Thermal Noise of LIGO Mirrors

- A Theoretical study using FdT -

1. Degree of Freedom
2. Fluctuation-Dissipation Theorem
3. Generalized Susceptibility
4. Mirror Fluctuation
5. Summary

Yasushi Mino (□ □□)
TAPIR, Caltech

G0900646-v1

Thermal Noise of LIGO Mirrors

1. Degree of Freedom

ex. Ideal gas equation of state $V = \frac{NkT}{P}$

Volume is the function of both temperature and pressure.
The state of ideal gas is characterized by two variables;

$$\{V, P\} \text{ or } \{V, T\} \text{ or } \{T, P\}$$

What characterize the state of the mirror?
It is the thermo-elastic system;

Elastic degree of freedom $u^i(x)$ $p^i(x) := \frac{1}{\rho}\dot{u}^i(x)$

Thermal degree of freedom $T(x)$

One may use the internal energy density, instead of the temperature.

All the physical coefficients of the mirror are the function of these variables. To scope out the noise property of the mirror, we need to know;

$$\langle u^i(x,t)u^j(x',t')\rangle \quad \langle u^i(x,t)T(x',t')\rangle \quad \langle T(x,t)T(x',t')\rangle$$

2. Fluctuation-Dissipation Theorem
- stochastic point of view -

The macroscopic evolution equation; $\frac{d}{dt} X^A = -\lambda^A{}_B X^B$

To have the microscopic system in the equilibrium, there must be a stochastic force induced by the heat bath;

$$\frac{d}{dt} X^A = -\lambda^A{}_B X^B + F^A$$

What is the statistical nature of the stochastic force?

It is markovian b/c the heat bath has no time scale.
This is determined by the detailed-balance.
It is made to be consistent with the equilibrium distribution;

$$\left\langle X^A(0) X^B(0) \right\rangle = \left[\beta^{-1}\right]^{AB} \longrightarrow \left\langle F^A(t) F^B(t') \right\rangle = \gamma^{AB} \delta(t - t')$$

$$\gamma^{AB} = \lambda^A{}_C \left[\beta^{-1}\right]^{CB}$$

2. Fluctuation-Dissipation Theorem (con't)

The result is summarized by using the GENERALIZED SUSCEPTIBILITY.

Susceptibility ; $$-i\omega\alpha^{AB} = -\lambda^{A}{}_{C}\alpha^{CB} + \frac{\gamma^{AB}}{kT}$$

Fluctuation ; $$\left\langle X^{A}(t)X^{B}(0)\right\rangle = \frac{1}{2\pi}\int d\omega e^{-i\omega t} S^{AB}(\omega)$$

Fluctuation-Dissipation Theorem ;

$$S^{AB}(\omega) = \frac{iT}{\omega}\left(\alpha^{BA*} - \alpha^{AB}\right)$$

We have 4x4 partial differential equations for 4x4 generalized susceptibility,

$$\alpha\left[u^{i}(x), u^{j}(x')\right] \quad \alpha\left[T(x), u^{j}(x')\right]$$
$$\alpha\left[u^{i}(x), T(x')\right] \quad \alpha\left[T(x), T(x')\right]$$

Equations for the generalized susceptibility;

$$\begin{aligned}
\delta^{ij}\delta(x-x') &= -\omega^2\rho_0\alpha[u^i(x),u^j(x')] + \alpha K\partial_i\alpha[T(x),u^j(x')] \\
&\quad -\left(K+\frac{1}{3}M\right)\partial_i\partial_k\alpha[u^k(x),u^j(x')] - M\partial_k\partial_k\alpha[u^i(x),u^j(x')]\,, \\
\frac{DT_0}{C_v^2}\frac{\alpha\mathbf{K}}{\bar{\mathbf{K}}+\mathbf{M}}\partial_i\delta(x-x') &= -i\omega\alpha[T(x),u^i(x')] - \frac{D}{C_v}\partial_j\partial_j\alpha[T(x),u^i(x')] \\
&\quad -i\omega\frac{T_0}{C_v}\alpha K\partial_j\alpha[u^j(x),u^i(x')]\,,
\end{aligned}$$

* These correspond to a generalization of Levin’s method of the pressure injection. One needs the heat injection for a consistent calculation.

$$\begin{aligned}
0 &= -\omega^2\rho_0\alpha[u^i(x),T(x')] + \alpha K\partial_i\alpha[T(x),T(x')] \\
&\quad -\left(K+\frac{1}{3}M\right)\partial_i\partial_j\alpha[u^j(x),T(x')] - M\partial_j\partial_j\alpha[u^i(x),T(x')]\,, \\
-\frac{DT_0}{C_v^2}\frac{\mathbf{K}+\frac{4}{3}\mathbf{M}}{\bar{\mathbf{K}}+\mathbf{M}}\partial_i\partial_i\delta(x-x') &= -i\omega\alpha[T(x),T(x')] - \frac{D}{C_v}\partial_i\partial_i\alpha[T(x),T(x')] \\
&\quad -i\omega\frac{T_0}{C_v}\alpha K\partial_i\alpha[u^i(x),T(x')]\,.
\end{aligned}$$

* These correspond to a generalization of Levin’s method of the heat injection.

3. Generalized Susceptibility

We consider the following effects;

1: both thermal dissipation and elastic friction

2: multi-layer coating

For an analytic calculation, we made following simplification;

1: half-infinite mirror

$$e^{-\frac{(\text{beam radius})^2}{(\text{mirror surface})}} \sim 10^{-20} << 1 \,, e^{-\frac{(\text{beam radius})}{(\text{mirror thickness})}} \sim 10^{-10} << 1$$

2: quasi-static elastic dynamics

$$\frac{\frac{(\text{mirror length scale})}{(\text{observation time})}}{(\text{sound velocity})} = \frac{(\text{mirror length scale})}{(\text{sound-crossing length})} \sim 10^{-4} << 1$$

3: short diffusion length

$$\frac{\sqrt{\frac{(\text{observation time})}{(\text{diffusion coefficient})}}}{(\text{beam size})} = \frac{(\text{diffusion length})}{(\text{beam size})} \sim 10^{-3} << 1$$

4: small expansion

$$(\text{thermo-elastic coupling}) \sim \frac{T_0}{C_v}\alpha^2 K \sim 10^{-4} << 1$$

