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Multi thin layer coating modeling

Title

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1 INTRODUCTION

In this note, the optical behavior of a multi thin layer coating is studied, and some of its characteristics are discussed to understand various features analytically. This work was performed in support of the LIGO core optics requirements definition.

The basic formulation is discussed in §2, and the result is used in §3 to analytically express the transmittance and the phase shift of the reflected wave when the layer thicknesses are slightly off from the design value of the quarter wave length. Three cases are studied for the small deviations of the layer thickness, 1) the uniform accordion case, in which all layers deviate by the same fractional amount from the design thickness, 2) the dual accordion case, in which those two groups of layers with high and low refractive index each show accordion structures, but the magnitude of the deviations are different, and 3) the random case, in which all layers have independent random deviations. §4 is devoted for the discussion about the relation between the reflectance measurement and phase shift. The possibility of using a HeNe laser metrology to infer phase uniformity at the Ar^+ laser wavelength is discussed in §5.

The main results derived in this note are: (1) the phase shift fluctuation is dominated by the surface profile nonuniformity of the coating surface, (2) the deviation of each layer, $\Delta(nd)/(nd)$, is limited by the allowable root mean square (rms) of the surface fluctuation, $\sigma_{coating}$, by the relationship $\Delta(nd)/(nd) < 0.1 \cdot \sigma_{coating} / \lambda$, 3) the phase shift fluctuation, which is mainly determined by the physical spatial thickness of the coating, may not be predictable by a center wavelength measurement, which is mainly sensitive to optical thickness, and 4) a HeNe laser will be a good tool, for either direct or cross-check purposes, for uniformity measurements. *Results (1) and (2) pose very stringent requirement on the core optics coating*.

2 BASICS OF THE MULTILAYER HIGH-REFLECTANCE COATING

As is shown in Appendix 1, where the basic formulas for the multi thin layer high-reflectance coating are given, the multi layer system is completely described by the characteristics matrix [M] defined by Eq. (A-10). Using this matrix and the refractive index of the substrate n_s, the reflectance, transmittance and their phases are given by Eq. (A-12), Eq. (A-13), Eq. (A-14) and Eq. (A-15). A C program was written to calculate these four quantities and all the numerical results below are calculated using this program.

The phases defined in Eq. (A-14) and Eq. (A-15) are those for the ratios E_r/E_i and E_t/E_i, i.e., the difference between outgoing and incoming fields on the surface of the coating (e.g., point [1] or [2] in Fig 1.). When studying the phase uniformity, the phase referenced to the same plane is needed (e.g., points [r] and [2] in Fig 1.). The phase shift of the reflected wave (phase difference at [r] and [i] in Fig 1.) is given by

$$\Phi_r = \phi_r + \phi_{vac}$$
$$\phi_{vac} = \frac{4\pi\Delta_z}{\lambda} \qquad (1)$$

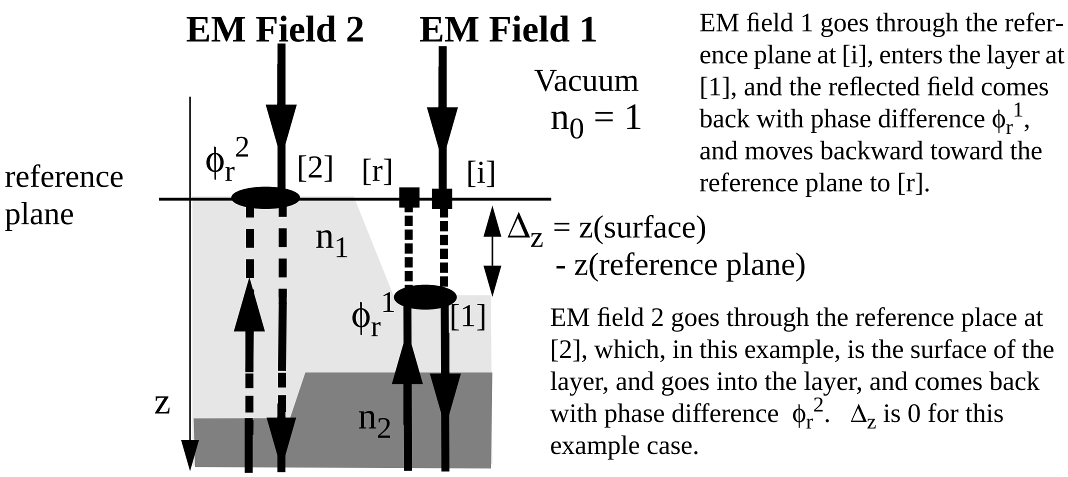

Fig 1. Phase difference of reflected field

where ϕ_r is given by (A-14), and ϕ_{vac} is the phase shift caused by the extra physical path between the reference place and the coating surface.

From this equation, the difference of phases between two points on the reference plane is given by

$$\Delta\Phi_{12} = (\phi_r^1 - \phi_r^2) + \frac{4\pi n_0 (z_1(\text{surface}) - z_2(\text{surface}))}{\lambda} \tag{2}$$

The first term comes from the difference of optical paths (schematically shown by dashed curve in Fig 1.) within the coating, and the second term comes from the difference of path lengths in vacuum (schematically shown by dotted curve in the same figure). For a typical coating to be used for LIGO, the signs of these two terms are the same.

