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Vibrational Transfer Function and the Sound Velocity in a Finite Elastic Body

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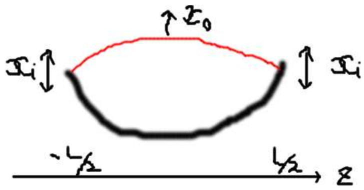

Figure 1: Tensioned string on a bow.

Abstract

When one measures the vibration transfer function by exciting one point and measure the response at the other point that is some distance away from the excitation point in an elastic body, a natural expectation is that there is an inherent time delay term in the transfer function, and that this delay is proportional to the distance over the velocity of the sound. This is not the case, however, when the elastic body has a finite dimension, as a standing wave is formed in a finite body as a result of the excitation.

1 Introduction

Suppose that you have an elastic body of a finite dimension, and you excite the vibration at one point and measure the response at another point that is away from the excitation point by L. Since the velocity of the sound in the medium is finite, a natural expectation is that the transfer function has a time delay term of L/v where v is the sound velocity.

This is not true however, as the excitation generates a standing wave pattern in the medium, and this standing wave pattern is just a linear superposition of the travelling wave from the infinite past. The only exception is the case where there is no standing wave (e.g. if the body in question is infinitely large, or if the medium is so lossy that the traveling wave passing the observation point is damped considerably before it is reflected by the boundary and comes back.)

In this document, two simple one-dimensional models are presented, one analytical and the other numerical. The latter is of more practical use as it is relatively easy to extend to 3-dimensional body. Since the puropose of this document is to show that there's no "time delay term" L/v, no attempt was made to model a realistic elastic body system like a cylindrical mirror. [1]

2 Analytical solution of a string

Let's start with the simplest example of a string on a bow (Fig.1). The length, tension, and the linear density of the string is L, T and ρ respectively. Your objective is to shake the bow in a direction perpendicular to the string by $x_i(t)$ and measure the transfer function from your excitation to the displacement of the string at the center, $x_o(t)$.

The transverse displacement of the string is represented by $x(z,t)$ where z is the coordinate along the length of the string, and we set the origin at the middle of the string.

The equation of motion of the string is

$$\rho \frac{d^2}{dt^2} x(z,t) = T \frac{d^2}{dz^2} x(z,t) \tag{1}$$

and its general solution is

$$x(z,t) = f(t - z/v) + g(t + z/v) \tag{2}$$

$$v = \sqrt{\frac{T}{\rho}} \tag{3}$$

where f and g are general real functions and v is the velocity of transverse wave in the string.

[1] However, readers should note that qualitative conclusions you can draw from one-dimensional model should still be applicable to 3-dimensional case.

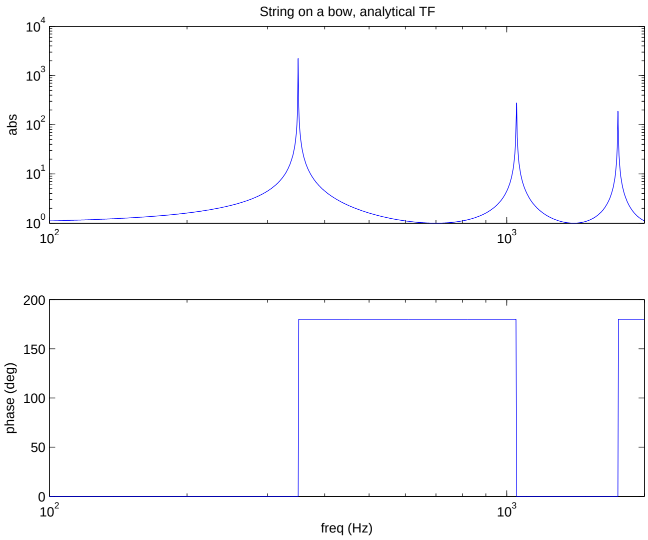

Figure 2: Transfer function from the vibration of a bow to the vibration of the string at the center. In this plot, the sound velocity v and the length of the string L was chosen so the first resonant frequency $v/2L$ becomes 350 Hz. No "delay" term appears in the transfer function despite the finite velocity of sound.