5: thin coating layer

$$\frac{(\text{coating thickness})}{(\text{diffusion length})} \sim 10^{-2} << 1$$

$$\alpha[u^z(x,z_m),u^z(x',z_n)] \rightarrow \frac{1}{\kappa}\frac{K^S+\frac{4}{3}M^S}{2\left(K^S+\frac{1}{3}M^S\right)M^S}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')},$$

$$\alpha[\partial_i u^i(x,z_m),u^z(x',z_n)] \rightarrow \begin{cases} -\frac{1}{K^S+\frac{1}{3}M^S}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for even } m \text{ and } m>n \\ -\frac{K^S+\frac{1}{3}M^S+M^T}{(K^S+\frac{1}{3}M^S)(K^T+\frac{4}{3}M^T)}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for odd } m \text{ and } m>n \\ -\frac{M^S}{(K^S+\frac{1}{3}M^S)(K^S+\frac{4}{3}M^S)}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for even } m \text{ and } m<n \\ -\frac{M^T}{(K^S+\frac{1}{3}M^S)(K^T+\frac{4}{3}M^T)}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for odd } m \text{ and } m<n \\ -\frac{K^S+\frac{7}{3}M^S}{2(K^S+\frac{1}{3}M^S)(K^S+\frac{4}{3}M^S)}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for even } m=n \\ -\frac{K^S+\frac{1}{3}M^S+2M^T}{2(K^S+\frac{1}{3}M^S)(K^T+\frac{4}{3}M^T)}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for odd } m=n \end{cases}$$

$$\alpha[\partial_z u^z(x,z_m),u^z(x',z_n)] \rightarrow \begin{cases} -\frac{1}{2(K^S+\frac{1}{3}M^S)}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for even } m \text{ and } m>n \\ -\frac{2K^S-K^T+\frac{2}{3}M^S+\frac{2}{3}M^T}{2(K^S+\frac{1}{3}M^S)(K^T+\frac{4}{3}M^T)}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for odd } m \text{ and } m>n \\ \frac{K^S-\frac{2}{3}M^S}{2(K^S+\frac{1}{3}M^S)(K^S+\frac{4}{3}M^S)}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for even } m \text{ and } m<n \\ \frac{K^T-\frac{2}{3}M^T}{2(K^S+\frac{1}{3}M^S)(K^T+\frac{4}{3}M^T)}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for odd } m \text{ and } m<n \\ -\frac{M^S}{2(K^S+\frac{1}{3}M^S)(K^S+\frac{4}{3}M^S)}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for} \\ -\frac{K^S-K^T+\frac{1}{3}M^S+\frac{2}{3}M^T}{2(K^S+\frac{1}{3}M^S)(K^T+\frac{4}{3}M^T)}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for} \end{cases}$$

$$\alpha[T(x,z_m),u^z(x',z_n)] \rightarrow \begin{cases} 0 & \text{for } m>n \\ \frac{T_0}{C_v^S}\frac{\alpha^S\mathbf{K}^S}{\mathbf{K}^S+\frac{4}{3}\mathbf{M}^S}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for } m<n \text{ and e} \\ \frac{T_0}{C_v^T}\frac{\alpha^T\mathbf{K}^T}{\mathbf{K}^T+\frac{4}{3}\mathbf{M}^T}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for } m<n \text{ and o} \\ \frac{T_0}{C_v^S}\frac{\alpha^S\mathbf{K}^S}{2(\mathbf{K}^S+\frac{4}{3}\mathbf{M}^S)}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for even } m= \\ \frac{T_0}{C_v^T}\frac{\alpha^T\mathbf{K}^T}{2(\mathbf{K}^T+\frac{4}{3}\mathbf{M}^T)}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for odd } m= \end{cases}$$

even n:

$$\alpha[u^z(x,z_m),\partial_j u^j(x',z_n)] \rightarrow \begin{cases} -\frac{M^S}{(K^S+\frac{1}{3}M^S)(K^S+\frac{4}{3}M^S)}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for } m>n \\ -\frac{1}{K^S+\frac{1}{3}M^S}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for } m<n \\ -\frac{K^S+\frac{7}{3}M^S}{2(K^S+\frac{1}{3}M^S)(K^S+\frac{4}{3}M^S)}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for } m=n \end{cases}, \quad (4.5)$$

$$\alpha[\partial_i u^i(x,z_m),\partial_j u^j(x',z_n)] \rightarrow \begin{cases} \left(2\kappa\frac{M^S}{(K^S+\frac{1}{3}M^S)(K^S+\frac{4}{3}M^S)}-\tilde{\kappa}\frac{T_0}{C_v^S}\frac{(\alpha^S)^2\mathbf{K}^S K^S}{(\mathbf{K}^S+\frac{4}{3}\mathbf{M}^S)(K^S+\frac{4}{3}M^S)}\right)\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for even } m \\ \left(2\kappa\frac{M^T}{(K^S+\frac{1}{3}M^S)(K^T+\frac{4}{3}M^T)}-\tilde{\kappa}\frac{T_0}{C_v^S}\frac{(\alpha^S)^2\mathbf{K}^S K^T}{(\mathbf{K}^S+\frac{4}{3}\mathbf{M}^S)(K^T+\frac{4}{3}M^T)}\right)\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for odd } m \end{cases} \quad (4.6)$$

$$\alpha[\partial_z u^z(x,z_m),\partial_j u^j(x',z_n)] \rightarrow \begin{cases} \left(-\kappa\frac{K^S-\frac{2}{3}M^S}{(K^S+\frac{1}{3}M^S)(K^S+\frac{4}{3}M^S)}-\tilde{\kappa}\frac{T_0}{C_v^S}\frac{(\alpha^S)^2\mathbf{K}^S K^S}{(\mathbf{K}^S+\frac{4}{3}\mathbf{M}^S)(K^S+\frac{4}{3}M^S)}\right)\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for even } m \\ \left(-\kappa\frac{K^T-\frac{2}{3}M^T}{(K^S+\frac{1}{3}M^S)(K^T+\frac{4}{3}M^T)}-\tilde{\kappa}\frac{T_0}{C_v^S}\frac{(\alpha^S)^2\mathbf{K}^S K^T}{(\mathbf{K}^S+\frac{4}{3}\mathbf{M}^S)(K^T+\frac{4}{3}M^T)}\right)\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for odd } m \end{cases} \quad (4.7)$$

$$\alpha[T(x,z_m),\partial_j u^j(x',z_n)] \rightarrow \begin{cases} -\tilde{\kappa}\frac{T_0}{C_v^S}\frac{\alpha^S\mathbf{K}^S}{\mathbf{K}^S+\frac{4}{3}\mathbf{M}^S}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for } m\neq n \\ \left(-\tilde{\kappa}\frac{T_0}{C_v^S}\frac{\alpha^S\mathbf{K}^S}{\mathbf{K}^S+\frac{4}{3}\mathbf{M}^S}+\frac{T_0}{C_v^S}\frac{\alpha^S\mathbf{K}^S}{\mathbf{K}^S+\frac{4}{3}\mathbf{M}^S}\frac{1}{l_s}\right)\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for } m=n \end{cases}. \quad (4.8)$$

odd n:

$$\alpha[u^z(x,z_m),\partial_j u^j(x',z_n)] \rightarrow \begin{cases} -\frac{M^T}{(K^S+\frac{1}{3}M^S)(K^T+\frac{4}{3}M^T)}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for } m>n \\ -\frac{K^S+\frac{1}{3}M^S+M^T}{(K^S+\frac{1}{3}M^S)(K^T+\frac{4}{3}M^T)}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for } m<n \\ -\frac{K^S+\frac{1}{3}M^S+2M^T}{2(K^S+\frac{1}{3}M^S)(K^T+\frac{4}{3}M^T)}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for } m=n \end{cases}, \quad (4.9)$$

