The $\mathcal{F}$-statistic and its implementation in `ComputeFStatistic_v2`

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Abstract

These notes represent a somewhat high-level documentation of `ComputeFStatistic_v2`, starting from a derivation and general discussion of the $\mathcal{F}$-statistic, down to expressions that very closely resemble what is actually implemented in the code.

Contents

1 The signal $h(t)$ measured at the detector

A plain gravitational wave $h_{\mu\nu}$ propagating along the unit-vector $-\hat{n}$ can be written in TT gauge as a purely spatial tensor $\underline{h}$, namely

$$\underline{h}(t,\vec{r}) = h_+(\tau)\,\underline{e}^+ + h_\times(\tau)\,\underline{e}^\times\,, \tag{1}$$

where $\tau = t + \hat{n}\cdot\vec{r}/c$ and the polarization tensors $\underline{e}^{+,\times}$ are defined as

$$\underline{e}^+ = \hat{u}\otimes\hat{u} - \hat{v}\otimes\hat{v}\,, \quad \underline{e}^\times = \hat{u}\otimes\hat{v} + \hat{v}\otimes\hat{u}\,, \tag{2}$$

in terms of the unit vectors $\hat{u},\hat{v}$, which form an orthonormal basis $\{\hat{u},\hat{v},-\hat{n}\}$ of the wave-frame. The choice of basis $\{\hat{u},\hat{v}\}$ in the wave-plane is arbitrary, but one often chooses preferred directions given either by the source-geometry or by the principal polarization axis of elliptically polarized waves. It is convenient to re-express this in a source-independent basis that only depends on the propagation direction $-\hat{n}$ of the wave and the choice of an SSB-fixed reference frame $\{\hat{x},\hat{y},\hat{z}\}$. Such a frame is given by the basis vectors $\hat{\xi} \equiv \hat{z}\times\hat{n}/|\hat{z}\times\hat{n}|$, $\hat{\eta} \equiv \hat{\xi}\times\hat{n}$ and $-\hat{n}$. We define the polarization angle ψ as the angle (anti-clockwise around $\hat{n}$) between $\hat{u}$ and $\hat{\xi}$, i.e. $\cos\psi = \hat{u}\cdot\hat{\xi}$. This allows us to express the polarization basis $\{\hat{u},\hat{v}\}$ in terms of the basis $\{\hat{\xi},\hat{\eta}\}$ by a simple rotation by ψ around $-\hat{n}$, namely

$$\hat{u} = \hat{\xi}\,\cos\psi + \hat{\eta}\,\sin\psi\,, \tag{3}$$
$$\hat{v} = -\hat{\xi}\,\sin\psi + \hat{\eta}\,\cos\psi\,. \tag{4}$$

Introducing polarization-independent basis tensors in the wave-frame,

$$\underline{\varepsilon}^+ \equiv \hat{\xi}\otimes\hat{\xi} - \hat{\eta}\otimes\hat{\eta}\,, \tag{5}$$
$$\underline{\varepsilon}^\times \equiv \hat{\xi}\otimes\hat{\eta} + \hat{\eta}\otimes\hat{\xi}\,, \tag{6}$$

we can now express the wave-basis $\underline{e}^{+,\times}$ as

$$\underline{e}^+ = \cos 2\psi\,\underline{\varepsilon}^+ + \sin 2\psi\,\underline{\varepsilon}^\times \tag{7}$$
$$\underline{e}^\times = \sin 2\psi\,\underline{\varepsilon}^+ + \cos 2\psi\,\underline{\varepsilon}^\times\,. \tag{8}$$

In the long-wavelength limit ($L \ll \lambda/2\pi$), the scalar response $h^X(t)$ of a detector X to a GW $\underline{h}$ is expressible simply in terms of its detector tensor $\underline{d}^X$, namely

$$h^X(t) = \underline{d}^X(t) : \underline{h}(\tau^X) = d^X_{ij}\,h^{ij}(\tau^X)\,, \tag{9}$$

where $\tau^X(t) = t + \hat{n}\cdot\vec{r}^X(t)/c$ is (neglecting relativistic corrections) the arrival time of a wavefront at the SSB, which arrives at the detector X (at position

$\vec{r}^{\mathrm{X}}$) at time t. This timing relation accounts for the Doppler effect due to the motion of the detector relative to the source. The detector tensor for an interferometer with arms along $\hat{n}_1$ and $\hat{n}_2$ is simply given by

$$\underline{d} = \frac{1}{2} \left(\hat{n}_1 \otimes \hat{n}_1 - \hat{n}_2 \otimes \hat{n}_2 \right) . \tag{10}$$

Using (1), we can write (9) in the form

$$h^{\mathrm{X}}(t) = F_+^{\mathrm{X}}(t) \, h_+(\tau^{\mathrm{X}}) + F_\times^{\mathrm{X}}(t) \, h_\times(\tau^{\mathrm{X}}) , \tag{11}$$

in terms of the so-called beam-pattern functions

$$F_+^{\mathrm{X}}(t; \hat{n}, \psi) \equiv \underline{d}^{\mathrm{X}}(t) : \underline{e}^+ , \quad F_\times^{\mathrm{X}}(t; \hat{n}, \psi) \equiv \underline{d}^{\mathrm{X}}(t) : \underline{e}^\times . \tag{12}$$

Changing to the polarization-independent basis $\underline{\varepsilon}^{+,\times}$ using (7), we find

$$\begin{aligned} F_+^{\mathrm{X}}(t; \hat{n}, \psi) &= a^{\mathrm{X}}(t; \hat{n}) \, \cos 2\psi + b^{\mathrm{X}}(t; \hat{n}) \, \sin 2\psi , & (13) \\ F_\times^{\mathrm{X}}(t; \hat{n}, \psi) &= b^{\mathrm{X}}(t; \hat{n}) \, \cos 2\psi - a^{\mathrm{X}}(t; \hat{n}) \, \sin 2\psi , & (14) \end{aligned}$$

where the antenna-pattern functions $a^{\mathrm{X}}, b^{\mathrm{X}}$ are defined as

$$a^{\mathrm{X}}(t; \hat{n}) \equiv \underline{d}^{\mathrm{X}}(t) : \underline{\varepsilon}^+(\hat{n}) , \quad b^{\mathrm{X}}(t; \hat{n}) \equiv \underline{d}^{\mathrm{X}}(t) : \underline{\varepsilon}^\times(\hat{n}) . \tag{15}$$

This formulation has the advantage of explicitly factoring out the polarization angle ψ. The sky-position $\hat{n}$ of the source is expressible in standard equatorial (or ecliptic) coordinates α (right ascension), and δ (declination) as

$$\hat{n} = (\cos \delta \, \cos \alpha, \cos \delta \, \sin \alpha, \sin \delta) , \tag{16}$$

and by the above definitions, the corresponding polarization-independent wave-plane basis $\hat{\xi}, \hat{\eta}$ is therefore expressible as

$$\begin{aligned} \hat{\xi} &= (-\sin \alpha, \, \cos \alpha, \, 0) , \\ \hat{\eta} &= (\cos \alpha \, \sin \delta, \, \sin \alpha \, \sin \delta, \, -\cos \delta) . \end{aligned} \tag{17}$$

The contractions (15) are explicitly given by

$$\underline{d} : \underline{\varepsilon} = d_{11}\varepsilon_{11} + d_{22}\varepsilon_{22} + d_{33}\varepsilon_{33} + 2 \left(d_{12}\varepsilon_{12} + d_{13}\varepsilon_{13} + d_{23}\varepsilon_{23} \right) , \tag{18}$$

where $\underline{\varepsilon}^{+,\times}$ are easily computed in SSB coordinates from (17), and the problem of computing a, b is therefore reduced to computing the detector tensor $\underline{d}^{\mathrm{X}}(t)$ as a function of time in this coordinate system.