In the rest of this section, the case is considered in which all but the cap layer have optical thickness around $\lambda/4$, while that of the cap is near $\lambda/2$. p is the number of high and low index layer pairs (HL), and q, the total number of layers, is 2p+2 (see Fig 2.). For convenience, the cap layer is treated as two layers, one with $\sim\lambda/4$ thickness (c1 in Fig 2.), and the other corresponding to the remainder (c in Fig 2.). Then the entire coating can be treated as p+1 (HL) pairs each having $\sim\lambda/4$ optical thickness, and a single low index layer on top.

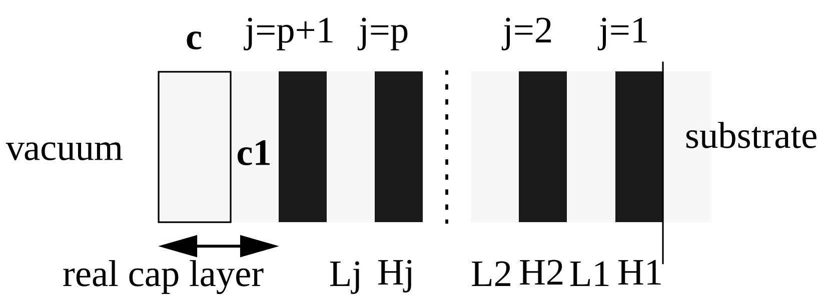

Fig 2. Numbering convention in Eq. (6)

The physical thickness corresponding to λ/4 in optical path depends on the reflective index, and is given for each layer by

$$(n_j d_j)_0 = \frac{\lambda_0}{4} \tag{3}$$

where λ_0 is the design wave length on resonance. If the coating were perfect, δ_j in Eq. (A-10) would be π/2 on resonance. To study behavior near resonance, ψ_j is introduced defined below:

$$\psi_j = \delta_j - \frac{\pi}{2} \tag{4}$$

This can be rewritten using the design parameters $(nd)_0$ and λ_0 as

$$\psi_j = \frac{\pi}{2} \cdot \left(\frac{\left(\frac{nd}{(nd)_0}\right)_j}{\frac{\lambda}{\lambda_0}} - 1 \right) \tag{5}$$

When one substitutes δ_j by $\pi/2+\psi_j$, Eq. (A-10) becomes:

$$\begin{aligned}
[M] &= [m_c] \prod_{j=1}^{p+1} [m_{Lj}]\,[m_{Hj}] \\
&= \begin{bmatrix} -\sin\psi_c & \frac{i\cos\psi_c}{n_c} \\ in_c\cos\psi_c & -\sin\psi_c \end{bmatrix} \prod_{j=1}^{p+1} \left(\begin{bmatrix} -\sin\psi_{Lj} & \frac{i\cos\psi_{Lj}}{n_{Lj}} \\ in_{Lj}\cos\psi_{Lj} & -\sin\psi_{Lj} \end{bmatrix} \cdot \begin{bmatrix} -\sin\psi_{Hj} & \frac{i\cos\psi_{Hj}}{n_{Hj}} \\ in_{Hj}\cos\psi_{Hj} & -\sin\psi_{Hj} \end{bmatrix} \right) \\
&= \begin{bmatrix} -s_c & \frac{ic_c}{n_c} \\ in_c c_c & s_c \end{bmatrix} \prod_{j=1}^{p+1} \begin{bmatrix} -\frac{n_{Hj}}{n_{Lj}} \cdot c_{Lj}c_{Hj} + s_{Lj}s_{Hj} & -i\left(\frac{1}{n_{Hj}} s_{Lj}c_{Hj} + \frac{1}{n_{Lj}} c_{Lj}s_{Hj}\right) \\ -i(n_{Lj}c_{Lj}s_{Hj} + n_{Hj}s_{Lj}c_{Hj}) & -\frac{n_{Lj}}{n_{Hj}} \cdot c_{Lj}c_{Hj} + s_{Lj}s_{Hj} \end{bmatrix}
\end{aligned} \tag{6}$$

where $[m_c]$ represents the partial cap layer, $s_k(c_k) = \sin(\cos)\psi_k$, and the product runs over the p+1 (HL) pairs as shown in Fig 2. All optical quantities can be calculated using this matrix.

When you substitute this expression for the matrix in the formula for the transmittance and reflectance (Eq. (A-12) - Eq. (A-15)), the following results can be derived as a consequence of the structure of Eq. (6) (see Appendix 2):

$$X(\vec{\psi}) = X_0(\vec{\psi}) \times (1 + F_{even}(\vec{\psi})) \tag{7}$$

where $\vec{\psi}$ is the vector of phase deviations, Eq. (5), for the 2p+2 layers, $X(\vec{\psi})$ is an optical quantity of interest, $X_0(\vec{\psi})$ is the leading component of X when $\vec{\psi}$ goes to zero (see, e.g., Eq. (12) and Eq. (15)), and $F_{even}(\vec{\psi})$ is a function of $\vec{\psi}$ which satisfies $F_{even}(\vec{\psi}) = F_{even}(-\vec{\psi})$. In other words, the quantity X will either be proportional only to even or odd powers of $\vec{\psi}$, but not mixtures of the two.