$$\alpha[\partial_i u^i(x,z_m),\partial_j u^j(x',z_n)] \rightarrow \begin{cases} \left(2\kappa\frac{M^T}{(K^S+\frac{1}{3}M^S)(K^T+\frac{4}{3}M^T)}-\tilde{\kappa}\frac{T_0}{C_v^S}\frac{\alpha^S\alpha^T K^S\mathbf{K}^T}{(K^S+\frac{4}{3}M^S)(\mathbf{K}^T+\frac{4}{3}\mathbf{M}^T)}\right)\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for even } m \\ \left(2\kappa\frac{(K^S+\frac{4}{3}M^S)(M^T)^2}{(K^S+\frac{1}{3}M^S)M^S(K^T+\frac{4}{3}M^T)^2}-\tilde{\kappa}\frac{T_0}{C_v^S}\frac{(\alpha^T)^2K^T\mathbf{K}^T}{(K^T+\frac{4}{3}M^T)(\mathbf{K}^T+\frac{4}{3}\mathbf{M}^T)}\right)\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for odd } m \end{cases} \quad (4.10)$$

$$\alpha[\partial_z u^z(x,z_m),\partial_j u^j(x',z_n)] \rightarrow \begin{cases} \left(-\kappa\frac{(K^S-\frac{2}{3}M^S)M^T}{(K^S+\frac{1}{3}M^S)M^S(K^T+\frac{4}{3}M^T)}-\tilde{\kappa}\frac{T_0}{C_v^S}\frac{\alpha^S\alpha^T K^S\mathbf{K}^T}{(K^S+\frac{4}{3}M^S)(\mathbf{K}^T+\frac{4}{3}\mathbf{M}^T)}\right)\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for even } m \\ \left(-\kappa\frac{(K^S+\frac{4}{3}M^S)(K^T-\frac{2}{3}M^T)M^T}{(K^S+\frac{1}{3}M^S)M^S(K^T+\frac{4}{3}M^T)^2}-\tilde{\kappa}\frac{T_0}{C_v^S}\frac{(\alpha^T)^2K^T\mathbf{K}^T}{(K^T+\frac{4}{3}M^T)(\mathbf{K}^T+\frac{4}{3}\mathbf{M}^T)}\right)\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for odd } m \end{cases} \quad (4.11)$$

G0900646-v1

$$\alpha[T(x,z_m),\partial_j u^j(x',z_n)] \rightarrow \begin{cases} -\tilde{\kappa}\frac{T_0}{C_v^S}\frac{\alpha^T\mathbf{K}^T}{\mathbf{K}^T+\frac{4}{3}\mathbf{M}^T}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for } m\neq n \\ \left(-\tilde{\kappa}\frac{T_0}{C_v^S}\frac{\alpha^T\mathbf{K}^T}{\mathbf{K}^T+\frac{4}{3}\mathbf{M}^T}+\frac{T_0}{C_v^T}\frac{\alpha^T\mathbf{K}^T}{\mathbf{K}^T+\frac{4}{3}\mathbf{M}^T}\frac{1}{l_n}\right)\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for } m=n \end{cases}. \quad (4.12)$$

(tentative results)

even n:

$$\alpha[u^z(x,z_m),\partial_z u^z(x',z_n)] \;\rightarrow\; \begin{cases} \frac{K^S-\frac{2}{3}M^S}{2\left(K^S+\frac{1}{3}M^S\right)\left(K^S+\frac{4}{3}M^S\right)}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for } m>n \\ -\frac{1}{2\left(K^S+\frac{1}{3}M^S\right)}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for } m<n \\ -\frac{M^S}{2\left(K^S+\frac{1}{3}M^S\right)\left(K^S+\frac{4}{3}M^S\right)}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for } m=n \end{cases}, \tag{4.13}$$

$$\alpha[\partial_i u^i(x,z_m),\partial_z u^z(x',z_n)] \;\rightarrow\; \begin{cases} \left(-\kappa\frac{K^S-\frac{2}{3}M^S}{\left(K^S+\frac{1}{3}M^S\right)\left(K^S+\frac{4}{3}M^S\right)}-\tilde{\kappa}\frac{T_0}{C_v^S}\frac{(\alpha^S)^2\mathbf{K}^S K^S}{\left(\mathbf{K}^S+\frac{4}{3}\mathbf{M}^S\right)\left(K^S+\frac{4}{3}M^S\right)}\right)\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for even } m \\ \left(-\kappa\frac{\left(K^S-\frac{2}{3}M^S\right)M^T}{\left(K^S+\frac{1}{3}M^S\right)M^S\left(K^T+\frac{4}{3}M^T\right)}-\tilde{\kappa}\frac{T_0}{C_v^S}\frac{(\alpha^S)^2\mathbf{K}^S K^T}{\left(\mathbf{K}^S+\frac{4}{3}\mathbf{M}^S\right)\left(K^T+\frac{4}{3}M^T\right)}\right)\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for odd } m \end{cases} \tag{4.14}$$

$$\alpha[\partial_z u^z(x,z_m),\partial_z u^z(x',z_n)] \;\rightarrow\; \begin{cases} \left(\kappa\frac{\left(K^S-\frac{2}{3}M^S\right)^2}{2\left(K^S+\frac{1}{3}M^S\right)M^S\left(K^S+\frac{4}{3}M^S\right)}-\tilde{\kappa}\frac{T_0}{C_v^S}\frac{(\alpha^S)^2\mathbf{K}^S K^S}{\left(\mathbf{K}^S+\frac{4}{3}\mathbf{M}^S\right)\left(K^S+\frac{4}{3}M^S\right)}\right)\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for even } m \\ \left(\kappa\frac{\left(K^S-\frac{2}{3}M^S\right)\left(K^T-\frac{2}{3}M^T\right)}{2\left(K^S+\frac{1}{3}M^S\right)M^S\left(K^T+\frac{4}{3}M^T\right)}-\tilde{\kappa}\frac{T_0}{C_v^S}\frac{(\alpha^S)^2\mathbf{K}^S K^T}{\left(\mathbf{K}^S+\frac{4}{3}\mathbf{M}^S\right)\left(K^T+\frac{4}{3}M^T\right)}\right)\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for odd } m \end{cases} \tag{4.15}$$