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2 Continuous Waves

The GW class of "continuous waves" is characterized by a signal model $h_{+,\times}(\tau)$ (in the SSB) of the form

$$h_+(\tau) = A_+ \cos \Phi(\tau), \quad h_\times(\tau) = A_\times \sin \Phi(\tau). \tag{19}$$

Assuming a slowly varying intrinsic signal frequency $2\pi f(\tau) \equiv d\Phi(\tau)/d\tau$, the phase $\Phi(\tau)$ can be expanded around the reference time τ_{ref}, namely $\Phi(\tau) = \phi_0 + \phi(\Delta\tau)$, with $\phi_0 = \Phi(\tau_{\mathrm{ref}})$ and

$$\phi(\Delta\tau) = 2\pi \sum_{s=0} \frac{f^{(s)}}{(s+1)!} \, [\Delta\tau]^{s+1} . \tag{20}$$

The detector-specific timing relation relevant for isolated neutron stars is

$$\Delta\tau^{\mathrm{X}}(t; \hat{n}) \equiv \tau^{\mathrm{X}} - \tau_{\mathrm{ref}} = t - \tau_{\mathrm{ref}} + \frac{\vec{r}^{\mathrm{X}}(t) \cdot \hat{n}}{c}, \tag{21}$$

where τ^{X} is the arrival-time in the SSB of the phase at the detector X at time t. The spin parameters $f^{(s)}$ are defined as

$$f^{(s)} \equiv \left. \frac{d^s f(\tau)}{d\tau^s} \right|_{\tau_{\mathrm{ref}}} . \tag{22}$$

We denote the set of "Doppler parameters" affecting the time evolution of the phase $\phi(\Delta\tau^{\mathrm{X}})$ as $\lambda \equiv \{\hat{n}, f^{(s)}\}$. Combining (11), (13) (19), we find

$$h^{\mathrm{X}}(t; \mathcal{A}, \lambda) = \sum_{\mu=1}^{4} \mathcal{A}^\mu \, h_\mu^{\mathrm{X}}(t; \lambda), \tag{23}$$

with the four amplitude parameters $\mathcal{A}^\mu$ given by

$$\begin{aligned}
\mathcal{A}^1 &= A_+ \cos\phi_0 \cos 2\psi - A_\times \sin\phi_0 \sin 2\psi, \\
\mathcal{A}^2 &= A_+ \cos\phi_0 \sin 2\psi + A_\times \sin\phi_0 \cos 2\psi, \\
\mathcal{A}^3 &= -A_+ \sin\phi_0 \cos 2\psi - A_\times \cos\phi_0 \sin 2\psi, \\
\mathcal{A}^4 &= -A_+ \sin\phi_0 \sin 2\psi + A_\times \cos\phi_0 \cos 2\psi,
\end{aligned} \tag{24}$$

which is a re-parametrization of the (detector-independent) signal-parameters $A_+, A_\times, \phi_0, \psi$. The (detector-dependent) wave-components $h_\mu^{\mathrm{X}}(t; \lambda)$ are

$$\begin{aligned}
h_1^{\mathrm{X}}(t) &= a^{\mathrm{X}}(t) \cos\phi(\Delta\tau^{\mathrm{X}}), \quad h_2^{\mathrm{X}}(t) = b^{\mathrm{X}}(t) \cos\phi(\Delta\tau^{\mathrm{X}}), \\
h_3^{\mathrm{X}}(t) &= a^{\mathrm{X}}(t) \sin\phi(\Delta\tau^{\mathrm{X}}), \quad h_4^{\mathrm{X}}(t) = b^{\mathrm{X}}(t) \sin\phi(\Delta\tau^{\mathrm{X}}).
\end{aligned} \tag{25}$$

We see from (13) that a change of the polarization-angle $\psi' = \psi + \Delta\psi$ changes the antenna-pattern to

$$
\begin{aligned}
F'_+ &= F_+ \cos 2\Delta\psi + F_\times \sin 2\Delta\psi \,, & (26) \\
F'_\times &= -F_+ \sin 2\Delta\psi + F_\times \cos 2\Delta\psi \,, & (27)
\end{aligned}
$$

There is some residual gauge-freedom in the amplitude-parameters $\{A_+, A_\times, \psi, \phi_0\}$, namely

- $\psi \to \psi + \pi/2$, and $\phi_0 \to \phi_0 + \pi$.

- $\psi \to \psi + \pi/4$, $\phi_0 \to \phi_0 - \pi/2$ and $A_+ \leftrightarrow A_\times$.

- $\phi_0 \to \phi_0 + \pi$ and $A_+ \to -A_+$, $A_\times \to -A_\times$

In the case of a triaxial NS, the signal-amplitudes $A_{+/\times}$ are expressible explicitly in terms of the wave-amplitude h_0 and the inclination angle ι with respect to the line-of-sight, namely

$$
A_+ = \frac{1}{2} h_0 \left(1 + \cos^2 \iota \right) \,, \quad A_\times = h_0 \cos \iota \,. \tag{28}
$$

and the wave-amplitude h_0 can is given by

$$
h_0 = \frac{4\pi^2 G}{c^4} \frac{\epsilon \, I_{zz} \, f^2}{d} \,, \tag{29}
$$

in terms of the triaxial ellipticity $\epsilon \equiv (I_{xx} - I_{yy})/I_{zz}$, and the distance d.

3   Noise and detection statistic

We follow the notation of [5, 1] by denoting vectors of detector-specific quantities in boldface, i.e. $\{\boldsymbol{x}\}^{\mathrm{X}} = x^{\mathrm{X}}$. We can now write the explicit dependencies of the signal-model (23) on the signal-parameters as

$$
\boldsymbol{s}(t; \mathcal{A}, \lambda) = \mathcal{A}^\mu \, \boldsymbol{h}_\mu(t; \lambda) \,, \tag{30}
$$

where here and in the following we implicitly sum over 'amplitude-indices' $\mu, \nu \in \{1, 2, 3, 4\}$.

If the data $x^{\mathrm{X}}(t)$ measured at different detectors X consists of stationary Gaussian noise $n^{\mathrm{X}}(t)$ and a signal with parameters $\mathcal{A}_{\mathrm{s}}$, λ_{s}, then we can write

$$
\boldsymbol{x}(t) = \boldsymbol{n}(t) + \boldsymbol{s}(t; \mathcal{A}_{\mathrm{s}}, \lambda_{\mathrm{s}}) \,, \tag{31}
$$

in terms of the signal-model (30).

In the general case of correlated noises $n^{\mathrm{X}}(t)$ (which can be relevant for the two LIGO detectors in Hanford, or for LISA), we define the (single-sided!) noise density matrix as

$$S^{\mathrm{XY}}(f) = 2 \int_{-\infty}^{\infty} \kappa^{\mathrm{XY}}(\tau)\, e^{-i2\pi f\tau}\, d\tau\,, \tag{32}$$

in terms of the correlation-functions

$$\kappa^{\mathrm{XY}}(\tau) \equiv E\left[n^{\mathrm{X}}(t+\tau)\, n^{\mathrm{Y}}(t)\right]\,. \tag{33}$$

We can now define the multi-detector scalar product (in analogy to [3]) as

$$(\boldsymbol{x}|\boldsymbol{y}) \equiv 4\,\Re \int_0^{\infty} \tilde{x}^{\mathrm{X}}(f)\, S_{\mathrm{XY}}^{-1}(f)\, \tilde{y}^{\mathrm{Y}*}(f)\, df\,, \tag{34}$$

where $\Re$ denotes the real part, and we use implicit summation over repeated upper and lower detector-indicies, and the inverse noise-matrix $S_{\mathrm{XY}}^{-1}\, S^{\mathrm{YZ}} = \delta_{\mathrm{X}}^{\mathrm{Z}}$. In the case of uncorrelated noises $S^{\mathrm{XY}} = S^{\mathrm{X}}\, \delta^{\mathrm{XY}}$, this scalar product reduces to

$$(\boldsymbol{x}|\boldsymbol{y}) = \sum_{\mathrm{X}} (x^{\mathrm{X}}|y^{\mathrm{X}})\,, \tag{35}$$

in terms of the usual single-detector scalar product

$$(x^{\mathrm{X}}|y^{\mathrm{X}}) \equiv 4\,\Re \int_0^{\infty} \frac{\tilde{x}^{\mathrm{X}}(f)\, \tilde{y}^{\mathrm{X}*}(f)}{S^{X}(f)}\, df\,. \tag{36}$$

