Corrigendum: The loudest event statistic: general formulation, properties and applications

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Abstract. The *loudest event statistic*, a method by which the rate at which events occur can be deduced from the significance of the most significant event (or *loudest* event), has been employed in several papers describing the search for gravitational waves produced by coalescing compact binaries in data from the LIGO and Virgo observatories. The paper "The loudest event statistic: general formulation, properties and applications" 2009 *Class. Quantum Grav.* **26** 175009 [1] presents a general formulation of the loudest event statistic and addresses topics on the estimation of rate intervals, on combining multiple experiments, and on marginalizing over uncertainties in parameters. A conceptual error in Sec. 5 of [1] led to invalid results regarding the marginalization over uncertainties in the averaged detection efficiency; specifically its Eqs. (23) and (24) are incorrect, as are its Eqs. (25) and (27). This *Corrigendum* presents a correct treatment of the marginalization of uncertainties in the estimated detection efficiency.

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During the internal collaboration review of the search for gravitational waves from low mass compact binary coalescence in LIGO's sixth science run and Virgo's science runs 2 and 3 [2], an inconsistency in results obtained from two different methods of marginalization over uncertainties in the measured value of the search's detection efficiency revealed there was an error in Sec. 5.1, Eqs. (23) and (24), of Ref. [1]. Here we provide a correct version of Sec. 5.1 of Ref. [1]. At the end we give the trivial correction to Sec. 5.2.

Our discussion begins with Eq. (11) of Ref. [1], which expresses the posterior distribution for the mean number of events μ with a ranking statistic above some largest observed loudness parameter $\hat{x}$ (the "loudest event") as

$$p(\mu \mid \hat{\epsilon}, \hat{\Lambda}) = \frac{p(\mu)(1+\mu\hat{\epsilon}\hat{\Lambda})\mathrm{e}^{-\mu\hat{\epsilon}}}{\int d\mu\, p(\mu)(1+\mu\hat{\epsilon}\hat{\Lambda})\mathrm{e}^{-\mu\hat{\epsilon}}} \qquad (1)$$

where $\hat{\epsilon} = \epsilon(\hat{x})$ is the average detection efficiency function evaluated at the ranking statistic value of the loudest event observed, and $\hat{\Lambda} = \Lambda(\hat{x})$ is a measure of the relative probability of the loudest event arising from a foreground signal versus such an event occurring due to the experimental background; a value $\hat{\Lambda} = 0$ results if the loudest event is definitely from the background, and $\hat{\Lambda} \to \infty$ in the limit that the loudest event is definitely from the foreground. The function $\Lambda(x)$ is defined by Eq. (12) of Ref. [1]. A Poisson-distributed foreground rate with mean number of events μ is assumed, while $p(\mu)$ is the prior unconditional probability distribution for this quantity. The detection efficiency $\epsilon(x)$ is a monotonic function of the ranking statistic x that describes the probability that a single foreground event provided by nature will have a ranking statistic at least as significant as the value x; therefore, it is natural to describe events in terms of the detection efficiency variable ϵ, and, in particular, the loudest event will have the value $\hat{\epsilon}$, which is the *smallest* value of ϵ that occurs during the experiment.

Equation (1) is obtained from the likelihood function,

$$p(x \mid \mu, B) = p_0(x)[1 + \mu\epsilon(x)\Lambda(x)]\mathrm{e}^{-\mu\epsilon(x)}, \tag{2}$$

via the application of Bayes's theorem,

$$p(\mu \mid \hat{\epsilon}, \hat{\Lambda}) = \frac{p(\mu)p(\hat{x} \mid \mu, B)}{\int d\mu\, p(\mu)p(\hat{x} \mid \mu, B)}, \tag{3}$$

where B indicates dependence of the probability distribution on the background. The function $p_0(x)$ is the unconditional probability distribution of the experimental *background* producing an event having a loudest event x, while the function $p(x \mid \mu, B)$ is the probability distribution for x in an experiment having both a background and a foreground as described above. All dependence of the posterior distribution on the background is contained in the value $\hat{\Lambda}$.