$$\alpha[T(x,z_m),\partial_z u^z(x',z_n)] \;\rightarrow\; \begin{cases} -\tilde{\kappa}\frac{T_0}{C_v^S}\frac{\alpha^S\mathbf{K}^S}{\mathbf{K}^S+\frac{4}{3}\mathbf{M}^S}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for } m\neq n \\ \left(-\tilde{\kappa}\frac{T_0}{C_v^S}\frac{\alpha^S\mathbf{K}^S}{\mathbf{K}^S+\frac{4}{3}\mathbf{M}^S}+\frac{T_0}{C_v^S}\frac{\alpha^S\mathbf{K}^S}{\mathbf{K}^S+\frac{4}{3}\mathbf{M}^S}\frac{1}{l_n}\right)\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for } m=n \end{cases}. \tag{4.16}$$

odd n:

$$\alpha[u^z(x,z_m),\partial_z u^z(x',z_n)] \;\rightarrow\; \begin{cases} \frac{K^T-\frac{2}{3}M^T}{2\left(K^S+\frac{1}{3}M^S\right)\left(K^T+\frac{4}{3}M^T\right)}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for } m>n \\ -\frac{2K^S-K^T+\frac{2}{3}M^S+\frac{2}{3}M^T}{2\left(K^S+\frac{1}{3}M^S\right)\left(K^T+\frac{4}{3}M^T\right)}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for } m<n \\ -\frac{K^S-K^T+\frac{1}{3}M^S+\frac{2}{3}M^T}{2\left(K^S+\frac{1}{3}M^S\right)\left(K^T+\frac{4}{3}M^T\right)}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for } m=n \end{cases}, \quad (4.1$$

$$\alpha[\partial_i u^i(x,z_m),\partial_z u^z(x',z_n)] \;\rightarrow\; \begin{cases} \left(-\kappa\frac{K^T-\frac{2}{3}M^T}{\left(K^S+\frac{1}{3}M^S\right)\left(K^T+\frac{4}{3}M^T\right)}-\tilde{\kappa}\frac{T_0}{C_v^S}\frac{\alpha^S\alpha^T K^S\mathbf{K}^T}{\left(K^S+\frac{4}{3}M^S\right)\left(\mathbf{K}^T+\frac{4}{3}\mathbf{M}^T\right)}\right)\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for even } m \\ \left(-\kappa\frac{\left(K^S+\frac{4}{3}M^S\right)\left(K^T-\frac{2}{3}M^T\right)M^T}{\left(K^S+\frac{1}{3}M^S\right)M^S\left(K^T+\frac{4}{3}M^T\right)^2}-\tilde{\kappa}\frac{T_0}{C_v^S}\frac{(\alpha^T)^2K^T\mathbf{K}^T}{\left(K^T+\frac{4}{3}M^T\right)\left(\mathbf{K}^T+\frac{4}{3}\mathbf{M}^T\right)}\right)\frac{1}{(2\pi} & \end{cases}$$

$$\alpha[\partial_z u^z(x,z_m),\partial_z u^z(x',z_n)] \;\rightarrow\; \begin{cases} \left(\kappa\frac{\left(K^S-\frac{2}{3}M^S\right)\left(K^T-\frac{2}{3}M^T\right)}{2\left(K^S+\frac{1}{3}M^S\right)M^S\left(K^T+\frac{4}{3}M^T\right)}-\tilde{\kappa}\frac{T_0}{C_v^S}\frac{\alpha^S\alpha^T K^S\mathbf{K}^T}{\left(K^S+\frac{4}{3}M^S\right)\left(\mathbf{K}^T+\frac{4}{3}\mathbf{M}^T\right)}\right)\frac{1}{(2\pi)} & \\ \left(-\kappa\frac{\left(K^S+\frac{4}{3}M^S\right)\left(K^T-\frac{2}{3}M^T\right)^2}{2\left(K^S+\frac{1}{3}M^S\right)M^S\left(K^T+\frac{4}{3}M^T\right)^2}-\tilde{\kappa}\frac{T_0}{C_v^S}\frac{(\alpha^T)^2K^T\mathbf{K}^T}{\left(K^T+\frac{4}{3}M^T\right)\left(\mathbf{K}^T+\frac{4}{3}\mathbf{M}^T\right)}\right)\frac{1}{(2} & \end{cases}$$

$$\alpha[T(x,z_m),\partial_z u^z(x',z_n)] \;\rightarrow\; \begin{cases} -\tilde{\kappa}\frac{T_0}{C_v^S}\frac{\alpha^T\mathbf{K}^T}{\mathbf{K}^T+\frac{4}{3}\mathbf{M}^T}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for } m\neq n \\ \left(-\tilde{\kappa}\frac{T_0}{C_v^S}\frac{\alpha^T\mathbf{K}^T}{\mathbf{K}^T+\frac{4}{3}\mathbf{M}^T}+\frac{T_0}{C_v^T}\frac{\alpha^T\mathbf{K}^T}{\mathbf{K}^T+\frac{4}{3}\mathbf{M}^T}\frac{1}{l_n}\right)\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for } m=n \end{cases}$$

even n:

$$\alpha[u^z(x,z_m),T(x',z_n)] \;\rightarrow\; \begin{cases} -\frac{T_0}{C_v^S}\frac{\alpha^S K^S\left(K^S+\frac{2}{3}M^S\right)}{\left(K^S+\frac{1}{3}M^S\right)\left(K^S+\frac{4}{3}M^S\right)}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for } m>n \\ -\frac{T_0}{C_v^S}\frac{2\alpha^S K^S}{K^S+\frac{4}{3}M^S}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for } m<n \\ -\frac{T_0}{C_v^S}\frac{3\alpha^S K^S\left(K^S+\frac{2}{3}M^S\right)}{2\left(K^S+\frac{1}{3}M^S\right)\left(K^S+\frac{4}{3}M^S\right)}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for } m=n \end{cases},$$

$$\alpha[\partial_i u^i(x,z_m),T(x',z_n)] \;\rightarrow\; \begin{cases} \tilde{\kappa}\frac{T_0}{C_v^S}\frac{\alpha^S K^S}{K^S+\frac{4}{3}M^S}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for even } m \\ \tilde{\kappa}\frac{T_0}{C_v^S}\frac{\alpha^T K^T}{K^T+\frac{4}{3}M^T}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for odd } m \end{cases},$$

$$\alpha[\partial_z u^z(x,z_m),T(x',z_n)] \;=\; \alpha[\partial_i u^i(x,z_m),T(x',z_n)],$$

$$\alpha[T(x,z_m),T(x',z_n)] \;\rightarrow\; \begin{cases} \tilde{\kappa}\frac{T_0}{C_v^S} & \text{for } m\neq n \\ \left(\tilde{\kappa}\frac{T_0}{C_v^S}+\frac{T_0}{C_v^S}\frac{1}{l_n}\right)\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for } m=n \end{cases}.$$

odd n:

$$\alpha[u^z(x,z_m),T(x',z_n)] \;\rightarrow\; \begin{cases} \left(-\frac{T_0}{C_v^S}\frac{\alpha^S K^S}{K^S+\frac{1}{3}M^S}-\frac{T_0}{C_v^T}\frac{\alpha^T K^T M^T}{\left(K^S+\frac{1}{3}M^S\right)\left(K^T+\frac{4}{3}M^T\right)}\right)\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for } m>n \\ \left(-\frac{T_0}{C_v^S}\frac{\alpha^S K^S}{K^S+\frac{1}{3}M^S}-\frac{T_0}{C_v^T}\frac{\alpha^T K^T\left(K^S+\frac{1}{3}M^S+M^T\right)}{\left(K^S+\frac{1}{3}M^S\right)\left(K^T+\frac{4}{3}M^T\right)}\right)\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for } m<n \\ \left(-\frac{T_0}{C_v^S}\frac{\alpha^S K^S}{K^S+\frac{1}{3}M^S}-\frac{T_0}{C_v^T}\frac{\alpha^T K^T\left(K^S+\frac{1}{3}M^S+2M^T\right)}{2\left(K^S+\frac{1}{3}M^S\right)\left(K^T+\frac{4}{3}M^T\right)}\right)\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for } m=n \end{cases}$$

$$\alpha[\partial_i u^i(x,z_m),T(x',z_n)] \;\rightarrow\; \begin{cases} \tilde{\kappa}\frac{T_0}{C_v^S}\frac{\alpha^S K^S}{K^S+\frac{4}{3}M^S}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for even } m \\ \tilde{\kappa}\frac{T_0}{C_v^S}\frac{\alpha^T K^T}{K^T+\frac{4}{3}M^T}\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for odd } m \end{cases},$$

$$\alpha[\partial_z u^z(x,z_m),T(x',z_n)] \;=\; \alpha[\partial_i u^i(x,z_m),T(x',z_n)],$$

$$\alpha[T(x,z_m),T(x',z_n)] \;\rightarrow\; \begin{cases} \tilde{\kappa}\frac{T_0}{C_v^S} & \text{for } m\neq n \\ \left(\tilde{\kappa}\frac{T_0}{C_v^S}+\frac{T_0}{C_v^T}\frac{1}{l_n}\right)\frac{1}{(2\pi)^2}e^{ik\cdot(x-x')} & \text{for } m=n \end{cases}.$$

(tentative results)

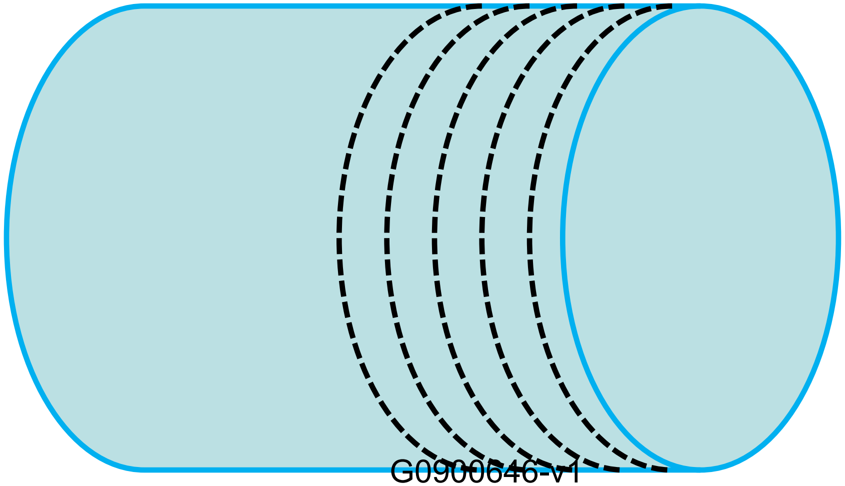
G0900646-v1

Thermal Noise of LIGO Mirrors

... for simplicity, here we only consider the fluctuation of

z-displacement	$u^z(x, y, z_n)$
Volume expansion	$\partial_i u^i(x, y, z_n)$
Thickness expansion	$\partial_z u^z(x, y, z_n)$
Temperature	$T(x, y, z_n)$

There are correlation b/w different layers.