A few important conclusions follow from this result:

A) When $\lambda = \lambda_0$ in Eq. (5), ψ_j in Eq. (7) is $\frac{\pi}{2}$(nd/(nd)$_0$-1). The effect due to a small deviation from the design value of thickness (i.e., nd=λ/4) is of second order in (Δ(nd)/nd). From the discussion in Appendix 3, the correction term is of the order $p \cdot (\Delta nd/nd)^2$.

B) When nd = (nd)$_0$ in Eq. (5), ψ_j in Eq. (7) is $\frac{\pi}{2}\lambda_0(1/\lambda - 1/\lambda_0)$, and the following result holds:

$$F_{even}\left(\frac{\pi}{2}\lambda_0\left(\frac{1}{\lambda_-} - \frac{1}{\lambda_0}\right)\right) = F_{even}\left(\frac{\pi}{2}\lambda_0\left(\frac{1}{\lambda_0} - \frac{1}{\lambda_+}\right)\right)$$
$$\text{if} \qquad \frac{1}{\lambda_-} - \frac{1}{\lambda_0} = \frac{1}{\lambda_0} - \frac{1}{\lambda_+} \qquad \text{or} \qquad \frac{1}{\lambda_0} = \frac{1}{2}\cdot\left(\frac{1}{\lambda_-} + \frac{1}{\lambda_+}\right) \tag{8}$$

In other words, the resonance wavelength may be deduced from measurement of the two wavelengths, $\lambda_- < \lambda_0 < \lambda_+$, at which F_{even} has the same magnitude. Eq. (8) is valid if no optical quantities, such as refractive indices, depend on the wavelength. In the case of LIGO core optics, the high index material, Ta_2O_5, exhibits a weak dispersion, and the λ dependence shifts the minimum position of the transmittance from the designed resonance wavelength downward by 1 nm. This will be discussed later.

C) For the accordion case, i.e., (nd/(nd)$_0$)$_j$ is the same for all layers, Eq. (8) is valid with the scaling

$$\lambda_0' \rightarrow \lambda_0 \cdot \frac{nd}{(nd)_0} \tag{9}$$

3 EXPRESSIONS FOR PHASE SHIFT AND TRANSMITTANCE

Using these results, the phase shift of the reflected wave and the amplitude transmittance will be discussed when the thickness of layers deviates slightly from the design value (λ/4n), i.e., $|\psi_j| \sim |\psi| \ll 1$.

3.1. Phase shift of reflected wave

In this limit, the phase shift, Eq. (A-14), is expressed as follows:

$$\tan\phi_r = \frac{-2n_L\xi^{2p+2}(\alpha-\beta)}{1+\beta^2 - n_L^2\xi^{4p+4}(1+\alpha^2)} \times (1 + O(p \times |\psi|^2)) \tag{10}$$

where

$$\xi = \frac{n_H}{n_L} \qquad (\xi^{2p+2} = 2.4 \times 10^5 \text{for p=16})$$

$$\alpha = \xi^{-2p-2}\psi_c + n_L \sum_{j=1}^{p+1} \left(\frac{\psi_{Lj}}{n_H} + \frac{\psi_{Hj}}{n_L}\right)\xi^{1-2j} \sim O(|\psi|) \tag{11}$$

$$\beta = \xi^{2p+2}\psi_c + \frac{1}{n_L}\sum_{j=1}^{p+1} (n_H\psi_{Lj} + n_L\psi_{Hj})\xi^{2j-1} \sim O(\xi^{2p}|\psi|)$$

By neglecting α compared to β due to the large value of ξ^{2p+2}, the phase may be approximated as

$$\tan\phi_r = [\tan\phi_r]_0 \times (1 + O(p \times |\psi|^2))$$
$$[\tan\phi_r]_0 \equiv -\frac{2\beta}{n_L\xi^{2p+2}} \sim O(\psi) \tag{12}$$

Dual accordion case

$[\tan\phi_r]_0$ is given by the following form for the dual accordion case (i.e., all layers with the same refractive index have the same fractional deviation from the design value, i.e., $\psi_{Lj}=\psi_L$ and $\psi_{Hj}=\psi_H$, but ψ_H may be different from ψ_L)

$$[\tan\phi_r]_0 = -\frac{2}{n_L}\frac{1}{\xi^2 - 1}[(2\xi^2 - 1)\psi_L + \xi\psi_H] \tag{13}$$

The reason that the phase change on the surface is of the order $|\psi|$, independent of the number of layers, is that only the last few layers (closest to the vacuum) contribute, as can be seen from Eq. (11).

As is given in Eq. (2), the net phase shift referred to a common plane also contains another term for vacuum propagation, which is proportional to the total algebraic sum of variations of thickness of all layers. If the thickness changes from the design value is of accordion type, i.e., all layers tend to have the same size (including the sign) of deviation from the design value, ($\psi_i \sim |\psi|$ for all layers), then the second term is of the order of p x $|\psi|$, which is larger than the first term (refer to Eq. (2)). A detailed discussion will be given in the next section.