The symmetry of the excitation means $x(-z,t) = x(z,t)$, thus Eq.2 becomes

$$x(z,t) = f(t - v/z) + f(t + z/v). \tag{4}$$

Using this, in the Fourier domain the excitation x_i and the displacement at the observation point x_o are written by

$$\begin{aligned} \widetilde{x_i}(\omega) &= \widetilde{f}(\omega)\left[\exp\left(-i\frac{L\omega}{2v}\right) + \exp\left(i\frac{L\omega}{2v}\right)\right] \\ &= 2\cos(L\omega/2v)\widetilde{f}(\omega), \qquad (5) \\ \widetilde{x_o}(\omega) &= 2\widetilde{f}(\omega). \qquad (6) \end{aligned}$$

The transfer function from the excitation to the output is therefore

$$H(\omega) = \frac{1}{\cos(L\omega/2v)}. \tag{7}$$

The resonant frequencies are defined by

$$f_n = \frac{2n+1}{2L}v \tag{8}$$

where n is a non-negative integer.

Note that this is a real function, and doesn't have any phase delay except the sign change over the resonant frequencies $\omega_n = (2n+1)\pi v/L$ $(n = 1, 2, ...)$ (See Fig.2).

2.1 Transfer function as a standing wave pattern due to interference

The reason why there is no apparent time delay term in the transfer function in Eq.7 is because of the interference of the acoustic wave in the string. The wave propagates back and force many times between the left and right end of the string, thus causing a standing wave pattern.

Suppose that our observation point is at $z < 0$ (not necessarily the center). We define two characteristic time constants, $\tau_0 = L/2v$ and $\tau_1(z) = |z|/v$. The wave generated by the excitation in the left end propagates to the right. It reaches the observation point at $\tau_0 - \tau_1$, continues to travel in the right direction, hits the right end and reflected with 180 degrees phase flip due to the boundary condition, then comes back to the observation point at $3\tau_0 + \tau_1$, hits the left end and reflected with 180 degrees phase flip etc., so the effect of the left excitation is represented by

$$\begin{aligned} x^{left}(z,t) &= x_i(t-\tau_0+\tau_1) - x_i(t-3\tau_0-\tau_1) + x_i(t-5\tau_0+\tau_1) - x_i(t-7\tau_0-\tau_1)... \\ &= \sum_{n=0}^{\infty}(-1)^n x_i\left[t-(2n+1)\tau_0+(-1)^n\tau_1\right]. \end{aligned} \tag{9}$$

The effect of the right excitation is obtained by flipping the sign, so the total becomes

$$x(z,t) = \sum_{n=0}^{\infty}(-1)^n \left\{x_i\left[t-(2n+1)\tau_0+(-1)^n\tau_1\right] + x_i\left[t-(2n+1)\tau_0-(-1)^n\tau_1\right]\right\}. \tag{10}$$

When the excitation is sinusoidal, e.g. $x_i = \sin\omega t$, the above equation becomes

$$\begin{aligned} x(z,t) &= \sum_{n=0}^{\infty}(-1)^n \left\{\sin\omega\left[t-(2n+1)\tau_0+(-1)^n\tau_1\right] + \sin\omega\left[t-(2n+1)\tau_0-(-1)^n\tau_1\right]\right\}. \\ &= 2\cos\omega\tau_1 \sum_{n=0}^{\infty}(-1)^n \sin\omega\left[t-(2n+1)\tau_0\right] \\ &= \frac{\cos\omega\tau_1(z)}{\cos\omega\tau_0}\sin\omega t \\ &= \frac{\cos(\omega z/v)}{\cos(\omega L/2v)}\sin\omega t. \end{aligned} \tag{11}$$

Note that this is the product of the standing wave pattern, which is dependent on the frequency, and the sinusoidal vibration term. The transfer function is therefore the standing wave pattern itself:

$$H(z,\omega) = \frac{\cos(\omega z/v)}{\cos(\omega L/2v).} \tag{12}$$

Clearly the speed of sound is not only related to the eigenfrequencies ($\omega_n = (2n+1)v/L$, $n = 0, 1, \cdots$) but also the shape of the standing wave.

Figure 3 shows the standing wave pettern for several different frequencies.