Section 5.1 of Ref. [1] was concerned with the situation in which the value of $\hat{\epsilon}$ is not known exactly‡. Suppose that the *true* value of efficiency is related to the measured value via a factor α so that

$$\epsilon(x) = \alpha\epsilon_{\text{meas}}(x). \tag{4}$$

Given an observed loudest event, and consequently a value of $\hat{\epsilon}_{\text{meas}} = \epsilon_{\text{meas}}(\hat{x})$, we wish to infer the value of the efficiency and also the posterior distribution on for μ. We suppose that, for a given $\hat{\epsilon}_{\text{meas}}$, the probability distribution for the true (but unknown) efficiency $\hat{\epsilon}$ is $p(\hat{\epsilon} \mid \hat{\epsilon}_{\text{meas}})$. For simplicity, and in keeping with Ref. [1], we choose the gamma-distribution,

$$p(\hat{\epsilon}; k, \theta) = \frac{1}{\theta^k\Gamma(k)}\, \hat{\epsilon}^{k-1}\mathrm{e}^{-\hat{\epsilon}/\theta}, \tag{5}$$

where the two parameters θ and k determine the mean, $k\theta$, and the variance, $k\theta^2$ [cf. Eq. (22) of Ref. [1]]. If we assume that the mean of this distribution is equal to the measured value of the efficiency $\hat{\epsilon}_{\text{meas}}$, then we fix the *scale parameter* to be $\theta = \hat{\epsilon}_{\text{meas}}/k$ to obtain $p(\hat{\epsilon} \mid \hat{\epsilon}_{\text{meas}}, k) = p(\hat{\epsilon}; k, \theta = \hat{\epsilon}_{\text{meas}}/k)$; the *shape parameter* k then specifies the width of the distribution. When k is large, the gamma-distribution will be sharply-peaked about its mean value. Equation (5) can be re-expressed as a probability distribution for the scale factor α in Eq. (4) as

$$p(\alpha; k) = \frac{1}{\Gamma(k)}\, k^k\alpha^{k-1}\mathrm{e}^{-k\alpha}. \tag{6}$$

‡ In the context of gravitational wave searches, such as the one described in [2], the search efficiency is estimated using Monte Carlo methods in which simulated signals from a hypothetical distribution are added to the detector noise, and the fraction of such signals with ranking statistics greater than loudest event is the measured value of $\hat{\epsilon}$; this value has a measurement uncertainty due to the finite number of trials performed.

The scale factor α is a random variable describing the uncertainty in the measurement of the efficiency. We must now derive a modified version of Eq. (1) that accounts for this additional random variable. We may either (i) construct a *marginal likelihood* by marginalizing over this uncertainty in the likelihood function and then proceed to use Bayes's theorem to obtain a posterior distribution for μ, or (ii) treat α as a nuisance parameter in the posterior distribution (see, e.g., Sec. 36.1.4.2 of Ref. [3]). These should yield the same result, but it is instructive to derive the result using both methods.

We employ method (i) first. We use Eqs. (2) and (4) to write the likelihood function

$$p(x \mid \mu, \alpha, B) = p_0(x)[1 + \mu\alpha\epsilon_{\text{meas}}(x)\Lambda(x)]\mathrm{e}^{-\mu\alpha\epsilon_{\text{meas}}(x)}. \tag{7}$$

We now construct a marginal likelihood, $p(x \mid \mu, k, B)$, by marginalizing over α,

$$p(x \mid \mu, k, B) = \int_0^\infty d\alpha\, p(\alpha; k)p(x \mid \mu, \alpha, B), \tag{8}$$

and using our gamma-distribution for α we find

$$\begin{aligned} p(x \mid \mu, k, B) &= p_0(x)\frac{k^k}{\Gamma(k)} \\ &\qquad \times \int_0^\infty d\alpha\, \alpha^{k-1}[1 + \mu\alpha\epsilon_{\text{meas}}(x)\Lambda(x)]\mathrm{e}^{-\alpha[k+\mu\epsilon_{\text{meas}}(x)]} \\ &= p_0(x)\frac{1 + \mu\epsilon_{\text{meas}}(x)[\Lambda(x) + 1/k]}{(1 + \mu\epsilon_{\text{meas}}(x)/k)^{k+1}}. \end{aligned} \tag{9}$$

Note that in the limit $k \to \infty$ we recover Eq. (2) with $\epsilon(x)$ replaced with $\epsilon_{\text{meas}}(x)$. We now obtain a posterior distribution for μ from Bayes's theorem,

$$p(\mu \mid k, \hat{\epsilon}_{\text{meas}}, \hat{\Lambda}) = \frac{p(\mu)p(\hat{x} \mid \mu, k, B)}{p(\hat{x} \mid k, B)}, \tag{10}$$

where the marginal probability for x is

$$p(x \mid k, B) = \int_0^\infty d\mu\, p(\mu)p(x \mid \mu, k, B). \tag{11}$$

For illustration we take a uniform prior, $p(\mu) = \text{const}$, cf. Eq. (13) of Ref. [1]. We have