With the definition (36) of the multi-detector scalar product, the likelihood-function for Gaussian stationary noise can be written simply as

$$P\left(\boldsymbol{n}(t)|S^{\mathrm{XY}}\right) = k\, e^{-\frac{1}{2}(\boldsymbol{n}|\boldsymbol{n})}\,, \tag{37}$$

where k is a normalization factor, which is independent of the noise $\boldsymbol{n}$. This expression can be used with (31) to obtain the likelihood-function of observing data $\boldsymbol{x}(t)$ *assuming* a signal $\{\mathcal{A}, \lambda\}$, namely

$$P(\boldsymbol{x}|\mathcal{A}, \lambda, S^{\mathrm{XY}}) = k\, e^{-\frac{1}{2}(\boldsymbol{x}|\boldsymbol{x})}\, \exp\left[(\boldsymbol{x}|\boldsymbol{s}) - \frac{1}{2}(\boldsymbol{s}|\boldsymbol{s})\right]\,. \tag{38}$$

Using Bayes' theorem, we obtain the posterior probability for a signal $\{\mathcal{A}, \lambda\}$ given the data $\boldsymbol{x}$ as

$$P(\mathcal{A}, \lambda|\boldsymbol{x}, S^{\mathrm{XY}}) = k'\, \exp\left[(\boldsymbol{x}|\boldsymbol{s}) - \frac{1}{2}(\boldsymbol{s}|\boldsymbol{s})\right]\, P(\mathcal{A}, \lambda)\,, \tag{39}$$

in terms of the prior probability of the signal $P(\mathcal{A}, \lambda)$, and the normalization factor k' which is independent of the signal.

Assuming a flat prior $P(\mathcal{A}, \lambda) = \text{const.}$ and substituting the signal-model (23) into (39), we can write the posterior probability as

$$\log P(\mathcal{A}, \lambda | \boldsymbol{x}, S^{\mathrm{XY}}) = \log P_0 + \mathcal{A}^\mu \, x_\mu(\lambda) - \frac{1}{2} \mathcal{A}^\mu \, \mathcal{M}_{\mu\nu}(\lambda) \, \mathcal{A}^\nu \,, \qquad (40)$$

with implicit summation over 'amplitude-indices', and where we defined

$$x_\mu(\lambda) \;\equiv\; (\boldsymbol{x} | \boldsymbol{h}_\mu) \,, \qquad (41)$$

$$\mathcal{M}_{\mu\nu}(\lambda) \;\equiv\; (\boldsymbol{h}_\mu | \boldsymbol{h}_\nu) \,. \qquad (42)$$

We can now analytically maximize the posterior (127) with respect to the four amplitudes $\mathcal{A}^\mu$, to obtain the partially-maximized (not marginalized!) posterior probabilty for the Doppler-parameters λ, which defines the so-called "$\mathcal{F}$-statistic", namely

$$2\mathcal{F}(\lambda | \boldsymbol{x}) = x_\mu \, \mathcal{M}^{\mu\nu} \, x_\nu \,, \qquad (43)$$

where $\mathcal{M}^{\mu\nu} \equiv \{\mathcal{M}^{-1}\}^{\mu\nu}$, i.e. $\mathcal{M}_{\mu\alpha} \mathcal{M}^{\alpha\nu} = \delta^\nu_\mu$. Note that we can consider the four vectors $\boldsymbol{h}_\mu$ as a basis on the 'amplitude-space' μ (for fixed λ), and $\mathcal{M}$ the corresponding metric on this space, i.e. we can use it to raise and lower amplitude-indices, and define the dual basis as

$$\boldsymbol{h}^\mu(t) \equiv \mathcal{M}^{\mu\nu} \, \boldsymbol{h}_\nu(t) \,, \qquad (44)$$

such that $(\boldsymbol{h}^\mu | \boldsymbol{h}_\nu) = \delta^\mu_\nu$. The corresponding "amplitude-components" of the data-vector $\boldsymbol{x}(t)$ with respect to the dual basis-vectors $\boldsymbol{h}^\mu(t)$ are

$$x^\mu = (\boldsymbol{x} | \boldsymbol{h}^\mu) = \mathcal{M}^{\mu\nu} \, x_\nu, \qquad (45)$$

as would be expected. With this notation the $\mathcal{F}$-statistic can be written even more compactly as

$$2\mathcal{F}(\lambda) = x^\mu \, x_\mu \,. \qquad (46)$$

The maximum-likelihood estimators for the four unknown amplitudes $\mathcal{A}^\mu$ are given by

$$\mathcal{A}^\mu_{\mathrm{MLE}} = \mathcal{M}^{\mu\nu} \, x_\nu \,. \qquad (47)$$

4   Practical computation

Assuming $x(t)$ and $y(t)$ to describe such narrow-band continuous signals, we can approximate the scalar product (34) by

$$(\boldsymbol{x} | \boldsymbol{y}) \approx 2 S_{\mathrm{XY}}^{-1}(f_{\mathrm{s}}) \int_0^T x^{\mathrm{X}} \, y^{\mathrm{Y}} \, dt \,. \qquad (48)$$

In order to simplify the following derivations and notation, we restrict ourselves to the more common case of uncorrelated detector-noises, i.e. we assume $S^{\mathrm{XY}} = S^{\mathrm{X}} \delta^{\mathrm{XY}}$. On the other hand, we want to generalize the above approach in the sense that we don't have to assume the noise to be exactly stationary.

In practice we will be computing the power-spectra $S^{\mathrm{X}}(f)$ over shorter time-periods T_{SFT}, and we can allow the noise to be slowly varying from one SFT α to the next, i.e. $S_{\alpha}^{\mathrm{X}}(f)$. Let us now define the (SFT-dependent) noise-weights

$$w_{\alpha}^{\mathrm{X}}(f) \equiv \frac{\left\langle S_{\alpha,k}^{\mathrm{X}} \right\rangle_{k}^{-1}}{\mathcal{S}^{-1}} \,, \tag{49}$$

where the average $\langle . \rangle_{k}$ is over the frequency-bins k of the SFT α of detector X. The quantity $\mathcal{S}^{-1}$ is in principle an arbitrary normalization factor, and for practical reasons we will chose it in such a way as to make the weights w^{X} roughly of order unity, namely (see `LALComputeMultiNoiseWeights()`)

$$\mathcal{S}^{-1} \equiv \left\langle S_{\alpha,k}^{\mathrm{X}} \right\rangle_{\alpha,k,\mathrm{X}}^{-1} \,, \tag{50}$$

where the average $\langle . \rangle_{\alpha,k,\mathrm{X}}$ is over the SFT frequency-bins k of all SFTs α of all detectors X. With this we can write the scalar product as

$$(\boldsymbol{x}|\boldsymbol{y}) = 2\, T\, \mathcal{S}^{-1} \, \langle x\,y \rangle_{S} \,, \tag{51}$$

where we defined the noise-weighted average

$$\langle x\,y \rangle_{S} \equiv \frac{1}{T} \sum_{\mathrm{X}} w^{\mathrm{X}} \int_{0}^{T} x^{\mathrm{X}}(t)\, y^{\mathrm{X}}(t)\, dt \,. \tag{52}$$

The scalar products encountered in the $\mathcal{F}$-statistic typically involve products of slowly-varying functions (such as the antenna-patterns $a(t), b(t)$), and phase-functions $\sin\phi(t)$ and $\cos\phi(t)$, which are (approximately) *periodic* on very short timescales $\tau \sim 1/f \ll T$.

In this approximation, the 4×4 matrix $\mathcal{M}_{\mu\nu}$ of (42), namely

$$\mathcal{M}_{\mu\nu} = (\boldsymbol{h}_{\mu}|\boldsymbol{h}_{\nu}) = \mathcal{S}^{-1} T\, \chi_{\mu\nu} \,, \tag{53}$$

reduces to the the block-form

$$\chi_{\mu\nu} = 2\, \langle h_{\mu} h_{\nu} \rangle_{S} \approx \begin{pmatrix} A & C & 0 & 0 \\ C & B & 0 & 0 \\ 0 & 0 & A & C \\ 0 & 0 & C & B \end{pmatrix} \,, \tag{54}$$

with the 3 independent components

$$A \equiv \left\langle a^2 \right\rangle_S \,, \quad B \equiv \left\langle b^2 \right\rangle_S \,, \quad C \equiv \left\langle a\,b \right\rangle_S \,. \tag{55}$$

and we define the determinant $D \equiv A\,B - C^2$.