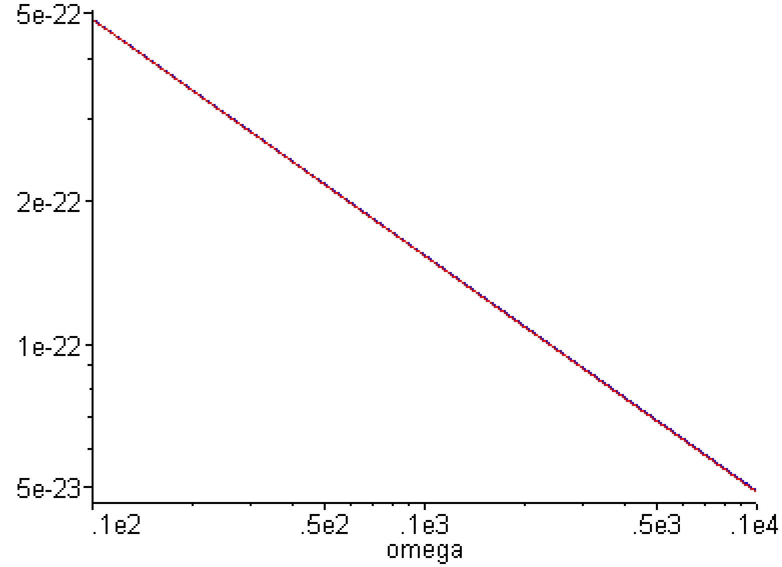


4. Mirror Fluctuation for Gaussian-shaped beam

$$
\begin{aligned}
S\{<u^z>_{2N+2},<u^z>_{2N+2}\} &= \frac{\sqrt{2}}{2\sqrt{\pi}}\frac{T_0}{\omega r_0} Im\left[(1-\nu_S^2)\frac{1}{Y^S}\right],\\
S\{<\partial_i u^i>_1,<u^z>_{2N+2})\} &= -\frac{1}{2\pi}\frac{T_0}{\omega r_0^2} Im\left[\frac{(1+\nu^S)(1-2\nu^S)(1-2\nu^T)}{1+\nu^T}\frac{1}{Y^S}\right],\\
S\{<\partial_z u^z>_1,<u^z>_{2N+2})\} &= \frac{1}{2\pi}\frac{T_0}{\omega r_0^2} Im\left[\frac{(1+\nu^S)(1-2\nu^S)\nu^T}{1-\nu^T}\frac{1}{Y^S}\right],\\
S\{<T>_1,<u^z>_{2N+2})\} &= \frac{1}{4\pi}\frac{T_0^2}{\omega r_0^2}\left\{-\frac{2}{3}\frac{\alpha^S}{C_v^S} m\left[\nu^S\right]-\frac{1}{3}\frac{\alpha^T}{C_v^T} Im\left[\frac{(1+\nu^S)(1-2\nu^S)}{1-\nu^T}\frac{Y^T}{Y^S}\right]\right\},\\
S\{<\partial_i u^i>_1,<\partial_j u^j>_1\} &= \frac{\sqrt{2}}{2\sqrt{\pi}}\frac{T_0}{\omega r_0^3} Im\left[\frac{(1-(\nu^S)^2)(1-2\nu^T)^2}{(1-\nu^T)^2}\frac{1}{Y^S}\right]\\
&\quad+\frac{\sqrt{2}}{36\pi}\frac{T_0^2}{\sqrt{\omega D^S C_v^S} r_0^2}\frac{(\alpha^T)^2(1+(\nu^T)^2)}{1-(\nu^T)^2},\\
S\{<\partial_z u^z>_1,<\partial_i u^i>_1\} &= -\frac{\sqrt{2}}{2\sqrt{\pi}}\frac{T_0}{\omega r_0^3} Im\left[\frac{(1-(\nu^S)^2)(1-2\nu^T)\nu^T}{(1-\nu^T)^2}\frac{1}{Y^S}\right]\\
&\quad+\frac{\sqrt{2}}{36\pi}\frac{T_0^2}{\sqrt{\omega D^S C_v^S} r_0^2}\frac{(\alpha^T)^2(1+(\nu^T)^2)}{1-(\nu^T)^2},\\
S\{<T>_1,<\partial_i u^i>_1\} &\to 0,\\
S\{<\partial_z u^z>_1,<\partial_z u^z>_1\} &= -\frac{\sqrt{2}}{2\sqrt{\pi}}\frac{T_0}{\omega r_0^3} Im\left[\frac{(1-(\nu^S)^2)(\nu^T)^2}{(1-\nu^T)^2}\frac{1}{Y^S}\right]\\
&\quad+\frac{\sqrt{2}}{36\pi}\frac{T_0^2}{\sqrt{\omega D^S C_v^S} r_0^2}\frac{(\alpha^T)^2(1+(\nu^T)^2)}{1-(\nu^T)^2},\\
S\{<T>_1,<\partial_z u^z>_1\} &\to 0,\\
S\{<T>_1,<T>_1\} &= -\frac{\sqrt{2}}{4\pi}\frac{T_0^2}{\sqrt{\omega D^S C_v^S} r_0^2}.
\end{aligned}
$$

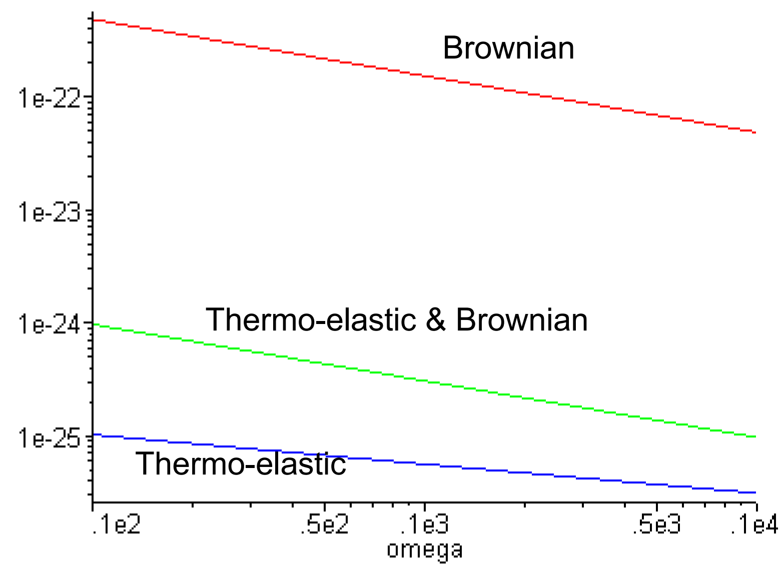
Brownian Noise from the interface and the sum of all the noise effect;

Basically, the Brownian noise from the interface is dominant and all other effects are almost negligible.



Brownian Noises from the interface and 'Thermo-elastic Noise' of the first layer ;

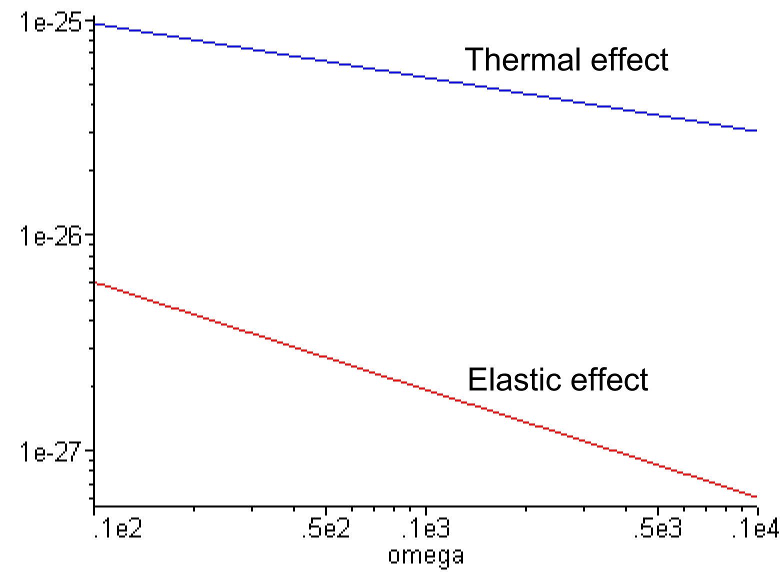
'Thermo-elastic Noise' from the first layer includes both elastic and thermo-elastic effects.



Elastic and thermal effects of “Thermo-elastic Noise” of the first layer;

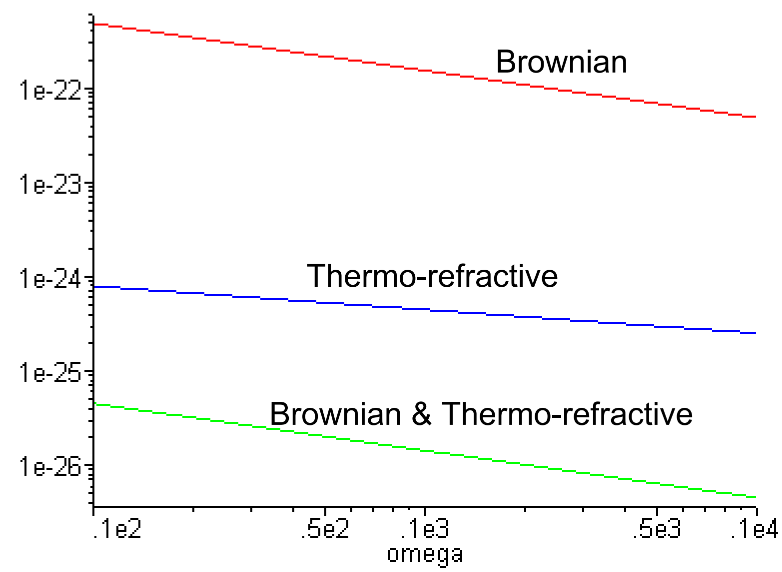
The thermal effect is dominant over the elastic effect.