$$\begin{aligned} p(x \mid k, B) &= \int_0^\infty d\mu\, p_0(x)\frac{1 + \mu\epsilon_{\text{meas}}(x)[\Lambda(x) + 1/k]}{[1 + \mu\epsilon_{\text{meas}}(x)/k]^{k+1}} \\ &= p_0(x)\frac{k}{k-1}\frac{1 + \Lambda(x)}{\epsilon_{\text{meas}}(x)} \end{aligned} \tag{12}$$

and therefore the posterior distribution is

$$p(\mu \mid k, \hat{\epsilon}_{\text{meas}}, \hat{\Lambda}) = \frac{k-1}{k}\frac{\hat{\epsilon}_{\text{meas}}}{1 + \hat{\Lambda}}\frac{1 + \mu\hat{\epsilon}_{\text{meas}}(\hat{\Lambda} + 1/k)}{(1 + \mu\hat{\epsilon}_{\text{meas}}/k)^{k+1}}. \tag{13}$$

Equation (13) is the correct version of Eq. (24) of Ref. [1].

An alternative, but equivalent, method of handling the parameter α is to treat it as a nuisance parameter in a posterior distribution $p(\mu, \alpha \mid k, \hat{\epsilon}_{\text{meas}}, \hat{\Lambda})$ and to integrate it out:

$$p(\mu \mid k, \hat{\epsilon}_{\text{meas}}, \hat{\Lambda}) = \int_0^\infty d\alpha\, p(\mu, \alpha \mid k, \hat{\epsilon}_{\text{meas}}, \hat{\Lambda}). \tag{14}$$

From Bayes's theorem, we see that the posterior with the nuisance parameter is given by

$$p(\mu, \alpha \mid k, \hat{\epsilon}_{\text{meas}}, \hat{\Lambda}) = \frac{p(\mu)p(\alpha; k)p(\hat{x} \mid \mu, \alpha, B)}{p(\hat{x} \mid k, B)}, \tag{15}$$

where

$$p(x \mid k, B) = \int_0^\infty d\mu\, p(\mu) \int_0^\infty d\alpha\, p(\alpha; k) p(x \mid \mu, \alpha, B) \tag{16}$$

is exactly the marginal probability for x given earlier, as can be seen by substituting Eq. (8) into Eq. (11). Finally, inserting Eq. (15) into Eq. (14) gives us

$$\begin{aligned} p(\mu \mid k, \hat{\epsilon}_{\text{meas}}, \hat{\Lambda}) &= \frac{p(\mu)}{p(\hat{x} \mid k, B)} \int_0^\infty d\alpha\, p(\alpha; k) p(\hat{x} \mid \mu, \alpha, B) \\ &= \frac{p(\mu) p(\hat{x} \mid \mu, k, B)}{p(\hat{x} \mid k, B)} \end{aligned} \tag{17}$$

which is the same as Eq. (10). As expected we obtain the same final result for the posterior distribution if we consider α to be a nuisance parameter that we integrate out or if we marginalize the likelihood over the parameter α.

We can now explain the mistake in Ref. [1]. Note that we had to modify Eq. (1) to account for the addition of the random variable α, while Ref. [1] used its Eq. (11) without modification. Thus the substitution of Eq. (11) into Eq. (23) in Ref. [1] is not equivalent to the fundamental definition of the marginalized posterior, given by Eq. (10) or (14) here.

The same mistake was made in Sec. 5.2 of Ref. [1]. Its Eq. (25) is the incorrect starting point. Instead, if we suppose $\Lambda(x) = \beta \Lambda_{\text{meas}}(x)$ where β is a random variable with expectation value of unity, then it is straightforward to get the correct result following the methods explained here. The easiest approach is to first compute the marginal likelihood, which in this case is obtained by substituting $\Lambda(x) = \beta \Lambda_{\text{meas}}(x)$ in Eq. (2), multiplying by $p(\beta)$, and integrating over β. The result has the exact same form with $\Lambda(x) \to \Lambda_{\text{meas}}(x)$. *It then follows that the correct version of Eq. (27) in Ref. [1] is its Eq. (14) with this same substitution.* Thus, the posterior marginalized over uncertainties in $\hat{\Lambda}_{\text{meas}}$ does not depend on the variance of this uncertainty at all, and Eqs. (26) and (28) in Ref. [1] are superfluous. This also means marginalizing over uncertainties in $\hat{\Lambda}_{\text{meas}}$ will not change upper limits on the rate at all. The conclusion of Sec. 5.2 of Ref. [1] is still correct: marginalization over uncertainties in $\hat{\Lambda}_{\text{meas}}$ can be neglected.

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