Accordion case ($\psi_H=\psi_L$)

For the accordion case, the variation of ϕ_r due to either wavelength shift or optical thickness change may be calculated from Eq. (13) as:

$$\frac{\partial \phi_r}{\partial\left(\frac{\lambda_0}{\lambda}\right)} = \frac{\partial \phi_r}{\partial\left(\frac{(nd)}{(nd)_0}\right)} = C_\phi \equiv -\pi \frac{2n_H - n_L}{n_L(n_H - n_L)} = -527° \tag{14}$$

i.e., the dependence on the optical thickness around the design value can be measured from the wavelength dependence.

3.2. Power transmittance

The transmittance may be expressed as

$$\begin{aligned} T &= [T]_0(1 + \tilde{T}_2) \\ [T]_0 &= \frac{4}{n_L}\left(\frac{n_L}{n_H}\right)^{2p+2} = 1.157 \times 10^{-5} \qquad (\lambda = \lambda_0) \end{aligned} \tag{15}$$

Dual accordion case

The higher order term $\tilde{T}_2$, which corresponds to F_{even} in Eq. (7), is given by the following expression for the dual accordion case:

$$\begin{aligned} \tilde{T}_2 &= (p+1)\frac{(\xi^2+1)(\psi_L^2+\psi_H^2)+4\xi\psi_L\psi_H}{\xi^2-1} \\ &= \tau\left(\frac{\lambda_0}{\lambda} - 1 + \frac{\varepsilon_{nd}^L + \varepsilon_{nd}^H}{2}\right)^2 \\ \tau &= 2(p+1)\frac{\xi+1}{\xi-1}\left(\frac{\pi}{2}\right)^2 \qquad \varepsilon_{nd}^{L,H} = \left(\frac{nd}{(nd)_0}\right)^{L,H} - 1 \end{aligned} \tag{16}$$

The higher order term is again proportional to p, the number of layers.

If the reflective index does not have any wavelength dependence, the transmittance has a minimum at $\lambda_0(1+(\varepsilon^L_{nd}+\varepsilon^H_{nd})/2)$. However, in our case, Ta_2O_5 is dispersive and, because of the large number of layers, $[T]_0$ has a significant wavelength dependence. The wavelength dependence of n_H is

$$\begin{aligned} n_H &= n_H^0(1 + n'(\lambda - \lambda_0)) \\ n_H^0 &= 2.101, n' = -1.0\times10^{-4} \end{aligned} \tag{17}$$

From Eq. (15), (16) and (17), the wavelength which gives the minimum of the transmittance is

$$\lambda_{min} = \lambda_0\left(1 + \frac{(p+1)n'\lambda_0}{\tau} + \frac{\varepsilon_{nd}^L + \varepsilon_{nd}^H}{2}\right)$$
$$= \lambda_0\left(1 + \frac{n'\lambda_0}{2}\frac{n_H^0 - n_L}{n_H^0 + n_L}\left(\frac{2}{\pi}\right)^2 + \frac{\varepsilon_{nd}^L + \varepsilon_{nd}^H}{2}\right) \quad (18)$$

The correction due to the dispersion is 0.2% or 1nm for $\lambda = 514.5$ nm.

4 RELATIONSHIP BETWEEN PHASE SHIFT, TRANSMITTANCE AND COATING THICKNESS ERRORS

Accordion and dual accordion cases

If the variation of the thickness of layers is of the dual accordion type, the reflectance measurement and phase shift are related as follows. The two components of the phase shift may be written as:

$$\phi_r = C_\phi^L \varepsilon_{nd}^L + C_\phi^H \varepsilon_{nd}^H$$
$$C_\phi^L = -\frac{\pi}{n_L}\frac{2\xi^2 - 1}{\xi^2 - 1} \qquad C_\phi^H = -\frac{\pi}{n_L}\frac{\xi}{\xi^2 - 1} \quad (19)$$
$$\phi_{vac} = -\frac{4\pi\Sigma}{\lambda} \qquad \Sigma = \frac{\lambda}{4}\left((p+2)\frac{\varepsilon_{nd}^L}{n_L} + (p+1)\frac{\varepsilon_{nd}^H}{n_H}\right)$$

and the total phase shift becomes by substituting numerical values:

$$\mathbf{\Phi} = \phi_r + \phi_d$$
$$= (-362\varepsilon_{nd}^L - 166\varepsilon_{nd}^H) + (-2220\varepsilon_{nd}^L - 1460\varepsilon_{nd}^H)$$
$$= -2580\varepsilon_{nd}^L - 1630\varepsilon_{nd}^H \quad (20)$$
$$= -4210\varepsilon_{nd}; \qquad \varepsilon_{nd}^L = \varepsilon_{nd}^H$$

where units are in degree and the last expression is for the accordion case. In order to have a uniformity of $\lambda/400$ (the core optics specification calls out for this quality of optical surface), the fractional change of the optical thickness ε_{nd} _must be in the order of_ $2 \cdot 10^{-4}$.