3 One-dimensional mass-spring system

Suppose that there are $N+1$ masses connected by N springs. Each of masses is m, the natural length of the spring is l and the spring constant k. The velocity of the sound in this system is represented by $v = l\sqrt{k/m}$. The displacement of the masses from their natural resting position is represented by $x_n(t)$ where n is the non-negative integer. We excite this system by exerting a force $F(t) = F_0\cos\omega t$ to one of the end masses, and observe the response of the other end mass.

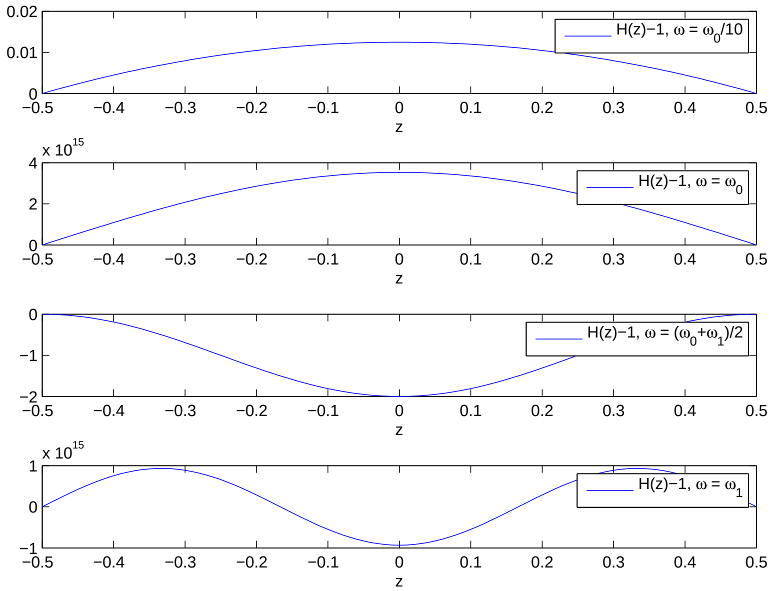

Figure 3: Standing wave pattern of the string for various frequencies. Note that $\omega = \omega_n$ $(n = 0, 1)$ in the plot just means that the frequency is very close to the resonance. Also note that $H(\omega) - 1$ instead of $H(z)$ is shown in these plots.

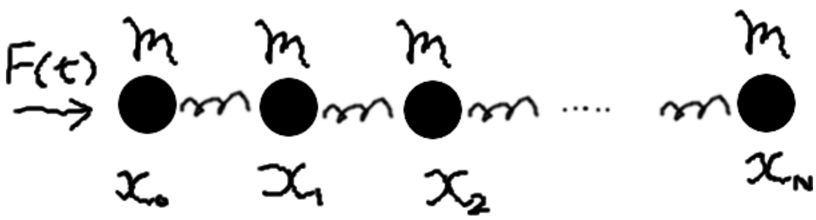

Figure 4: One-dimensional mass-spring system with $N + 1$ masses.

Number of masses	$N+1$=101
Each mass m	1 kg
Spacing between the neighboring masses l	1 m
Spring constant of each spring k	100 N/m
Time step	2 ms
Velocity of sound	$l\sqrt{k/m} = 10$ m/s
Total length	$lN = 100$ m
Sound transit time	100/10=10 s

Table 1: Parameters used for the simulation.

The equation of motion in the frequency domain is represented by the following:

$$A\widetilde{x} = B\widetilde{F}(\omega) \tag{13}$$

$$\widetilde{x} \equiv \begin{pmatrix} \widetilde{x_0}(\omega) \\ \widetilde{x_1}(\omega) \\ \widetilde{x_2}(\omega) \\ \vdots \\ \widetilde{x_N}(\omega) \end{pmatrix} \tag{14}$$

$$A = \begin{pmatrix} -m\omega^2+k & -k & 0 & \cdots & & 0 \\ -k & -m\omega^2+2k & -k & 0 & & \vdots \\ 0 & -k & -m\omega^2+2k & -k & \ddots & \\ \vdots & 0 & -k & \ddots & \ddots & 0 \\ & & \ddots & \ddots & -m\omega^2+2k & -k \\ 0 & \cdots & & 0 & -k & -m\omega^2+k \end{pmatrix} \tag{15}$$