We can now write the $\mathcal{F}$-statistic defined in (43) more explicitly as

$$2\mathcal{F} = x^\mu \, \mathcal{M}^{-1}_{\mu\nu} \, x^\nu = \frac{1}{T\mathcal{S}^{-1}D} \left[B\left(x_1^2 + x_3^2 \right) + A\left(x_2^2 + x_4^2 \right) - 2C\left(x_1 x_2 + x_3 x_4 \right) \right] \,, \tag{56}$$

where (see (41) and (51))

$$x_\mu \equiv (\boldsymbol{x}|\boldsymbol{h}_\mu) = 2\,\mathcal{S}^{-1} \sum_X \int_0^T w^X \, x^X \, h_\mu^X \, dt \,. \tag{57}$$

Now, in the presence of pure noise, the $\mathcal{F}$ expectation-value is $E[2\mathcal{F}] = 4$. For practical and numerical convenience, we want to make sure that all quantities involved in computing $\mathcal{F}$ are roughly of order $\mathcal{O}(1)$. This is already the case for the antenna-pattern functions A, B, C, the only quantities of wildly different scale are the input data $x^X(t)$, which are of scale

$$E[x(t)] \sim h_0 \sim 10^{-20} \,, \tag{58}$$

or thereabouts. Therefore we want to re-normalize the data first by the 'noise-floor' h_0 such that its expectation-value is of order $\mathcal{O}(1)$. By the Wiener-Khintchine theorem we can estimate the (single-sided!) PSDs $S_\alpha^X(f)$ as

$$T_{\mathrm{SFT}}\, S_\alpha^X(f) \sim 2E\left[\left| \widetilde{x}_\alpha^X(f) \right|^2 \right] \,, \tag{59}$$

where $\widetilde{x}_\alpha^X(f)$ is the "Short Fourier transform" (SFT)

$$\widetilde{x}_\alpha^X(f) = \int_0^{T_{\mathrm{SFT}}} x_\alpha^X(t)\, e^{-i2\pi ft}\, dt = T_{\mathrm{SFT}} \left\langle x_\alpha^X\, e^{-i2\pi ft} \right\rangle \sim T_{\mathrm{SFT}}\, h_0 \,, \tag{60}$$

and the PSD is therefore found to be of the order

$$S_\alpha^X(f) \sim 2T_{\mathrm{SFT}}\, h_0^2 \,. \tag{61}$$

Therefore, if we re-normalize the data as

$$y_\alpha^X(t) \equiv \frac{x_\alpha^X(t)}{\sqrt{T_{\mathrm{SFT}} S_\alpha^X/2}} \,, \tag{62}$$

we have $E[y_\alpha^X(t)] \sim \mathcal{O}(1/T_{\mathrm{SFT}})$ and $E[\widetilde{y}_\alpha^X] \sim \mathcal{O}(1)$.

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Note that in the special `--SignalOnly` case, the code (`ComputeFStatistic_v2`) does not try to normalize the data and instead *assumes* the (single-sided) noise-power to be $S^X = 1$. Furthermore, in this case therefore have $T_{\mathrm{SFT}} \mathcal{S}^{-1} = T_{\mathrm{SFT}}$. Given that the data does not get normalized in this case, there is now a "missing" normalization-factor of $\sqrt{T_{\mathrm{SFT}}/2}$, which is applied to F_a, F_b and instead.

The projected data-components x_μ can be expressed as

$$x_\mu = \sqrt{2\gamma}\, \hat{x}_\mu \,, \tag{63}$$

where we defined

$$\hat{x}_\mu \equiv \sum_X \int \sqrt{w^X} y^X h^X_\mu \, dt \,, \tag{64}$$

and the overall noise-normalization factor

$$\gamma \equiv \mathcal{S}^{-1}\, T_{\mathrm{SFT}} \,. \tag{65}$$

Therefore,

$$x_1 - i\, x_3 = \sqrt{2\gamma} \sum_X \int \sqrt{w^X} y^X a^X e^{-i\phi^X} \, dt \,, \tag{66}$$

$$x_2 - i\, x_4 = \sqrt{2\gamma} \sum_X \int \sqrt{w^X} y^X b^X e^{-i\phi^X} \, dt \,. \tag{67}$$

Comparing this to the antenna-pattern factors (55), we see that we can absorb the noise-weighting factors w^X_α completely into the antenna-pattern functions by defining

$$\widehat{a}^X(t) \equiv \sqrt{w^X(t)}\, a^X(t) \,, \quad \widehat{b}^X(t) \equiv \sqrt{w^X(t)}\, b^X(t) \,, \tag{68}$$

where strictly speaking, $\widehat{a}$ and $\widehat{b}$ are now functions of frequency, but for most practical purposes (if the frequency-band is not too large) we should be able to approximate $w^X_\alpha(f) \approx w^X_\alpha(f_0)$ at some intermediate frequency f_0.

This allows us to write

$$A = \frac{1}{T} \sum \int \widehat{a}^X \widehat{a}^X \, dt, \quad B = \frac{1}{T} \sum \int \widehat{b}^X \widehat{b}^X \, dt \quad C = \frac{1}{T} \sum \int \widehat{a}^X \widehat{b}^X \, dt \,. \tag{69}$$

We now define the quantities

$$F^X_a \equiv \int_0^T y^X \widehat{a}^X \, e^{-i\phi^X} \, dt, \quad F^X_b \equiv \int_0^T y^X \widehat{b}^X \, e^{-i\phi^X} \, dt \,. \tag{70}$$

and

$$F_a \equiv \sum_{\mathrm{X}} F_a^{\mathrm{X}}, \quad F_b \equiv \sum_{\mathrm{X}} F_b^{\mathrm{X}}, \tag{71}$$

and we see from (66) that $F_{a,b}$ is related to $\hat{x}_\mu$ simply via

$$\hat{x}_1 = \Re(F_a), \quad \hat{x}_2 = \Re(F_b), \quad \hat{x}_3 = -\Im(F_a), \quad \hat{x}_4 = -\Im(F_b). \tag{72}$$

We can now express the $\mathcal{F}$-statistic (56) in a more suitable form for numerical evaluation, namely

$$2\mathcal{F} = \left(\frac{T_{\mathrm{SFT}}}{T}\right) \frac{2}{D} \left[B|F_a|^2 \right] + A|F_b|^2 - 2C\Re\left(F_a\, F_b^* \right) \right] \tag{73}$$

The integrals (70) have the structure of a Fourier-transformation and one should evaluate them using an efficient FFT-algorithm. For historical reasons, however, we first consider a more "direct" method of calculation which is more readily implementable in the existing codes.