As has been discussed in the last section, Eq. (8) and Eq. (9), one can calculate the center wave length by measuring spectral reflectance by using the following equation

$$\frac{1}{\lambda_c} = \frac{1}{2}\left(\frac{1}{\lambda_-} + \frac{1}{\lambda_+}\right) \quad (21)$$

where λ_- and λ_+ are the wavelengths with the same value of reflectance (the exact value is unimportant). This has been confirmed by explicit numerical calculations using the half height of the maximum of the reflectance and also using twice the value of minimum of transmittance. The wavelength at the real minimum of the transmittance is slightly away from the design value, as was shown in Eq. (18), but the shift is independent of ε_{nd} and may be neglected when considering the variation of the position of extrema. As has been mentioned before, the variation of ε_{nd} must be limited to the order of 10^{-4}. To infer such an accuracy from the transmittance measurement, the center wavelength must be measured to better accuracy (i.e., better than 0.1 nm).

Accordion type errors are worst case, because all the variations contribute coherently, as can be seen from Eq. (11). For example, the dependence on deviation for dual accordion type error is:

$$\Phi = -2580\varepsilon_{nd}^{L} - 1630\varepsilon_{nd}^{H}$$
$$\lambda_c = \lambda_0\left(1 + \frac{\varepsilon_{nd}^{L} + \varepsilon_{nd}^{H}}{2}\right) \qquad (22)$$

For the accordion case, i.e., $\varepsilon_{nd}^{L} = \varepsilon_{nd}^{H}$, there is a linear relationship between phase shift and center wavelength. For the dual accordion case, when $\varepsilon_{nd}^{L} \neq \varepsilon_{nd}^{H}$, the ratios of their coefficients are different and measurement of the variation of the center wavelength does not permit anything to be inferred about phase shift variation.

Random case

If accordion type errors are sufficiently reduced, random variations among layer thickness will become the dominant contribution. In order to analyze these effects, a Monte Carlo analysis was performed. The thickness of each layer was generated following a Gaussian distribution with a standard deviation $\sigma(nd)$ for the fractional thickness error. Many sample runs were used to determine the variation of the phase shift with wavelength. Fig 3. summarizes these results. From this figure, one can see that, to have a uniformity of $\lambda/400$, or 0.9°, $\sigma(nd)$ must be less than 0.09%, or the rms of the variation of the center wave length must be less than 0.12 nm. These numbers are less stringent than those for the accordion case, but are still quite tight.

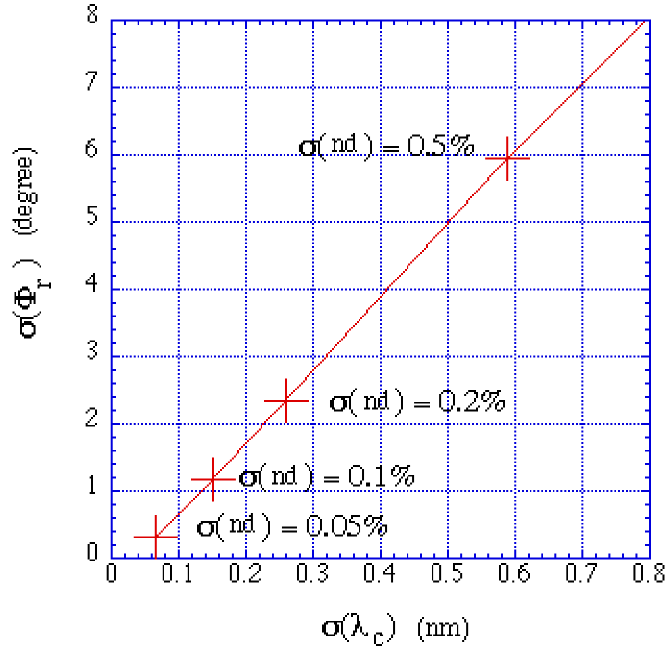

Fig 3. Phase shift and center wavelength

5 $Ar^{+}(514.5\ nm)$ VS $HeNe(632.8\ nm)$ CALIBRATION

The possibility of deducing the phase shift at 514.5 nm by using measurements at the HeNe wavelength of 632.8 nm is studied.

Accordion case

For the accordion case, the phase shifts at the Ar (ϕ_{Ar}) and that at the HeNe wavelength (ϕ_{HeNe}) are related by the following equation, where $\delta\phi$ is the deviation of the measured phase shift from the design value:

$$\delta\phi_{Ar} = 0.51\delta\phi_{HeNe} \qquad (23)$$

Dual Accordion case

In this case, there is no one-to-one mapping between phase shifts at the two wavelengths, and Eq. (23) is valid only on average. The ambiguity of ϕ_{Ar} for a measured ϕ_{HeNe} depends strongly on the nature of the dual accordion structure and size. The following case is studied to understand the magnitude of ambiguity: each type of layer was assigned an error $\Delta(nd)/(nd)_{L,H}$, and this error was chosen uniformly between -ε and ε. After generating many samples, it was found that $\delta\phi_{Ar}$

was within ± 400·ε ° around the average value in Eq. (23). This ambiguity is small enough to be used for the cross calibration for ε ≤ 0.2%.