$$B = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \tag{16}$$

The solution is obtained by

$$\widetilde{x} = A^{-1}B\widetilde{F}(\omega) \tag{17}$$

Because of the simplicity of the matrix A, it is possible to solve this analytically as far as N is finite. However, for a large N, it's much more practical to solve this system in time-domain using a computer simulation. In this case, the equation of motion is the following:

$$m\frac{d^2}{dt^2}x_0 = k(x_1 - x_0) + F(t) \tag{18}$$

$$m\frac{d^2}{dt^2}x_n = k(x_{n-1} - 2x_n + x_{n+1}) \tag{19}$$

$$m\frac{d^2}{dt^2}x_N = k(x_{N-1} - x_N). \tag{20}$$

3.1 Simulation for a system with 101 masses

In this document, a system comprising 101 masses and 100 sprinngs was studied using a simple simulation code. Table 1 shows the parameters used.

In the simulation, at $t = 0$ all of the masses were at rest. To see the low-frequency response of the system, at $t = 0$ a sinusoidal force of 5 mHz was applied to the excitation point (x_0), and the time evolution of the system was tracked for 6000 seconds. Because of the "sound transit time" of 10 seconds, if there

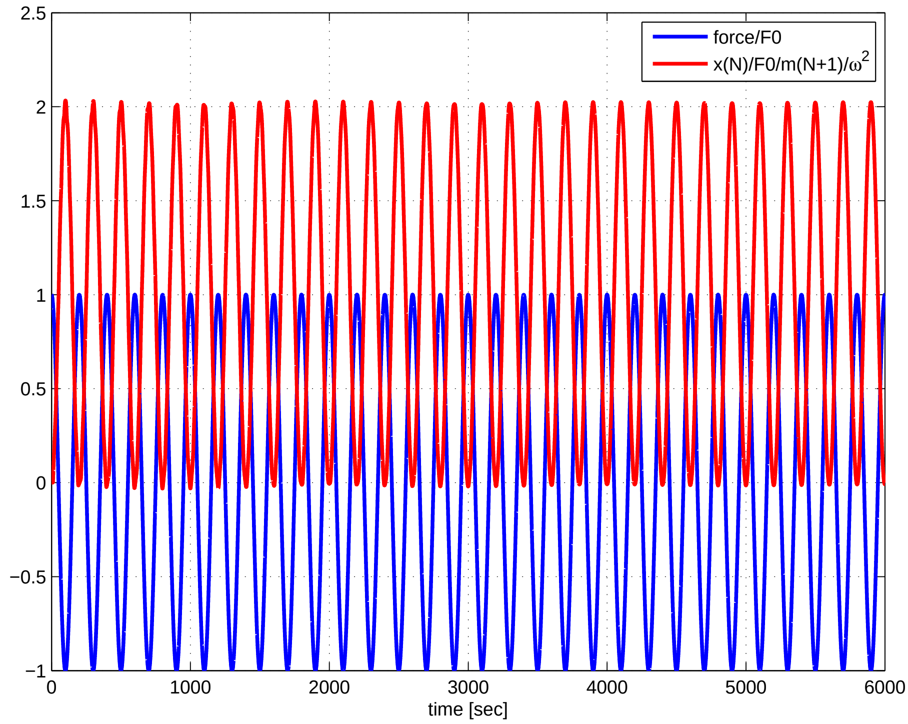

Figure 5: The entire simulation. The blue trace is the force applied to x_0 and the red trace is the displacement of x_N normalized by the response of a point mass with a mass of $m(N+1)$.

is a delay associated with the velocity of sound, after everything settled down one should be able to see $10 \text{ s} \times 5 \text{ mHz} \times 360 = 18 \text{ degrees}$ of phase delay.

One of the problem in such a simulation is that one has to include some loss to avoid numerical instability and to damp all of the higher order modes excited by the initial "kick" of the excitation. For this purpose, a loss corresponding to 10^{-2} of loss angle (at the excitation frequency) was added to the system. The phase shift of 10^{-2} radian or about 0.6 degrees caused by this loss is much smaller than the one that might be caused by the sound transit time, and therefore doesn't make it any more difficult to see if there is such a time delay.

Figure 5 shows the excitation and the displacement of the observation point for the entire simulation.

To draw any conclusion, however, we have to look closer, first at the end of the simulation and then at the beginning.

