LASER INTERFEROMETER GRAVITATIONAL WAVE OBSERVATORY
— LIGO —
CALIFORNIA INSTITUTE OF TECHNOLOGY
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Educational Activity **LIGO-T080159-v1-G** **6 November 2008**

‘What Time is It?’:
An Algebra Based Derivation of Time Dilation

Amber L. Stuver

For Mr. R. C. Bowman
Hempfield Area High School, Greensburg, PA
Thank you for showing me the beauty of Relativity with this derivation!

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INTRODUCTION:

This activity is appropriate for **high school** students with skills in **algebraic manipulation** and basic **geometry**. While this activity focuses on mathematics, it also makes connections to physics and physical science by deriving concepts from Einstein's Special Relativity and considering their consequences (as appropriate for high school science). Such interdisciplinary connections offer opportunities to introduce concepts, such as Relativity, that are not normally a part of the curriculum.

Mathematics Concepts:
- Velocity (speed) equation
- Algebra
- Pythagorean Theorem

Physics Connections:
- Velocity (speed)
- Vector addition (although the term vector is not explicitly used)
- Introduction to Special Relativity

Suggested Louisiana GLEs

Mathematics:

Grade 9

Algebra
8. Use order of operations to simplify or rewrite variable expressions (A-1-H) (A-2-H)
9. Model real-life situations using linear expressions, equations, and inequalities (A-1-H) (D-2-H) (P-5-H)
12. Evaluate polynomial expressions for given values of the variable (A-2-H)
13.

Grade 10

Number and Number Relations
1. Simplify and determine the value of radical expressions (N-2-H) (N-7-H)
2. Predict the effect of operations on real numbers (e.g., the quotient of a positive number divided by a positive number less than 1 is greater than the original dividend) (N-3-H) (N-7-H)

Geometry
9. Construct 2- and 3-dimensional figures when given the name, description, or attributes, with and without technology (G-1-H)
12. Apply the Pythagorean theorem in both abstract and real-life settings (G-2-H)

Grades 11–12

Number and Number Relations
2. Evaluate and perform basic operations on expressions containing rational exponents (N-2-H)

Algebra
6. Analyze functions based on zeros, asymptotes, and local and global characteristics of the function (A-3-H)

Science:

Grades 9–12

Science as Inquiry
The Abilities Necessary to Do Scientific Inquiry
5. Utilize mathematics, organizational tools, and graphing skills to solve problems (SI-H-A3)

Physics
(Recommended for Grades 11/12)

Physical Science
Forces and Motion
12. Model scalar and vector quantities (PS-H-E2)
13. Solve for missing variables in kinematic equations relating to actual situations (PS-H-E2)

Name ____

What Time is It?

Einstein's Theory of Relativity is centered on the postulate that the speed of light will be the same no matter how it is measured. We will call the speed of light **c**.

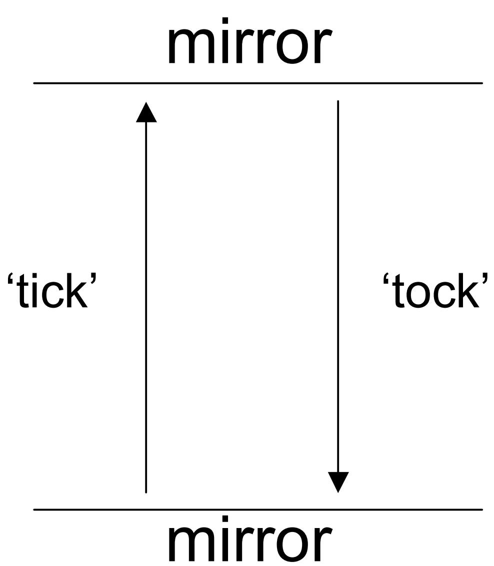
That means that if you are standing still and turn on a flashlight, you will measure the speed of light to be **c**. Now, if you are standing on a train moving at a constant speed, you will also measure the speed of light out of the flashlight to be **c**. But, if you move past a friend who can see your flashlight on the train, what is the speed of light the friend measures? Will they measure the speed of light to be **c** or **c** plus the speed of the train?