Note that in practice (see next section), the integrals will be computed using discrete sums over SFTs α, and it will be more convenient to use "discretized" versions of A, B, C of (69), so we define

$$\hat{A} \equiv \sum_{\mathrm{X},\alpha} \widehat{a}_\alpha^{\mathrm{X}} \widehat{a}_\alpha^{\mathrm{X}}, \quad \hat{B} \equiv \sum_{\mathrm{X},\alpha} \widehat{b}_\alpha^{\mathrm{X}} \widehat{b}_\alpha^{\mathrm{X}}, \quad \hat{C} \equiv \sum_{\mathrm{X},\alpha} \widehat{a}_\alpha^{\mathrm{X}} \widehat{b}_\alpha^{\mathrm{X}}, \tag{74}$$

which are related to A, B, C via

$$A = \frac{T_{\mathrm{SFT}}}{T}\, \hat{A}, \quad B = \frac{T_{\mathrm{SFT}}}{T}\hat{B}, \quad C = \frac{T_{\mathrm{SFT}}}{T}\, \hat{C}, \tag{75}$$

and therefore

$$D = \left(\frac{T_{\mathrm{SFT}}}{T}\right)^2 \hat{D}, \tag{76}$$

where $\hat{D} \equiv \hat{A}\,\hat{B} - \hat{C}^2$, such that (73) can now equivalently be expressed as

$$2\mathcal{F} = \frac{2}{\hat{D}} \left[\hat{B}\,|F_a|^2 \right] + \hat{A}\,|F_b|^2 - 2\hat{C}\,\Re\left(F_a\, F_b^* \right) \right]. \tag{77}$$

Introducing the 'discretized' version $\hat{\mathcal{M}}_{\mu\nu}$ as

$$\hat{\mathcal{M}}_{\mu\nu} \equiv \begin{pmatrix} \hat{\chi} & 0 \\ 0 & \hat{\chi} \end{pmatrix}, \quad \text{where} \quad \hat{\chi} \equiv \begin{pmatrix} \hat{A} & \hat{C} \\ \hat{C} & \hat{B} \end{pmatrix}, \tag{78}$$

we see from (53) that this is related to $\mathcal{M}_{\mu\nu}$ via

$$\mathcal{M}_{\mu\nu} = \gamma\, \hat{\mathcal{M}}_{\mu\nu}. \tag{79}$$

In terms of these quantities, we can express the maximum-likelihood estimators for the amplitudes (24) explicitly as

$$
\mathcal{A}^{\mu}_{\mathrm{MLE}} = \mathcal{M}^{\mu\nu} x_\mu = \frac{2}{\sqrt{2\gamma}} \hat{\mathcal{M}}^{\mu\nu} \hat{x}_\nu = \frac{2}{\hat{D}\sqrt{2\gamma}}
\begin{pmatrix}
\hat{B}\,\hat{x}_1 - \hat{C}\,\hat{x}_2 \\
-\hat{C}\,\hat{x}_1 + \hat{A}\,\hat{x}_2 \\
\hat{B}\,\hat{x}_3 - \hat{C}\,\hat{x}_4 \\
-\hat{C}\,\hat{x}_3 + \hat{A}\,\hat{x}_4
\end{pmatrix} , \qquad (80)
$$

4.0.1 The Williams-Schutz approximation ("LALDemod")

(This section is originally based on Xavie's LALDemod-notes
`http://www.lsc-group.phys.uwm.edu/~siemens/demod.pdf`.)
Note that the multi-detector components of $\mathcal{F}$ entering (73), namely F_a, F_b of (70), and A, B, C of (69) are all simply *sums* over detector-specific quantities $F_a^{\mathrm{X}}, F_b^{\mathrm{X}}$ and $A^{\mathrm{X}}, B^{\mathrm{X}}, C^{\mathrm{X}}$, which can therefore be calculated independently for each detector. In the following we focus on calculating these individual detector pieces, and will therefore completely drop the detector-index for simplicity of notation.
Based on the approximation-scheme suggested in [7], we now re-index the (normalized) data timeseries y_i with respect to time-stretches α of duration T_{SFT}, and perform a DFT on each of those short-duration signals (an "SFT").

$$
y_{\alpha,j} = y(t_{\alpha,j}) , \quad \text{where} \quad t_{\alpha,j} = \alpha\, T_{\mathrm{SFT}} + j\Delta t , \qquad (81)
$$

and $j = 0, ..., N-1$, and $\alpha = 0, ..., M-1$ and $T_{\mathrm{SFT}} = N\Delta t$. Furthermore we define the SFTs as

$$
\widetilde{y}_{\alpha;k} \equiv \Delta t \sum_{j=0}^{N-1} y_{\alpha,j}\, e^{-i2\pi kj/N} , \qquad (82)
$$

which is *exactly* what is stored in an SFT following the "SFT-v2" specification (`LIGO-T040164-01-Z`). Note, however, that in practice we only store the first $\lfloor N/2 \rfloor$ frequency-bins, as for real y_j we have $\widetilde{y}_{N-k|N} = \widetilde{y}_k^*$.
The inverse operation to (82) is

$$
y_{\alpha,j} = \Delta f \sum_{k=0}^{N-1} \widetilde{y}_{\alpha;k}\, e^{i2\pi kj/N} . \qquad (83)
$$

We write the discretized version of (70) using the SFT-indexing (81), which results in

$$
F_a^{\mathrm{X}} = \Delta t \sum_{\alpha=0}^{M-1} \sum_{j=0}^{N-1} y_{\alpha,j}\, \widehat{a}_{\alpha,j}\, e^{-i2\pi\varphi_{\alpha,j}} , \qquad (84)
$$

where we defined $2\pi\varphi_{\alpha,j} \equiv \phi(t_{\alpha,j})$. (Note that the factor of 2π is taken out for convenience in the later evaluations of this sum).

The typical SFT-durations are less than an hour, say, and we can therefore approximate the antenna-pattern functions as nearly constant over this period (e.g. picking the SFT-midpoints), so $\widehat{a}_{\alpha,j} \approx \widehat{a}_{\alpha}$. Using this and the inverse DFT (83), we can write this as

$$F_a^{\mathrm{X}} = \Delta f \Delta t \sum_{\alpha=0}^{M-1} \widehat{a}_{\alpha} \sum_{j=0}^{N-1} e^{-i2\pi\varphi_{\alpha,j}} \sum_{k=0}^{N-1} \widetilde{y}_{\alpha;k} e^{i2\pi jk/N} . \tag{85}$$

The phase-evolution of a typical continuous pulsar-signal is dominated by the linear term $\phi(t) \approx 2\pi f t$, and we approximate it by a first-order expansion around each SFT-midpoint, namely $t_{\alpha,\frac{1}{2}} \equiv (\alpha + \frac{1}{2})T_{\mathrm{SFT}}$, for which we find

$$\varphi_{\alpha,j} = \varphi_{\alpha,\frac{1}{2}} + \dot{\varphi}_{\alpha,\frac{1}{2}} \, T_{\mathrm{SFT}} \left(\frac{j}{N} - \frac{1}{2} \right) + \mathcal{O}(2) . \tag{86}$$

Using this expansion we obtain

$$F_a^{\mathrm{X}} = \Delta t \, \Delta f \sum_{\alpha=0}^{M-1} \widehat{a}_{\alpha} e^{-i2\pi\lambda_{\alpha}} \sum_{k=0}^{N-1} \widetilde{y}_{\alpha,k} \sum_{j=0}^{N-1} e^{-i2\pi\kappa(\alpha;k)\,j/N} , \tag{87}$$

where we defined

$$\lambda_{\alpha} \equiv \varphi_{\alpha,\frac{1}{2}} - \frac{1}{2}\dot{\varphi}_{\alpha,\frac{1}{2}} \, T_{\mathrm{SFT}} , \tag{88}$$

$$\kappa(\alpha;k) \equiv \dot{\varphi}_{\alpha,\frac{1}{2}} \, T_{\mathrm{SFT}} - k . \tag{89}$$

The last sum is simply a geometrical series, and so we find

$$\begin{aligned}
\sum_{j=0}^{N-1} e^{-i2\pi\kappa(\alpha;k)j/N} &= \frac{1 - e^{-i2\pi\kappa}}{1 - e^{-i2\pi\kappa/N}} \\
&\overset{N\gg 1}{\approx} \frac{N}{2\pi} \left(\frac{\sin 2\pi\kappa}{\kappa} + i\frac{\cos 2\pi\kappa - 1}{\kappa} \right) \\
&\equiv \frac{N}{2\pi} P\left(\kappa(\alpha;k)\right) = \frac{N}{2\pi} P_{\alpha;k} .
\end{aligned} \tag{90}$$