3.1.1 Transfer coefficient

Later in the simulation, when most of the eigenmodes are damped down, we can compare the actuation force and the displacement of the observation point to make a transfer coefficient or a transfer function at this excitation frequency.

Figure 6 shows the last 800 seconds or 4 actuation cycles of the simulation. No time delay as large as $L/v = 10 \text{ sec}$ is observed.

3.1.2 Transient response

Figure 7 shows the first 120 seconds of the simulation. In the top panel, an oscillation of 20 seconds period is observed. This is interpreted as the initial excitation of eigenmodes and the process of the summation of travelling waves (similar to Eq.10) going on at the same time.

The bottom panel shows the ratio of the displacement of the observation point and the excitation point. The displacement initially propagates with the speed of sound, and it takes about the "sound transit time" of 10 seconds for the observation point to catch up the motion of the excitation point, satisfying the causality

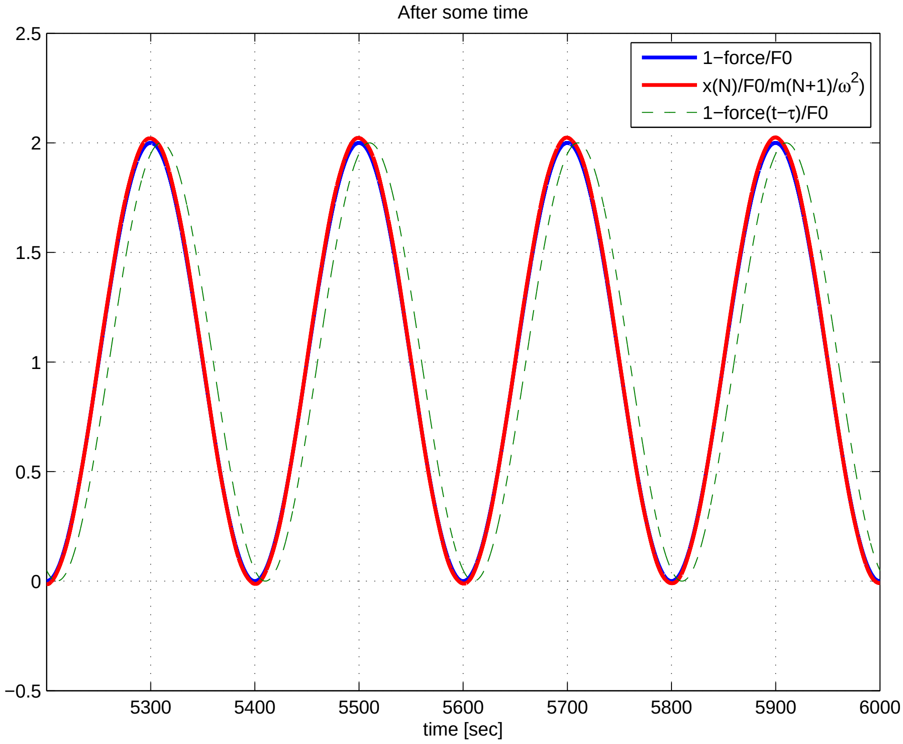

Figure 6: Last 800 seconds of the simulation. Blue trace is the excitation force, red is the displacement of the observation point, and the green is the excitation delayed by the sound transit time. Note that the excitation traces (blue and green) are phase-flipped for easier phase comparison.

in a loose sense in that the bulk of the energy injected into the system by the excitation was transmitted with the velocity of the sound.

Note that the term "causality" is sometimes used more strictly to mean "no information can be transmitted faster than a certain velocity". In this strict sense of causality, our "certain velocity" is not the speed of sound, but rather the speed of interaction between the masses (which ultimately is determined by the interaction between the atoms in the spring). Indeed, if you look at Fig.7 very closely, you'll find that the displacement of the observation point starts to change earlier than $t = 10$ sec. In this simulation, since no relativity is included, the time it takes for any information to be transmitted from x_0 to x_N is only limited by the time step, the number of masses and the numerical accuracy of the simulation.

4 Discussion

Two models, one analytical and one numerical, were shown to demonstrate that there is no reason to include a time delay $\tau = L/v$ in a model of mechanical transfer function of a finite elastic body even though the sound velocity is finite.