It turns out that even though it seems that the friend would measure the speed of light to be **c** plus the speed of the train, they would still measure the speed of light to be **c**!

Because of this, some interesting things happen when light is used to measure time…

The Light-Clock

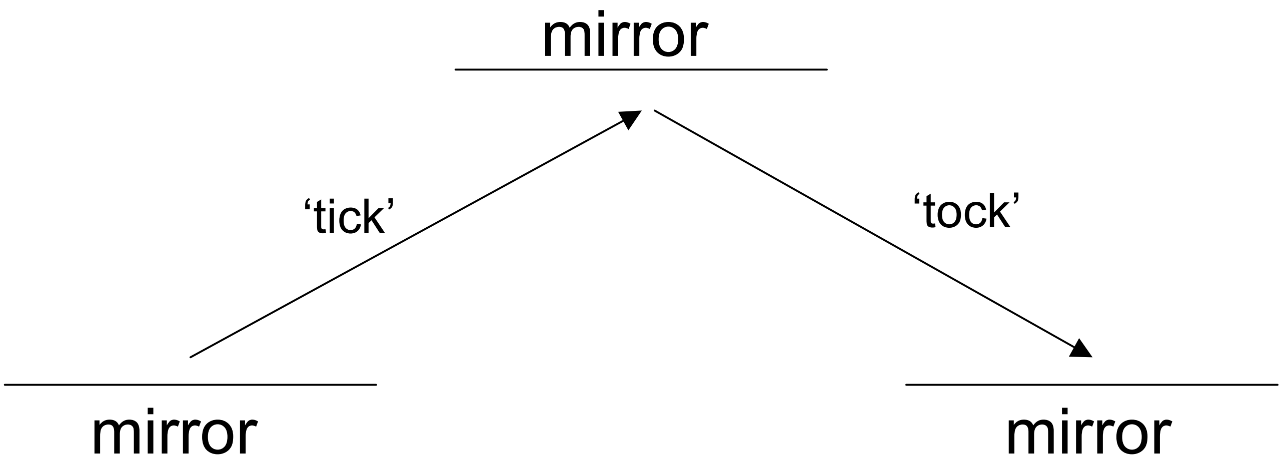
Consider a light-clock made of two mirrors separated by a distance **d**. We will measure time by the interval it takes light to travel from one mirror to the other:



1. How long does it take for light to make the 'tick' trip? Represent this time as $\mathbf{t_o}$.
(Hint: speed = distance/time)

***** $\mathbf{t_o}$**: *This is the time measured by all* <u>*stationary*</u> *(at rest) clocks.***

Now, let us consider a light-clock that is traveling with a speed **v** moving to the right across the paper:



2. How long does it take for the 'tick' trip? Represent this time as **t**.

(Hint: the distance light travels between mirrors is not simply **d** anymore. The mirrors have moved a distance **x** to the right from the time light leaves the bottom mirror to the time it reaches the top mirror [the vertical distance between the mirrors is still **d**]. Use the relation in the last hint to find **x** (in terms of **v** and **t**) and then use the Pythagorean theorem to find the distance of the 'tick' trip.)

*** t:** ***This is the time as measured by a <u>moving</u> clock as observed by an observer at rest.***

3. What is the relationship between $\mathbf{t_o}$ and $\mathbf{t}$?
(Hint: solve the equation found in question 1 for **d** and substitute this into the relationship for **t** [found in question 2].)

4. Simplify this relationship and solve for **t**:

What you have just derived is the equation for *time dilation*. This means that you (as an observer at rest) will always measure the passage of time to be constant ($\mathbf{t_o}$) but if you observe a clock moving by you with speed $\mathbf{v}$, you will measure the moving clock ticking slower ($\mathbf{t}$) than yours.