The function $P(x)$ is sometimes called "Dirichlet kernel", and it has the property of being strongly peaked around $x = 0$, and so we can truncate the sum over k in (87) to a few terms Δk (referred to as "Dterms" in the code) on either side of k^*, corresponding the to maximum of $P(\kappa)$, namely

$$k^* \equiv \mathrm{round}\left[\dot{\varphi}_{\alpha;\frac{1}{2}} \, T_{\mathrm{SFT}} \right] = \mathrm{round}\left[\frac{\hat{f}(t_{\alpha,\frac{1}{2}})}{\Delta f} \right] , \tag{91}$$

where $\hat{f}(t)$ is the "effective" signal-frequency in the detector frame (the time-derivative $\dot{\varphi}$ refers to the time in the detector-frame!), which shows that generally we'll have $k^* \gg 1$. With this approximation we find

$$F_a^{\mathrm{X}} \approx \frac{1}{2\pi} \sum_{\alpha=0}^{M-1} \widehat{a}_\alpha e^{-i2\pi\lambda(\alpha)} \sum_{k=k^*-\Delta k}^{k^*+\Delta k} \widetilde{y}_{\alpha;k} \, P_{\alpha;k} \,. \tag{92}$$

We'll also need explicit expressions for $\varphi_{\alpha,\frac{1}{2}}$ and $\dot{\varphi}_{\alpha,\frac{1}{2}}$ in order to compute $\lambda(\alpha)$ and $\kappa(\alpha; k)$, defined in (88). For this, we only need the timing-function $\tau(t)$ translating detector-arrival times to SSB, which is in the simplest purely Newtonian approximation is $\tau(t) = t + \vec{n} \cdot \vec{r}(t)/c$. Given this function, we define

$$\Delta\tau_\alpha \;\equiv\; \tau(t_{\alpha,\frac{1}{2}}) - \tau_{\mathrm{ref}} \,, \tag{93}$$

$$\dot{\tau}_\alpha \;\equiv\; \frac{d\tau}{dt}(t_{\alpha,\frac{1}{2}}) \approx 1 + \frac{\boldsymbol{v} \cdot \vec{n}}{c} \,, \tag{94}$$

(which are called `SSBtimes` in the code), and so the (full) phase-model (20) yields

$$\varphi_{\alpha,\frac{1}{2}} \;=\; \sum_s \frac{f^{(s)}}{(s+1)!} \Delta\tau_\alpha^{s+1} \,, \tag{95}$$

$$\dot{\varphi}_{\alpha,\frac{1}{2}} \;=\; \dot{\tau}_\alpha \sum_s \frac{f^{(s)}}{s!} \Delta\tau_\alpha^s \,. \tag{96}$$

4.0.2 Efficient computation of F_a^{X}

The computation of (92) will be the most time-consuming part in this code, in particular the "hot loop" which is the sum over k. It is therefore important to compute these sums in the most efficient way possible.

First it will be convenient to relabel this sum using $l \equiv k - k_0$ with $k_0 \equiv k^* - \Delta k$, and $\mathcal{N} \equiv 2\Delta k$, and so we find

$$\kappa(\alpha, l) = \kappa_\alpha - l \,, \tag{97}$$

where

$$\kappa_\alpha \equiv \mathrm{rem}\left(\dot{\varphi}_{\alpha,\frac{1}{2}} T_{\mathrm{SFT}}\right) + \Delta k \,, \tag{98}$$

and where we defined the "remainder"

$$\mathrm{rem}(x) \equiv x - \mathrm{round}\,[x] \,. \tag{99}$$

Next we note that

$$\sin 2\pi\kappa(\alpha, l) = \sin 2\pi\kappa_\alpha \equiv s_\alpha \tag{100}$$

$$\cos 2\pi\kappa(\alpha, l) - 1 = \cos 2\pi\kappa_\alpha - 1 \equiv c_\alpha \,, \tag{101}$$

and so the Dirichlet-kernel (90) has the form

$$P_{\alpha;k} = \frac{s_\alpha}{\kappa_\alpha - l} + i\frac{c_\alpha}{\kappa_\alpha - l} \,. \tag{102}$$

Now let us look at the "hot loop" in (92), which we can express as

$$\mathrm{XP}_\alpha \equiv \sum_{k=k_0}^{k_0+\mathcal{N}} \widetilde{y}_{\alpha;k} P_{\alpha;k} = [s_\alpha\, U_\alpha - c_\alpha\, V_\alpha] + i\, [c_\alpha\, U_\alpha + s_\alpha\, V_\alpha] \,, \tag{103}$$

where the two sums we need to evaluate are

$$U_\alpha \equiv \sum_{l=0}^{\mathcal{N}} \frac{u_l}{p_l} \,, \quad V_\alpha \equiv \sum_{l=0}^{\mathcal{N}} \frac{v_l}{p_l} \,, \tag{104}$$

with the further definitions

$$p_l \equiv \kappa_\alpha - l \,, \tag{105}$$

$$u_l \equiv \Re(\widetilde{y}_{\alpha;k_0+l}) \,, \tag{106}$$

$$v_l \equiv \Im(\widetilde{y}_{\alpha;k_0+l}) \,. \tag{107}$$

The above sums (104) are numerically not efficient as they consist of many divisions, which are slower than multiplications. This can be remedied with a clever algorithm suggested by Fekete Ãkos: bringing the sums on a common denominator $q_{\mathcal{N}}$, we get

$$U_\alpha = \frac{S_{\mathcal{N}}}{q_{\mathcal{N}}} \,, \quad V_\alpha = \frac{T_{\mathcal{N}}}{q_{\mathcal{N}}} \,, \tag{108}$$

where

$$S_{\mathcal{N}} = u_0\, p_1 p_2...p_{\mathcal{N}} + p_0\, u_1\, p_1...p_{\mathcal{N}} + ... + p_0 p_1...p_{\mathcal{N}-1}\, u_{\mathcal{N}} \,, \tag{109}$$

$$T_{\mathcal{N}} = v_0\, p_1 p_2...p_{\mathcal{N}} + p_0\, v_1\, p_1...p_{\mathcal{N}} + ... + p_0 p_1...p_{\mathcal{N}-1}\, v_{\mathcal{N}} \,, \tag{110}$$

$$q_{\mathcal{N}} = p_0 p_1...p_{\mathcal{N}} \,, \tag{111}$$

reducing the $2\mathcal{N}$ divisions to only 2. The required three components $S_{\mathcal{N}}$, $T_{\mathcal{N}}$ and $q_{\mathcal{N}}$ can be computed efficiently using the following recurrence:

$$S_n = p_n S_{n-1} + q_{n-1} u_n \,, \tag{112}$$

$$T_n = p_n T_{n-1} + q_{n-1} v_n \,, \tag{113}$$

$$p_n = p_{n-1} - 1 \,, \tag{114}$$

$$q_n = p_n\, q_{n-1} \,, \tag{115}$$

and the starting conditions

$$S_0 = u_0 \,, \tag{116}$$

$$T_0 = v_0 \,, \tag{117}$$

$$p_0 = \kappa_\alpha \,, \tag{118}$$

$$q_0 = p_0 \,. \tag{119}$$

The number of floating-point operations per iteration is 8, so in total we need $8\mathcal{N} + 8$ operations (not counting one sin/cos), of which only 2 are divisions. In the previous "LALDemod" algorithm (e.g `ComputeFstat.c:1.19`) XP_α was computed more directly resulting in $12\mathcal{N}$ floating point operations, of which $\mathcal{N}$ are divisions!

