \frac{0.02}{200} \approx 10^{-4} & \text{measurement,} \\ C\sqrt{2\epsilon_{\max}} \approx 0.02 \times 0.1 \approx 2 \times 10^{-3} & \text{detection.} \end{cases}$$

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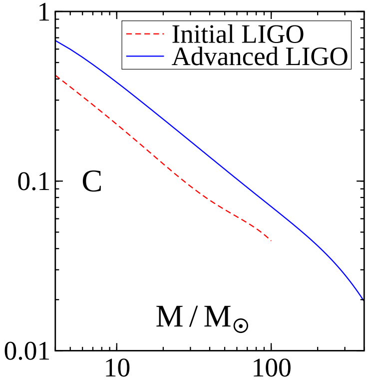


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- These requirements can be enforced as conditions on local-in-time bounds of the amplitude and phase errors:

$$\frac{||\delta h_m(t)||}{||h_m(t)||} \leq \sqrt{\frac{\int_{-\infty}^{\infty} A_m^2 \left(\delta\mu_\chi^2 G_\chi^2 + \delta\mu_\Phi^2 G_\Phi^2\right) dt}{\int_{-\infty}^{\infty} A_m^2 dt}} \lesssim \begin{cases} C/2\rho & \text{measurement} \\ C\sqrt{2\epsilon_{max}} & \text{detection} \end{cases}$$

# Summary and Questions

- A set of accuracy standards now exist for detector calibration, $\sqrt{\overline{\delta\chi_R}^2 + \overline{\delta\Phi_R}^2} < 1/(2\rho_{\max})$, and for model waveforms,

$$\sqrt{\overline{\delta\chi_m}^2 + \overline{\delta\Phi_m}^2} < \begin{cases} 1/(2\rho_{\max}) & \text{measurement,} \\ \sqrt{2\epsilon_{\max}} - 1/(2\rho_{\max}) & \text{detection.} \end{cases}$$

- These standards are difficult (impossible?) to enforce directly, so easier to enforce conditions have been derived, for calibration $\sqrt{\max[(\delta\chi_R)^2 + (\delta\Phi_R)^2]} < 1/(2\rho_{\max})$, and for waveforms:

$$\frac{||\delta h_m(t)||}{||h_m(t)||} \leq \sqrt{\frac{\int_{-\infty}^{\infty} A_m^2 \left(\delta\mu_\chi^2 G_\chi^2 + \delta\mu_\Phi^2 G_\Phi^2\right) dt}{\int_{-\infty}^{\infty} A_m^2 dt}} \lesssim \begin{cases} C/(2\rho_{\max}) & \text{measure,} \\ C\sqrt{2\epsilon_{\max}} & \text{detection.} \end{cases}$$

- Do the calibration and search template accuracies currently being used by LIGO satisfy these requirements?
- Do the waveforms produced by various NR groups satisfy the Advanced LIGO versions of these accuracy requirements?