Let's use the time dilation equation (found in question 4) to see this effect…

5. Imagine you observe a clock traveling at half the speed of light ($\mathbf{v} = 0.5\mathbf{c}$). Solve for $\mathbf{t}$ in terms of $\mathbf{t_o}$ (this is how many more $\mathbf{t_o}$ ticks there are between $\mathbf{t}$ ticks):

6. Now, imagine that you observe a clock traveling at 99% the speed of light ($\mathbf{v} = 0.99\mathbf{c}$). Now how much slower are the clocks at rest ticking compared to yours?

7. What happens to the time clocks traveling at the speed of light ($\mathbf{v} = \mathbf{c}$) keep?

Does time dilation really happen?

This phenomenon of time dilation really happens in our everyday world. However, the speed of light is very large – 299,792,458 meters per second or 186,282 miles per second! That means that when you are traveling in a car going 70 miles per hour on the interstate, you are only traveling at 0.000000104**c**. Because you are still moving so slow compared to the speed of light, you will not notice the very, very small difference between a clock in the car and a clock beside the road (the effect is too small for a calculator to register).

Also, nothing that has mass can ever travel the speed of light and nothing (including light) can ever travel faster than **c**. Because of this, **c** is sometimes referred to as the 'universal speed limit.'

Here is something to think about… If the measurement of time is not constant between clocks moving at different velocities, is the measurement of length constant? (Answer: No! This is called the *Lorenz contraction* and has a familiar form to the time dilation equation we just derived. With the Lorenz contraction, objects seem to be compressed only in a direction of motion.)

Name KEY

What Time is It?

Einstein's Theory of Relativity is centered on the postulate that the speed of light will be the same no matter how it is measured. We will call the speed of light **c**.

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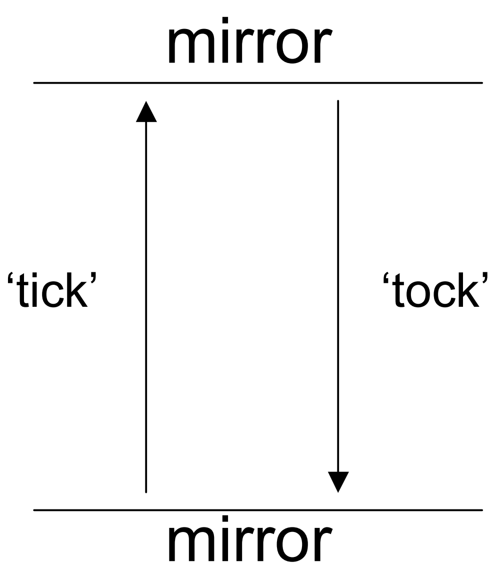
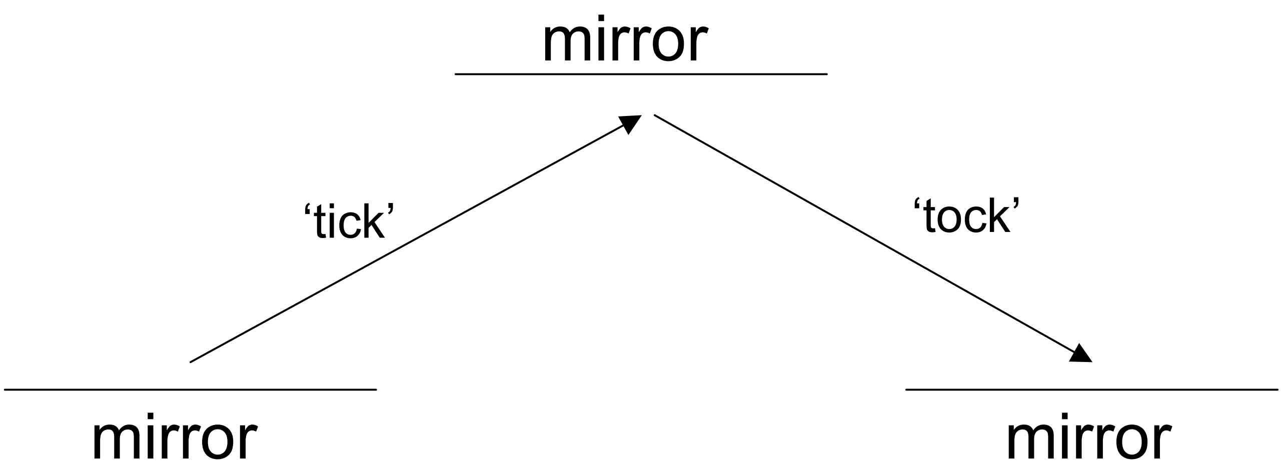