5 $\mathcal{F}$-statistic of perfectly matched signal

Let us first assume there is a signal present in the data which is perfectly matched by the search-template $\{\mathcal{A}_{\mathrm{s}}, \lambda_{\mathrm{s}}\}$, i.e.

$$\boldsymbol{x}(t) = \boldsymbol{n}(t) + \boldsymbol{s}(t; \mathcal{A}_{\mathrm{s}}, \lambda_{\mathrm{s}}) \,, \tag{120}$$

and the four amplitude-components x_μ defined in (41) are therefore

$$x_\mu(\mathcal{A}_{\mathrm{s}}, \lambda_{\mathrm{s}}) = n_\mu(\lambda_{\mathrm{s}}) + s_\mu(\mathcal{A}_{\mathrm{s}}, \lambda_{\mathrm{s}}) \,, \tag{121}$$

where naturally $n_\mu \equiv (\boldsymbol{n}|\boldsymbol{h}_\mu)$ and $s_\mu \equiv (\boldsymbol{s}|\boldsymbol{h}_\mu)$. With the above assumptions about the noise and the definition of the scalar product (34), one can show the identities

$$E\left[n_\mu\right] = 0 \,, \quad \text{and} \quad E\left[n_\mu\, n_\nu\right] = \mathcal{M}_{\mu\nu} \,, \tag{122}$$

where $E[.]$ denotes the expectation-value. This allows us to obtain further

$$E[x_\mu] = s_\mu \,, \quad \text{and} \quad E[x_\mu\, x_\nu] = \mathcal{M}_{\mu\nu} + s_\mu\, s_\nu \,, \tag{123}$$

i.e. the four random variables x_μ have mean s_μ and covariance $\mathcal{M}_{\mu\nu}$. Using these relations, we can find the expectation value of $2\mathcal{F}$ in the perfectly-matched case as

$$E[2\mathcal{F}] = 4 + \rho^2(0) \,, \tag{124}$$

where $\rho(0)$ is the "optimal" signal-to-noise ratio (SNR), expressible as

$$\rho^2(0) \equiv (\boldsymbol{s}|\boldsymbol{s}) = s_\mu\, \mathcal{M}^{\mu\nu}\, s_\nu = \mathcal{A}_{\mathrm{s}}^\mu\, \mathcal{M}_{\mu\nu}\, \mathcal{A}_{\mathrm{s}}^\nu \,. \tag{125}$$

Combining (24) and (53), this can be written more explicitly as

$$\rho^2(0) = h_0^2 \left(\alpha_1\, A + \alpha_2\, B + 2\alpha_3\, C\right) T\mathcal{S}^{-1} \,. \tag{126}$$

[Note, the difference to (68) of [6] is the use of single-sided noise PSD]. The functions $\alpha_i(\cos\iota, \psi)$ are defined as (with $\eta \equiv \cos\iota$):

$$\alpha_1(\eta, \psi) \equiv \frac{1}{4}(1+\eta^2)^2 \cos^2 2\psi + \eta^2 \sin^2 2\psi \,, \tag{127}$$

$$\alpha_2(\eta, \psi) \equiv \frac{1}{4}(1+\eta^2)^2 \sin^2 2\psi + \eta^2 \cos^2 2\psi \,, \tag{128}$$

$$\alpha_3(\eta, \psi) \equiv \frac{1}{4}(1-\eta^2)^2 \sin 2\psi \, \cos 2\psi \,. \tag{129}$$

5.1 Average SNR^2

It is often useful to compute averaged quantities over the amplitude parameters $\cos\iota, \psi$ and sky-position $\vec{n}$ of the signal. It is easy to see that averaging over $\cos\iota, \psi$ with flat priors, i.e.

$$\langle Q \rangle_{\cos\iota,\psi} \equiv \frac{1}{2} \int_{-1}^{1} d\cos\iota \, \frac{1}{\pi/2} \int_{-\pi/4}^{\pi/4} d\psi \, Q(\cos\iota, \psi) \,, \tag{130}$$

yields

$$\langle \alpha_1 \rangle_{\cos\iota,\psi} = \langle \alpha_2 \rangle_{\cos\iota,\psi} = \frac{2}{5} \,, \quad \langle \alpha_3 \rangle_{\cos\iota,\psi} = 0 \,. \tag{131}$$

The sky-average of A, B, C is a little more involved. From (55) we see that these antenna-pattern coefficients are time-averages and noise-weighted detector averages of a^2, b^2, and $a\,b$ respectively, with the antenna-pattern functions $a(t; \vec{n})$ and $b(t; \vec{n})$ defined in (15). We can therefore change the order of isotropic sky-averaging, defined as

$$\langle Q \rangle_{\vec{n}} \equiv \frac{1}{4\pi} \int_0^{2\pi} d\alpha \int_{-1}^1 d\sin\delta \, Q(\vec{n}) \,, \tag{132}$$

and time- and detector-averaging, i.e. $\langle A \rangle_{\vec{n}} = \langle\langle a^2 \rangle_{\vec{n}}\rangle_S$, so we only need to compute the sky-averages over a^2, b^2 and ab, which will be independent of time and of detector. Without loss of generality we can therefore assume a detector with arms $\hat{n}_1 = (1,0,0)$ and $\hat{n}_2 = (0,1,0)$, and combining (10), (7) and (15), we find

$$2a_0 = \hat{\xi}^1\hat{\xi}^1 - \hat{\xi}^2\hat{\xi}^2 - \hat{\eta}^1\hat{\eta}^1 + \hat{\eta}^2\hat{\eta}^2 \,, \quad b_0 = \hat{\xi}^1\hat{\eta}^1 - \hat{\xi}^2\hat{\eta}^2 \,. \tag{133}$$

Inserting the explicit expressions (17) for $\hat{\xi}, \hat{\eta}$ as a function of skyposition, we obtain

$$a_0^2 = \frac{1}{4}\left(\sin^2\alpha - \cos^2\alpha\right)^2 \left(1 + \sin^2\delta\right)^2 \,,$$
$$b_0^2 = \sin^2 2\alpha \, \sin^2\delta \,, \tag{134}$$

which are easily integrated, and we find

$$\begin{aligned}
\langle A \rangle_{\vec{n}} &= \left\langle a_0^2 \right\rangle_{\vec{n}} = \frac{7}{30} \,, \\
\langle B \rangle_{\vec{n}} &= \left\langle b_0^2 \right\rangle_{\vec{n}} = \frac{1}{6} \,, \\
\langle C \rangle_{\vec{n}} &= \left\langle a_0 \, b_0 \right\rangle_{\vec{n}} = 0 \,.
\end{aligned} \tag{135}$$

and so $\langle A \rangle_{\vec{n}} + \langle B \rangle_{\vec{n}} = \frac{2}{5}$.

Equipped with these averages, we now obtain from (126)

$$\left\langle \rho^2 \right\rangle_{\cos \iota, \psi} = \frac{2}{5} h_0^2 \, (A + B) \, T \mathcal{S}^{-1} \,, \tag{136}$$

$$\left\langle \rho^2 \right\rangle_{\vec{n}} = h_0^2 \left(\frac{7}{30} \alpha_1 + \frac{1}{6} \alpha_2 \right) T \mathcal{S}^{-1} \,, \tag{137}$$

$$\left\langle \rho^2 \right\rangle_{\vec{n}, \cos \iota, \psi} = \frac{4}{25} \, h_0^2 \, T \mathcal{S}^{-1} \,, \tag{138}$$

in agreement with Eq.(93) in [4].

It is sometimes convenient to express the instantaneous "strength" of a signal in the detectors, independently of the observation time and noise floor, and following [2] we define h_{rms} as the rms of the signal strain, averaged over time and detectors, i.e.

$$h_{\mathrm{rms}}^2 \equiv \left\langle s^2 \right\rangle_S = \mathcal{A}_{\mathrm{s}}^{\mu} \left\langle h_\mu h_\nu \right\rangle_S \mathcal{A}_{\mathrm{s}}^{\nu} = \frac{1}{2} \mathcal{A}_{\mathrm{s}}^{\mu} \, \chi_{\mu\nu} \, \mathcal{A}_{\mathrm{s}}^{\nu} \,, \tag{139}$$

in terms of the antenna-pattern matrix $\chi_{\mu\nu}$ defined in (54). Using h_{rms}, the optimal SNR (125) can be written as

$$\rho^2 = (s|s) = 2 T \mathcal{S}^{-1} \, h_{\mathrm{rms}}^2 \,. \tag{140}$$

Comparing this to (126), we find

$$h_{\mathrm{rms}}^2 = \frac{1}{2} h_0^2 \, (\alpha_1 \, A + \alpha_2 \, B + 2\alpha_3 \, C) \,, \tag{141}$$

and averaging over all sky-positions and orientation angles $\cos \iota, \psi$ at fixed amplitude h_0, we get

$$\left\langle h_{\mathrm{rms}}^2 \right\rangle_{\cos \iota, \psi, \vec{n}} = \frac{2}{25} \, h_0^2 \,, \tag{142}$$

in agreement with the expression found in [2].