This seems to suggest the coherent cancelling of Thermo-optic Noise, however, the calculation shows that the cross-correlation of the “Thermo-elastic noise” and Thermo-refractive noise (T) vanishes.



Brownian Noise and Thermo-refractive Noise of the first layer;

The cross-correlation is small.



5. Summary

We tentatively have the approximation result of mirror fluctuations.

- We consider the coupled dynamics of the thermal and elastic degree of freedom.
- We consider the cross-correlation of the thermal and elastic degree of freedom.
- We consider the multi-layer coating structure of the mirror.

- Brownian Noise from the interface b/w the substrate and the coating is dominant.

- Noise originates from the coating is generally small.

It is necessary to calculate the form factor of the laser phase with the multi-layer coating.

>>>>> Optimization of the mirror coating

Brownian thermal noise in the coated mirrors : 3-Dimentional consideration

Ting Hong, Huan Yang, Yanbei Chen
Caltech

GWADW, Florida, May 2009

G0900646-v1

Mirror thermal noise

Brownian noise;

Thermal elastic noise;

Thermal optic noise;

Thermal refractive noise…

Three types of dissipation:

- Direct expansion/contraction of coating thickness
- Transverse expansion/contraction of coating bending the substrate-coating interface
- Shear distortion in coating layers

Correlation between different components of brownian noise?

Can we cancel or decrease the Brownian thermal noise by careful design of coating?

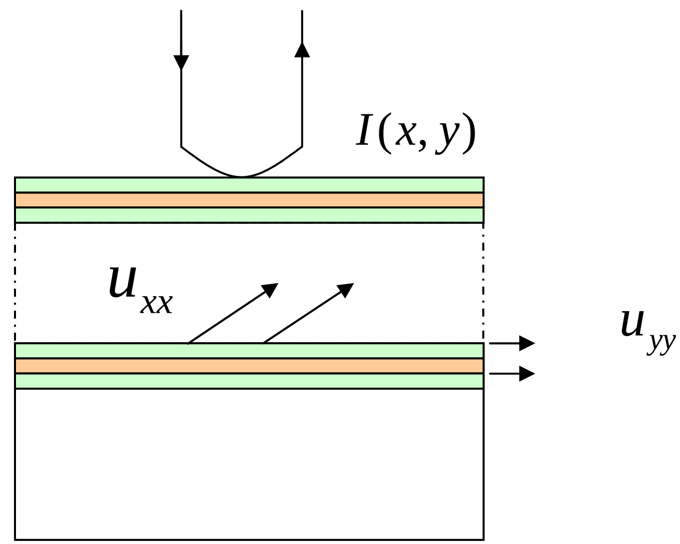
Mirror thermal noise

Previous result (G. Harry et.al)

$$S(f)=\frac{2k_BTd}{\pi^2 f\omega^2 Y_{sub}}\left(\frac{Y_{coat}}{Y_{sub}}\phi_{\parallel}+\frac{Y_{sub}}{Y_{coat}}\phi_{\perp}\right)$$

→ One dimentional motion

Penetration of the light, u_{xx} u_{yy}



Assumption in our calculation:

- Temperature is constant
- Half infinite mirror
- Thin coating

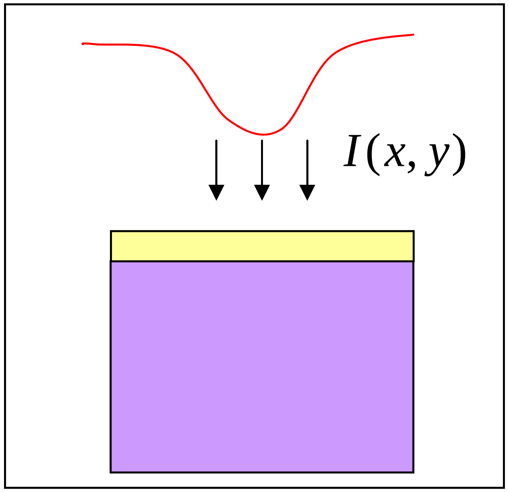
Formulation

Coating thickness change

$$\xi(t) = x_s(t) + \sum_{j=1}^{N} \delta l_j(t) + \sum_{j=1}^{N} \frac{n_j \delta l_j(t) + l_j \delta n_j(t)}{2} \mathrm{Im}\left(\frac{\partial \log \rho}{\partial \phi_j}\right)$$

Coating/substrate interface movement

Phase modification due to light penetration into coating layers



$$\delta n = \frac{\partial n}{\partial \log v}\Big|_T (u_{xx} + u_{yy} + u_{zz})$$

Volume fluctuation also contributes to noise

$$\bar{\xi} = \iint dxdy I(x,y) \xi(t)$$

Formulation

$$\overline{\xi} = \iint dxdy[u_{sz}I(x,y) - \sum_i a_i l_i (\frac{\partial I(x,y)}{\partial x} u_{xi} + \frac{\partial I(x,y)}{\partial y} u_{yi}) + \sum_i b_i I(x,y)\delta l_i]$$

$$W_{diss} = W_{elastic} \times \phi$$

$$W_{elastic} = \int_V dV \frac{1}{2} \tilde{u}_{ij} T_{ij}$$

Fluctuation dissipation theorem:

$$S_q(\omega) = \frac{8 k_B T}{\omega^2} \frac{W_{diss}}{F_0^2}$$

Poisson's ratio:0

$$W_{dis} = W_{//} + W_{\perp} + W_{\Diamond}$$

$$W_{//} \propto \phi_{//} \frac{Y_{coat}}{Y_{sub}^2} d \iint dk_x dk_y |\tilde{I}(k_x,k_y)|^2 [(1+\overline{a_i} k_x d)^2 + (1+\overline{a_i} k_y d)^2]$$

$$W_{\Diamond} \propto \phi_{\Diamond} \frac{1}{Y_{coat}} d \iint dk_x dk_y |\tilde{I}(k_x,k_y)|^2 (k_x^2 + k_y^2) d^2$$

$$W_{\perp} \propto \phi_{\perp} \frac{1}{Y_{coat}} d \iint dk_x dk_y \overline{b_i^2} |\tilde{I}(k_x,k_y)|^2$$

Area and interface fluctuation

$$S_{//} \propto \phi_{//} \frac{Y_{coat}}{Y_{sub}^2} d \iint dk_x dk_y \, |\tilde{I}(k_x, k_y)|^2 [(1+\overline{a}_i k_x d)^2 + (1+\overline{a}_i k_y d)^2]$$

For typical LIGO mirror coating

$$d \sim 0.01mm$$

$$k \sim (10cm)^{-1}$$ ← Beam size

$$kd \sim 0.0001 << 1$$

So usual light detection are not sensitive to area fluctuations. This also means that it's not plausible to cancel $S_{//}$ by using usual gaussian beams. But for other beam shape/coating configuration, it may be possible to reduce $S_{//}$.