1. How long does it take for light to make the 'tick' trip? Represent this time as $\mathbf{t_o}$.
(Hint: speed = distance/time)

$$\boxed{\begin{aligned} \text{speed} &= c \\ \text{time} &= t_0 \\ c &= \frac{d}{t_0} \\ t_0 &= \frac{d}{c} \end{aligned}}$$

* $\mathbf{t_o}$: ***This is the time measured by all <u>stationary</u> (at rest) clocks.***

Now, let us consider a light-clock that is traveling with a speed **v** moving to the right across the paper:



2. How long does it take for the 'tick' trip? Represent this time as **t**.

(Hint: the distance light travels between mirrors is not simply **d** anymore. The mirrors have moved a distance **x** to the right from the time light leaves the bottom mirror to the time it reaches the top mirror [the vertical distance between the mirrors is still **d**]. Use the relation in the last hint to find **x** (in terms of **v** and **t**) and then use the Pythagorean theorem to find the distance of the 'tick' trip.)

$$v = \frac{x}{t}$$
$$x = vt$$
$$\text{'tick' distance} = \sqrt{(vt)^2 + d^2}$$
$$t = \frac{\text{'tick' distance}}{c}$$
$$t = \frac{\sqrt{(vt)^2 + d^2}}{c}$$

If the speed of light were not constant regardless of the speed of the source, we would need to perform vector addition to combine **c** and **v** in the **t** equation. The resultant combination of **v** and **c** would then produce **t** = $\mathbf{t_o}$ and there would be no effect!

RELATIVE MOTION:
It may also be useful to introduce students to the concept of relative motion (this is what is meant by relativity in the Theory of Relativity). When there are no reference frames, there is no way to distinguish between an observer at rest and an observer moving with a constant velocity. For example, one can use the Earth as a reference frame to define when an object is at rest and when it is moving. But, if two objects were in a void universe (no stars or anything else to use as a reference frame) then there would be no way to tell if you are moving with a constant velocity **v** past an object or if you are at rest and the object is moving past you with a constant velocity **v** in the opposite direction. Therefore, later (in questions 5-7) when time dilation is calculated for different speeds, you can ask the students how an observer moving with the clocks that we originally defined to be "moving" would measure the time kept by the clocks we originally defined to be "stationary." (Answer: The "moving" clocks would measure the same time dilation for our clocks that we measured for theirs while they would not observe any dilation in their clocks!)

*** t:** ***This is the time as measured by a <u>moving</u> clock as observed by an observer at rest.***

3. What is the relationship between $\mathbf{t_o}$ and **t**?

(Hint: solve the equation found in question 1 for **d** and substitute this into the relationship for **t** [found in question 2].)

$$d = ct_0$$

$$t = \frac{\sqrt{(vt)^2 + (ct_0)^2}}{c}$$

4. Simplify this relationship and solve for **t**:

$$t^2 = \frac{(vt)^2 + (ct_0)^2}{c^2} = \frac{v^2t^2}{c^2} + \frac{c^2t_0^2}{c^2} = \frac{v^2t^2}{c^2} + t_0$$

$$t^2 - \frac{v^2t^2}{c^2} = t_0^2$$

$$t^2\left(1 - \frac{v^2}{c^2}\right) = t_0^2$$

$$t^2 = \frac{t_0^2}{1 - \frac{v^2}{c^2}}$$

$$t = \frac{t_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

*** t:** ***This is the time as measured by a <u>moving</u> clock as observed by an observer at rest.***

What you have just derived is the equation for *time dilation*. This means that you (as an observer at rest) will always measure the passage of time to be constant (**t_o**) but if you observe a clock moving by you with speed **v**, you will measure the moving clock ticking slower (**t**) than yours.