Random case

The layer thicknesses were generated using a Gaussian distribution. Fig 4. shows the relation between the HeNe phase shift and the corresponding Ar^+ phase shift. The solid line is the average of Ar phase shift, with slope of 0.51. The various dashed lines are ± 2 σ bounds for the cases where the standard deviation of the thickness fluctuation, Δ(nd)/nd, is 0.5%, 0.2%, or 0.1%. As has been discussed previously, when random fluctuations are dominant contribution to the phase shift, the level of the fluctuation must be less than 0.1%, and then metrology with a HeNe laser is possible to cross check the uniformity.

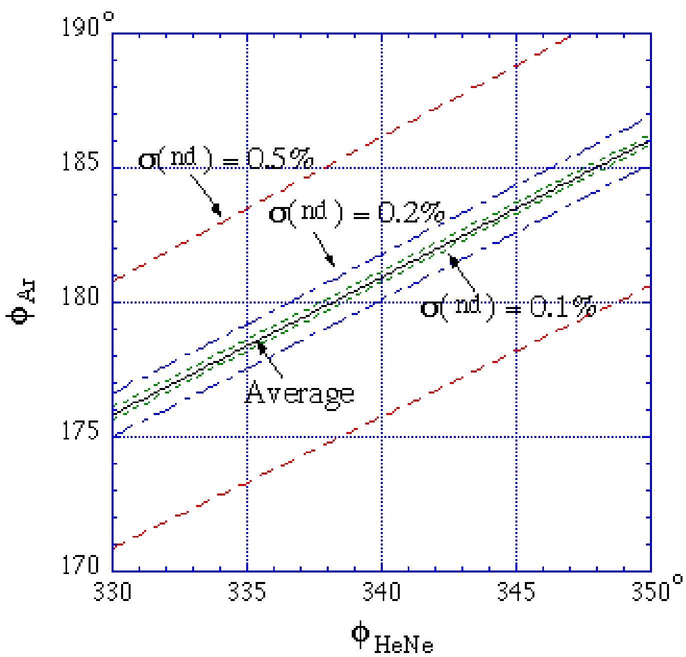

Fig 5. Ar vs HeNe phase shift

Appendix 1. Derivation of basic formulas for the multilayer coating problem

The detailed derivation of the formulas used in this note may be found in “Thin film optical filters” by H. A. Macleod (published by Adam Hilger Ltd), and the notations and conventions of this reference are used in this note. Electromagnetic fields are expressed by complex numbers for the convenience of calculation. The real part corresponds to the measurable quantity. In this appendix, the reflectance and transmittance are derived for a single layer case, and are then generalized to the multilayer case. Only normal incidence is addressed in this note, and the surface of the material is assumed to be flat.

At the boundary of two materials with different refractive indices (Fig A-1.), the transverse components (perpendicular to the direction of propagation of the wave) satisfy the following condition:

$$\begin{gathered} E_i + E_r = E_t \\ H_i + H_r = H_t \Rightarrow n_0 E_i - n_0 E_r = n_1 E_t \end{gathered} \tag{A-1}$$

where i, r and t represent incoming, reflected and transmitted, respectively. From these equations, the ratios of outgoing waves and incoming wave (Fresnel amplitude reflection and transmission coefficients) are derived as

$$\begin{gathered} \rho = \frac{E_r}{E_i} = \frac{n_0 - n_1}{n_0 + n_1} \\ \tau = \frac{E_t}{E_i} = \frac{2n_0}{n_0 + n_1} \end{gathered} \tag{A-2}$$

The energy flow is calculated as

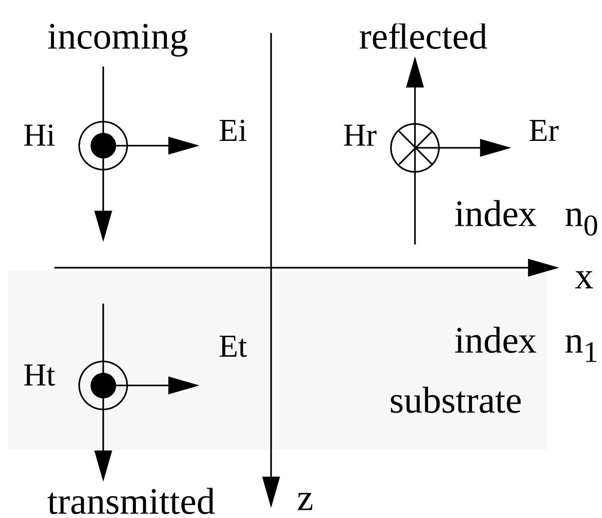

Fig A-1. Electricmagnetic fields at an interface

$$I = \frac{1}{2} \cdot Re(EH^*) = \frac{1}{2} \cdot Re(n^*EE^*) \tag{A-3}$$

and the reflectance and transmittance (the ratio of energy flows) are expressed as follows:

$$R = \rho\rho^* = \left|\frac{n_0 - n_1}{n_0 + n_1}\right|^2$$
$$T = Re\left(\frac{n_1}{n_0}\tau\tau^*\right) = \frac{4n_0 Re(n_1)}{|n_0 + n_1|^2} \tag{A-4}$$