Shear fluctuation & thickness fluctuation

$$S_{\Diamond} \propto \phi_{\Diamond} \frac{1}{Y_{coat}} d \iint dk_x dk_y \,|\tilde{I}(k_x, k_y)|^2 (k_x^2 + k_y^2) d^2$$

Same magnitude as $\phi_{\square}, \phi_{\perp}$?

Suppression factor

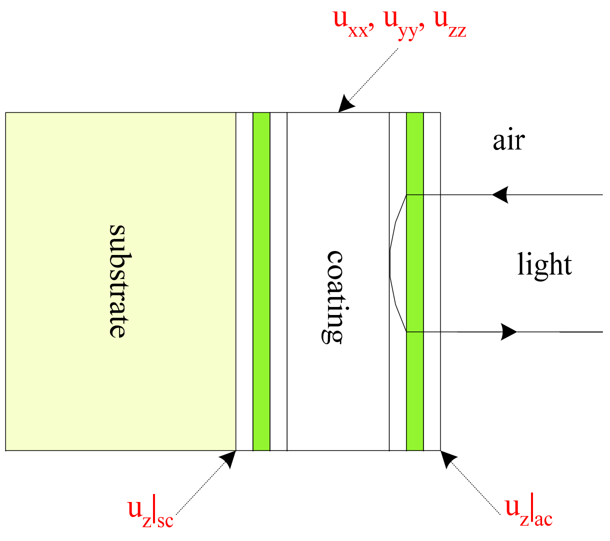
$$S_{\perp} \propto \phi_{\perp} \frac{1}{Y_{coat}} d \iint dk_x dk_y \,|\tilde{I}(k_x, k_y)|^2$$

Noise proportional to total thickness

Thickness fluctuations are independent in normal direction

Correlation functions

Total Brownian thermal noise depends on u_{xx}, u_{yy}, u_{zz}, $u_z|_{sc}$



1. u_{zz} in different location is uncorrelated

$$< u_{zz}|_a, u_{zz}|_b >= 0$$

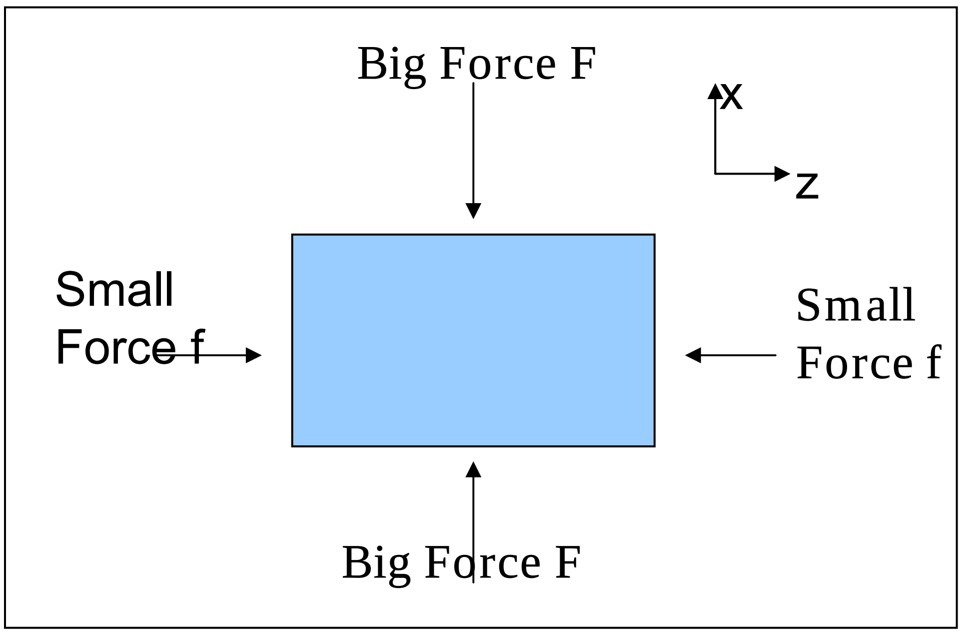
2. $u_z|_{sc}$ is not correlated with u_{zz};
3. $<d_i, d_j>=0$
4. At same (x,y), $u_{//}$ is not completed correlated

$$S_{u_{z|ac}} = S_{u_{z|sc}} + \sum_i S_{d_i}$$

When Poisson's ratio is not 0

$\phi_{\parallel}, \phi_{\perp}$ May not be well defined unless they are equal:

$u_{zz}T_{zz}$ may be negative on some case. If we have following set-up



$$T_{xx}u_{xx} \propto Fu_{xx}$$

$$T_{zz}u_{zz} \propto -f\,\frac{\sigma}{1-\sigma}u_{xx}$$

Different sign!

$$\phi_{\Sigma}, \phi_{\Theta} \qquad U = \frac{K}{2}\theta^2 + \mu\sum{}^2$$

More consistent definition

When Poisson's ratio is not 0

- If loss angles are the same, coating air interface motion and coating thickness fluctuation are not correlated.
- If bulk loss angle doesn't equal to shear loss angle, coating substrate interface motion and coating thickness fluctuation are correlated.

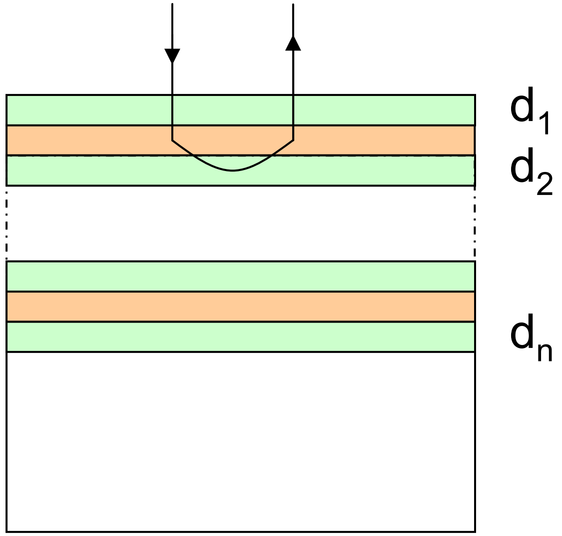
bulk fluctuation : positively correlated

shear fluctuation : negatively correlated

$$d_1 + d_2 + d_3 \ldots\ldots\ldots + d_n + u_z$$

$$\alpha_1 d_1 + \alpha_2 d_2 + \alpha_3 d_3 + \ldots..$$

Can only cancel the noise in the first few layers



Summary

- For usual gaussian beam detection, u_{xx},u_{yy} fluctuation noise are not important.
- Coating thickness fluctuations are independent, so canceling thickness fluctuation is not obvious.
- When Possion's ratio is not 0, the conventional defined $\phi_{\parallel}, \phi_{\perp}$ may not be appropriate
- There is correlation between the thickness fluctuation and interface movement when Possion's ratio is not zero.
- While canceling of thickness fluctuation is still difficult.