The only exception is when the losses in the system are so large that no standing wave pattern is formed. This condition is met when the product of the wavelength of the sound λ and the effective Q-value Q_{eff} of the medium and/or boundary at the excitation frequency is on the order of or smaller than the characteristic dimension L of the elastic body:

$$\lambda Q_{\text{eff}} = \frac{\lambda}{\phi} \lesssim L \tag{21}$$

where ϕ is the loss angle at the excitation frequency.

In the case of fused silica mirror, the velocity of the longitudinal and sheer sound wave are about 6 km/s and 4 km/s respectively.[1] At 100 Hz, the sound wavelengths are therefore about 60 and 40 meters. On the other hand, the intrinsic loss angle of the fused silica is on the order of 10^{-7} down to several Hz[2]. Even

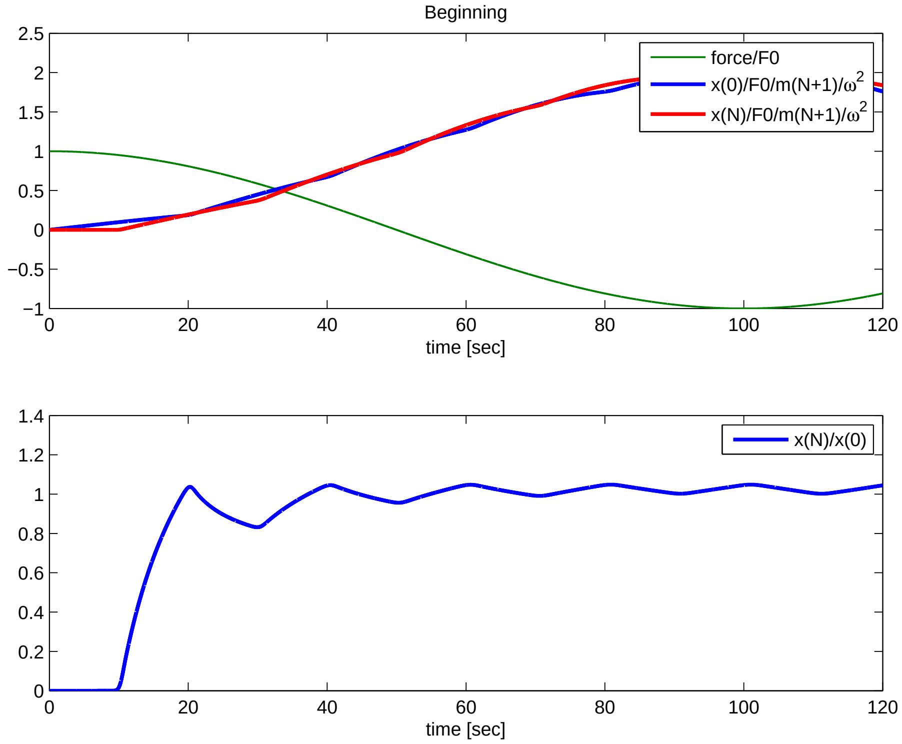

Figure 7: The first 120 seconds of the simulation. Top panel shows the excitation force (green), the displacement of the excitation point (blue) and the observation point (red). The bottom panel shows the ratio of the displacement of the two ends.

if one includes all of the other losses, it is difficult to imagine that the loss angle is much larger than 10^{-4} even at 100 Hz[3]. If you assume that the upper limit of the loss angle is 10^{-3}, the lower limit of λQ_{eff} is about 40 km.

This means that, for a fused silica mirror with a typical size of several tens of centimeters, there's always going to be a standing wave formed at 100 Hz, and therefore there is no time delay L/v in the transfer function.

5 Update: 3-D FEM

After this document was initially written, the author asked Phil Willems to perform a full 3-D FEM analysis of LIGO mirrors. His results, detailed in T080190[4], confirmed the conclusion of this document: No time-delay effect was observed.

References

[1] See e.g. CRC Handbook of Chemistry and Physics, 14-38.

[2] A. Heptonstall et. al, Phys. Lett. A 354 (2006) 353.

[3] K. Numata, et. al, Phys. Rev. Lett. 91 (2003) 260602.

[4] P. Willems, LIGO-T080190 (2008).