The characteristics of a single layer coating is calculated as follows (see Fig A-2.). There are two components of the fields, one propagating toward the positive (E^+, H^+) and one toward the negative z direction (E^-, H^-) with phase factors shown in Fig. A-2. The fields on the two boundaries (a and b) in a film with refractive index n_1 are expressed as follows using these two components,

$$E_j = E_{1j}^+ + E_{1j}^- \qquad (j = a, b) \tag{A-5}$$
$$H_j = H_{1j}^+ + H_{1j}^- = n_1 E_{1j}^+ - n_1 E_{1j}^-$$

The fields on surfaces a and b are related by the following equations due to the change in phase with spatial separation d:

$$E_{1a}^+ = E_{1b}^+ e^{i\delta}$$
$$E_{1a}^- = E_{1b}^- e^{-i\delta} \tag{A-6}$$

δ is the phase difference between the two surfaces, i.e.,

$$\delta = \frac{2\pi n_1 d}{\lambda} \tag{A-7}$$

From these equations, the fields on these two surfaces are related by the following equations.

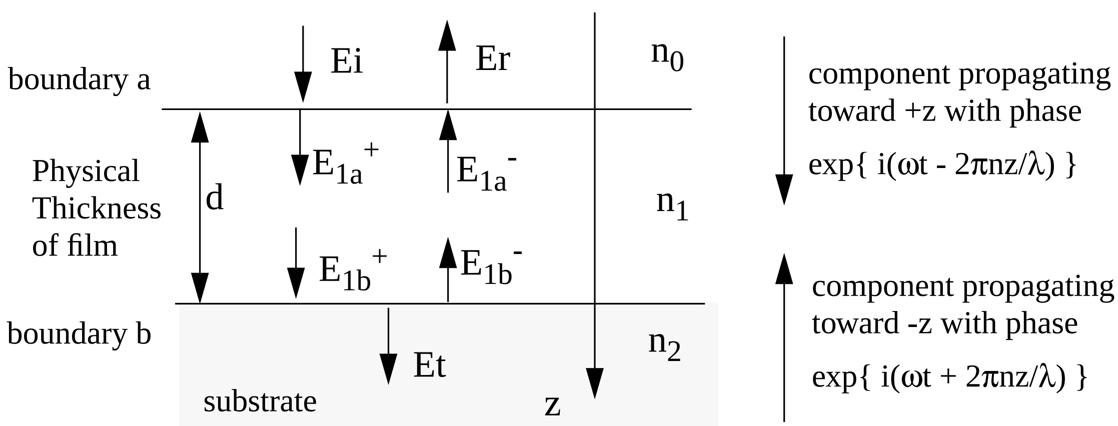

Fig A-2. Electric field on a thin film

$$\begin{bmatrix} E_a \\ H_a \end{bmatrix} = \begin{bmatrix} \cos\delta & \dfrac{i\sin\delta}{n_1} \\ in_1\sin\delta & \cos\delta \end{bmatrix} \cdot \begin{bmatrix} E_b \\ H_b \end{bmatrix} \tag{A-8}$$

This result for a single layer can be easily generalized for a multilayer case (see Fig A-3.), and the field at boundary a is related to that at boundary b as follows:

$$\begin{bmatrix} E_i + E_r \\ H_i + H_r \end{bmatrix} = \begin{bmatrix} M_{11} & iM_{12} \\ iM_{21} & M_{22} \end{bmatrix} \begin{bmatrix} E_t \\ H_t \end{bmatrix} \tag{A-9}$$

where

$$\begin{bmatrix} M_{11} & iM_{12} \\ iM_{21} & M_{22} \end{bmatrix} = [m_q][m_{q-1}]\dots[m_2][m_1]$$

$$[m_j] = \begin{bmatrix} \cos\delta_j & \dfrac{i\sin\delta_j}{n_j} \\ in_j\sin\delta_j & \cos\delta_j \end{bmatrix} \qquad \delta_j = \frac{2\pi n_j d_j}{\lambda} \tag{A-10}$$