6 Parameter estimation of the signal

From the expression (80) for the maximum-likelihood amplitudes $\mathcal{A}^\mu$ in terms of the measured F_a, F_b, we can infer the signal-parameters A_+, $A_\times$ (or equivalently h_0, $\cos\iota$) and ψ, ϕ_0, by using (24) and (28), mostly following Yousuke's notes. We compute the two quantities

$$A_s^2 \;\equiv\; \sum_{\mu=1}^{4} (\mathcal{A}^\mu)^2 = A_+^2 + A_\times^2 \,, \tag{143}$$

$$D_a \;\equiv\; \mathcal{A}^1\mathcal{A}^4 - \mathcal{A}^2\mathcal{A}^3 = A_+\,A_\times \,, \tag{144}$$

which can easily be solved for A_+, $A_\times$, namely

$$2A_{+,\times}^2 = A_s^2 \pm \sqrt{A_s^4 - 4D_a^2}\,, \tag{145}$$

where our convention here is $|A_+| \geq |A_\times|$, cf. (28), and therefore the '+' solution is A_+, and the $'-'$ is $A_\times$. The sign of A_+ is always positive by convention (28), while the sign of $A_\times$ is given by the sign of D_a, as can be seen from (144). Note that the discriminant in (145) is also expressible as

$$\mathrm{disc} \equiv \sqrt{A_s^4 - 4\,D_a^2} = A_+^2 - A_\times^2 \geq 0\,. \tag{146}$$

Having computed A_+, $A_\times$, we can now also obtain ψ and ϕ_0, namely defining $\beta \equiv A_\times/A_+$, and

$$b_1 \;\equiv\; \mathcal{A}^4 - \beta\mathcal{A}^1 \,, \tag{147}$$

$$b_2 \;\equiv\; \mathcal{A}^3 + \beta\mathcal{A}^2 \,, \tag{148}$$

$$b_3 \;\equiv\; \beta\mathcal{A}^4 - \mathcal{A}^1 \,, \tag{149}$$

we easily find

$$\psi = \frac{1}{2}\mathrm{atan}\left(\frac{b_1}{b_2}\right) \,. \tag{150}$$

and

$$\phi_0 = \mathrm{atan}\left(\frac{b_2}{b_3}\right) \,. \tag{151}$$

The amplitudes $\mathcal{A}^\mu$ are seen from (24) to be invariant under the following gauge-transformation, namely *simultaneously* $\{\psi \to \pi/2,\ \phi_0 \to \phi_0 + \pi\}$. Applying this twice, and taking account of the trivial gauge-freedom by 2π, this also contains the invariance $\psi \to \psi + \pi$. Note that there is still an overall

sign-ambiguity in the amplitudes $\mathcal{A}^\mu$, which can be determined as follows: compute a 'reconstructed' $\mathcal{A}_r^1$ from (24) using the estimates $A_{+,\times}$ and ψ, ϕ_0, and compare its sign to the original estimate $\mathcal{A}^1$ of (80). If the sign differs, the correct solution is simply found by replacing $\phi_0 \to \phi_0 + \pi$.

In order to fix a unique gauge, we restrict the quadrant of ψ to be $\psi \in [-\pi/4, \pi/4]$ (in accord with the TDS convention), which can always be achieved by the above gauge-transformation, while ϕ_0 remains unconstrained in $\phi_0 \in [0, 2\pi)$.

Converting $A_+, A_\times$ into h_0 and $\mu \equiv \cos\iota$ is done by solving (28), which yields

$$h_0 = A_+ + \sqrt{A_+^2 - A_\times^2}\,, \tag{152}$$

where we only kept the '+' solution, as we must have $h_0 > A_+$ (which can be seen from (28)). Finally, $\mu = \cos\iota$ is simply given by $\cos\iota = A_\times/h_0$.

6.0.1 Errors in parameter-estimation

From earlier considerations, we know that the errors dx_μ satisfy (assuming Gaussian noise):

$$E[dx_\mu\,dx_\nu] = \mathcal{M}_{\mu\nu}\,. \tag{153}$$

As a consequence of (80), we therefore obtain the covariance-matrix of the estimation-errors $d\mathcal{A}^\mu$ as

$$E[d\mathcal{A}^\mu\,d\mathcal{A}^\nu] = \mathcal{M}^{\mu\nu}\,, \tag{154}$$

which corresponds to the Cramér-Rao bound, and $\mathcal{M}^{\mu\nu}$ is seen to be the inverse Fisher-matrix.

For any functions $f_i(\mathcal{A}^\mu)$ of the four amplitudes, we therefore find the error-covariances

$$E[df_i\,df_j] = \frac{\partial f_i}{\partial \mathcal{A}^\mu}\frac{\partial f_j}{\partial \mathcal{A}^\nu}\,E[d\mathcal{A}^\mu\,d\mathcal{A}^\nu] = \frac{\partial f_i}{\partial \mathcal{A}^\mu}\,\mathcal{M}^{\mu\nu}\,\frac{\partial f_j}{\partial \mathcal{A}^\nu}\,. \tag{155}$$

In particular, the absolute errors of f_i are given by $\Delta f_i = \sqrt{E[(df_i)^2]}$.

Generally, let us consider the "output" parameters $B^\mu \equiv (A_+, A_\times, \phi_0, \psi)$ (or alternatively $\hat{B}^\mu \equiv (h_0, \cos\iota, \phi_0, \psi)$). We can then easily obtain from (24) the explicit derivatives

$$J^\mu{}_\nu \equiv \frac{\partial \mathcal{A}^\mu}{\partial B^\nu}\,, \tag{156}$$

and by *numerical* inversion we have $\partial B^\nu / \partial \mathcal{A}^\mu = J^{-1\nu}{}_\mu$. We can therefore directly compute the full covariance matrix of errors dB^μ by using (155), namely

$$E[dB^\mu\,dB^\nu] = J^{-1\mu}{}_\alpha\,J^{-1\nu}{}_\beta\,\mathcal{M}^{\alpha\beta}\,. \tag{157}$$

For the sake of verification of the implementation, we explicitly write the derivatives for the choice of output-parameters $B^\mu = (A_+, A_\times, \phi_0, \psi)$, namely

$$J^\mu{}_\nu = \begin{pmatrix} \cos\phi_0\,\cos 2\psi & -\sin\phi_0\,\sin 2\psi & \mathcal{A}^3 & -2\,\mathcal{A}^2 \\ \cos\phi_0\,\sin 2\psi & \sin\phi_0\,\cos 2\psi & \mathcal{A}^4 & 2\,\mathcal{A}^1 \\ -\sin\phi_0\,\cos 2\psi & -\cos\phi_0\,\sin 2\psi & -\mathcal{A}^1 & -2\,\mathcal{A}^4 \\ -\sin\phi_0\,\sin 2\psi & \cos\phi_0\,\cos 2\psi & -\mathcal{A}^2 & 2\,\mathcal{A}^3 \end{pmatrix} \tag{158}$$

Similarly, for the choice of output-variables $\hat{B}^\mu = (h_0, \cos\iota, \phi_0, \psi)$, we find, using $A_+ = \frac{1}{2}h_0(1 + \cos^2\iota)$ and $A_\times = h_0\,\cos\iota$:

$$\frac{\partial\mathcal{A}^\mu}{\partial h_0} = \frac{\mathcal{A}^\mu}{h_0}, \quad \frac{\partial\mathcal{A}^\mu}{\partial\cos\iota} = \hat{A}^\mu, \tag{159}$$

where

$$\hat{A}^\mu \equiv \mathcal{A}^\mu \,|\, \{A_+ \to A_\times,\ A_\times \to h_0\}, \tag{160}$$

and so we find

$$\hat{J}^\mu{}_\nu = \begin{pmatrix} \mathcal{A}^1/h_0 & \hat{A}^1 & \mathcal{A}^3 & -2\,\mathcal{A}^2 \\ \mathcal{A}^2/h_0 & \hat{A}^2 & \mathcal{A}^4 & 2\,\mathcal{A}^1 \\ \mathcal{A}^3/h_0 & \hat{A}^3 & -\mathcal{A}^1 & -2\,\mathcal{A}^4 \\ \mathcal{A}^4/h_0 & \hat{A}^4 & -\mathcal{A}^2 & 2\,\mathcal{A}^3 \end{pmatrix} \tag{161}$$

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