Let's use the time dilation equation (found in question 4) to see this effect…

5. Imagine you observe a clock traveling at half the speed of light (**v** = 0.5**c**). Solve for **t** in terms of **t_o** (this is how many more **t_o** ticks there are between **t** ticks):

$$t = \frac{t_0}{\sqrt{1-\frac{0.5^2c^2}{c^2}}} = \frac{t_0}{\sqrt{1-\frac{1}{4}}} = \frac{t_0}{\sqrt{\frac{3}{4}}}$$

$$t = \sqrt{\frac{4}{3}}t_0 \approx 1.155t_0$$

The moving clock (measuring time **t**) appears to be ticking about 15% slower then the stationary clock. Therefore, after traveling for one minute (60 seconds) the moving clock will appear to have only measured about 52 seconds.

6. Now, imagine that you observe a clock traveling at 99% the speed of light (**v** = 0.99**c**). Now how much slower are the clocks at rest ticking compared to yours?

$$t = \frac{t_0}{\sqrt{1-0.99^2}}$$

$$t \approx 7.09t_0$$

The moving clock (**t**) appears to be ticking nearly 610% slower than the stationary clock (**t_o**). That means, that it takes a little over 7 seconds in **t_o** time to tick off one second in **t** time (or for every minute of **t_o** time, only ~8.5 seconds have passed in **t** time).

7. What happens to the time clocks traveling at the speed of light (**v** = **c**) keep?

$$t = \frac{t_0}{\sqrt{1-1}} = \frac{t_0}{0}$$

$$\lim_{v\to c}\frac{t_0}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \infty$$

Time as measured by the moving clock (**t**) appears to have stopped. To interpret time as measured by **t** stopping, explain that there would need to be an infinite number of 'ticks' in **t_o** time to have 1 tick in **t** time. Observing the moving clock with a speed **c** would make it seem like time came to a stop for the moving clock.

Most students are told that the solution to dividing any number by zero is undefined. However, it is worthwhile to introduce them to the concept of a *limit*. To illustrate, take the inverse of increasingly smaller numbers (e.g. 1/1 = 1, 1/0.1 = 10, 1/0.01 = 100, etc.) and ask what the solution is tending to do (it is getting larger). Therefore, if you use the smallest number (zero) the solution will tend to the largest number (infinity).

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Here is something to think about… If the measurement of time is not constant between clocks moving at different velocities, is the measurement of length constant? (Answer: No! This is called the *Lorenz contraction* and has a familiar form to the time dilation equation we just derived. With the Lorenz contraction, objects seem to be compressed only in a direction of motion.)

As a result of the Lorenz contraction, the length of an object is compressed only along the direction of motion as observed from an observer at rest. The expression for this contraction is:

$$\boxed{l = l_0\sqrt{1-\frac{v^2}{c^2}}}$$

where **l** is the length measured by the stationary observer and $\mathbf{l_o}$ is the length of the object when it is at rest.

Therefore, if Superman were flying with his stomach parallel to the ground as he is traditionally pictured, he would appear to be shorter from fingertip to toes but his girth would be unchanged since that length is perpendicular to the direction of motion.

Mass is also changed when traveling near the speed of light. As measured by an observer at rest, a particle moving with speed **v** would appear to have and increased mass as given by:

$$\boxed{m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}}$$

where **m** is the mass measured by the observer at rest and $\mathbf{m_o}$ is the mass of the object when it is at rest.