For convenience, B, C and Y are defined as follows (note that $H_t = n_s E_t$),

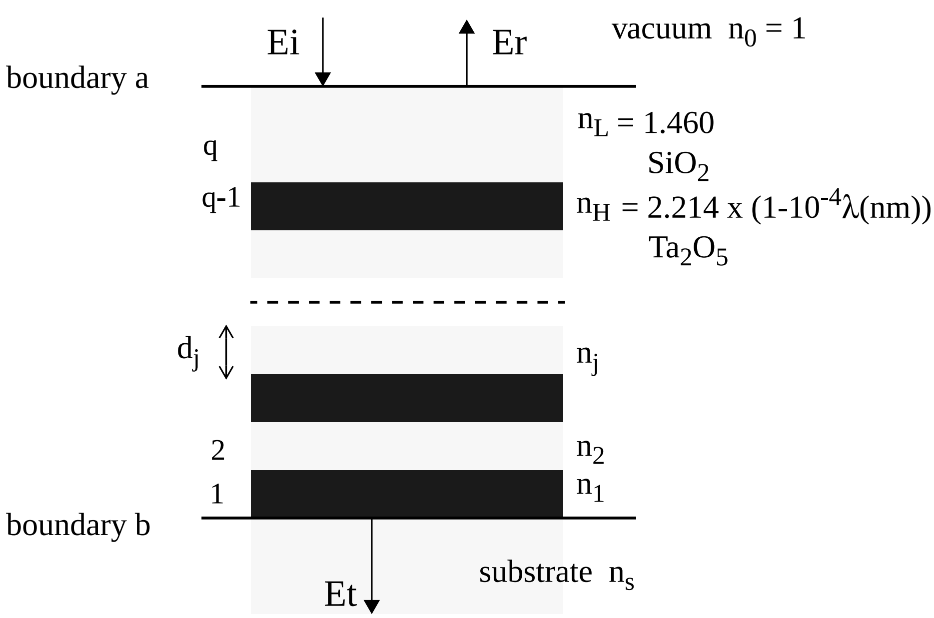

Fig A-3. Multi thin layer coating

$$\begin{bmatrix} B \\ C \end{bmatrix} \equiv \begin{bmatrix} M_{11} & iM_{12} \\ iM_{21} & M_{22} \end{bmatrix} \begin{bmatrix} 1 \\ n_s \end{bmatrix} \qquad Y \equiv \frac{H_i + H_r}{E_i + E_r} = \frac{C}{B} \tag{A-11}$$

By applying the same conditions as Eq. (A-1) on boundary a, the reflectance and transmittance are given by the following equations using the optical admittance Y defined above:

$$R = \left| \frac{n_0 - Y}{n_0 + Y} \right|^2 \tag{A-12}$$

$$T = \frac{4 n_0 Re(Y)}{\left| n_0 + Y \right|^2} \tag{A-13}$$

The phase difference of the reflected and transmitted field with respect to the incoming field are also derived as:

$$\tan\phi_r = -\frac{2 n_0 Im(Y)}{n_0^2 - |Y|^2} \tag{A-14}$$

$$\tan\phi_t = -\frac{Im(n_0 B + C)}{Re(n_0 B + C)} \tag{A-15}$$

Appendix 2. Proof of Eq. (7) $\quad X(\vec{\psi}) = X_0(\vec{\psi}) \times (1 + F_{even}(\vec{\psi}))$

When the multiplication of the p+1 matrices is performed, Eq. (6) has the following form:

$$\begin{aligned} \begin{bmatrix} M \end{bmatrix} &= \begin{bmatrix} -s_c & \dfrac{ic_c}{n_c} \\ in_c c_c & s_c \end{bmatrix} \cdot \begin{bmatrix} \tilde{F}^e_{11} & i\tilde{F}^o_{12} \\ i\tilde{F}^o_{21} & \tilde{F}^e_{22} \end{bmatrix} \\ &= \begin{bmatrix} F^o_{11} & iF^e_{12} \\ iF^e_{21} & F^o_{22} \end{bmatrix} \end{aligned} \tag{A-16}$$

where $F^{o(e)}$ (same for $\tilde{F}^{o(e)}$) are odd (even) real functions of $\vec{\psi}$, i.e.,

$$\begin{aligned} F^o_{ij}(\vec{\psi}) &= -F^o_{ij}(-\vec{\psi}) \\ F^e_{ij}(\vec{\psi}) &= F^e_{ij}(-\vec{\psi}) \end{aligned} \tag{A-17}$$

By substituting this into Eq. (A-11), one can see that transmittance and reflectance, Eq. (A-12) - Eq. (A-15), are even or odd functions of $\vec{\psi}$. For example, the transmittance and the reflectance phase are expressed as follows.

$$T = \frac{4n_S(F^o_{11}F^o_{22} + F^e_{12}F^e_{21})}{(F^{o2}_{11} + F^{e2}_{21}) + 2n_S(F^o_{11}F^o_{22} + F^e_{12}F^e_{21}) + n^2_S(F^{e2}_{12} + F^{o2}_{22})} \quad \text{(A-18)}$$

$$\tan\phi_r = \frac{-2(F^o_{11}F^e_{21} - n^2_S F^e_{12}F^o_{22})}{(F^{o2}_{11} - F^{e2}_{21}) + n^2_S(F^{e2}_{12} - F^{o2}_{22})} \quad \text{(A-19)}$$

As is clear from these expressions, transmittance is a even function of $\vec{\psi}$ and the reflectance phase is a odd function of $\vec{\psi}$.

Appendix 3. Dependence on the number of coating layers

When one expands $\tilde{F}^{o(e)}$ in Eq. (A-16) keeping up to second order in $\vec{\psi}$, all coefficients have one of the following forms:

$$X_1 = \sum_{i=1}^{p} \psi_i^2 \sim p\psi^2 \quad \text{(A-20)}$$

$$X_2 = \sum_{i=1}^{p} \sum_{j=i+1}^{p} \left(\frac{n_H}{n_L}\right)^{2(i-j)} \psi_i \psi_j \sim (A_1 p + A_2)\psi^2 \quad \text{(A-21)}$$

where the leading behavior is estimated by substituting $\psi_i=\psi$. In the second sum, there is no term proportional to the square of the number of layers. This is because, since $n_H/n_L > 1$, $(n_H/n_L)^{2(i-j)} \rightarrow 0$ as $j-i \rightarrow +\text{large}$. Physically, the interference phenomenon in multi thin layer coating occurs as a series of pair-wise layer interactions.